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Research Article

Evolution of Equity Norms in Small-World Networks

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The topology of interactions has been proved very influential in the results of models based on learning and evolutionary game theory. This paper is aimed at investigating the effect of structures ranging from regular ring lattices to random networks, including small-world networks, in a model focused on property distribution norms. The model considers a fixed and finite population of agents who play the Nash bargaining game repeatedly. Our results show that regular networks promote the emergence of the equity norm, while less-structured networks make possible the appearance of fractious regimes. Additionally, our analysis reveals that the speed of adoption can also be affected by the network structure.

1. Introduction

The emergence, prevalence, and collapse of social norms in groups have attracted scientists from a wide range of disciplines [1–5]. Social norms are appealing because they can act as mechanisms for regulating individual behaviour without the need of a central authority. Social norms are indeed rules that are socially enforced [6, 7].

In the field of Economics, there are plenty of transactions that are commonly regulated by means of habits, repeated interaction, community enforcement, social pressure, trust or reputation, rather than by formal contracts [8]. In particular, among the set of economic interactions that are often modulated by social norms, we can find those that govern the distribution of property within a group. In this context, social norms sometimes favour equitable distributions, but this is not always the case. Social norms can also contribute to the persistence of discriminatory allocations, often supported by observable differences in individual characteristics or group membership, such as gender, race, ethnicity, age, and caste.

Learning game theory provides a useful framework to analyse this type of norms formally [9–12]. Social interactions are modelled as games played by actors that use the history of the game to form expectations or beliefs about the other players' behaviour, and consequently select an appropriate strategy. In general, not all conceivable groups of players within a population will be equally likely to interact, that is, the population may be somewhat structured. In such cases, networks are particularly useful to describe the (sub)set of interactions that may take place: a player can only directly interact in the game with his neighbours in the network.

Relaxing the assumption of global interaction and using sophisticated learning rules usually reduces the analytic tractability of the models and accentuates the relative usefulness of computer simulation for exploration and analysis. Given the explicit correspondence between players in the model and computational entities in the simulation, those players are naturally implemented as agents in an agent-based model [13, 14]. This approach is increasingly used in social and economic models [15–20].

Concretely, in the case of property distribution norms, interactions are often modelled as Nash bargaining games (also known as Nash demand games) [21]. This game consists of two players that have to divide a sum of money among them. Each player demands a share without knowing the demand of the other. If the sum of their individual demands does not exceed the total, the payoff for each player is the amount of money they asked for; however, if the sum of the two demands exceeds the total, they both obtain nothing. Based on this game and its posterior evolutionary version [22], Axtell et al. [23] designed an agent-based model (henceforth AEY's model) to understand the transient and the asymptotic dynamics of the Nash bargaining game in a finite population. In their model, they assumed that the players can make three possible demands only: low (L), medium (M), and high(H) and agents play a noisy best reply to their past experience. The model shows that several persistent regimes different from the equity norm can appear and perpetuate under several learning rules and combination of parameters [23, 24].

AEY's model has been extended to understand the effect of spatial structure. In particular it has been analysed in regular square lattices with a fixed finite population of tagged agents [25]. This study revealed that that the mesoscopic properties of the interaction networks have a significant impact on the diffusion of strategies. However, real networks usually differ from the regular lattice topology [26]. To get deeper insights on the effect of social structure in the diffusion of norms, we analyse AEY's model in networks that may present the so-called *small-world effect* [27], that is, networks where the average distance between agents is relatively short.

The scientific origin of small-world research is attributed to the pioneering work of Pool and Kochen [28] and Milgram [29]. Nevertheless, the puzzle of how to explain the evidence that several real networks are highly clustered (as lattices, e.g.), and at the same time show the small-world effect (like, e.g., random networks), was not envisioned until the seminal work of Watts and Strogatz [30]. In their work they proved that both properties of real networks could be embodied in a simple mathematical network algorithm that interpolates between order and randomness. In the transition, they found a class of networks, *small-world networks*, displaying high clustering and the small-world effect simultaneously.

Models of dynamical systems embedded in small-world networks display different global behaviour due to enhanced signal-propagation speed, computational power, and synchronizability [30, 31]. The effect of this type of topology has been investigated by the academic community, examples of which include the analysis of iterated games such as Hawk-Dove [32, 33], Prisoner's Dilemma [34, 35], Minority Game [36], or Ultimatum Game [37] but also in diffusion models [38].

In this paper we have extended the analysis of dynamic norm diffusion in a population considering AEY's model as a framework. We have analysed the influence of the small-world topology on the results of the game. To this aim, we have organized the paper as follows: first, we briefly explain the extensions and modifications that we have performed on AEY's original model and the main properties of the network generator mechanism based on the Watts-Strogatz algorithm [30]. Next, we analyse the Markovian properties of the unperturbed and perturbed model. Subsequently we characterize the equity norm from an agent's perspective and define the concepts and mechanisms used to analyse the dynamics of the model. In the results section we design and discuss a set of experiments to analyse the frequency of states, the diffusion of the equity norm and the effect of the size of the population. We then finish with the conclusions of this work.

2. The Model

The model proposed in this paper is based on AEY's model [23]. In their abstraction, agents are *randomly paired up* to play a Nash Demand Game [21]. Agents play a game in which each of them can demand three possible portions of a virtual cake (which is a metaphor for a piece of available property): a low (L; 30%), a medium (M; 50%), or a high (H; 70%) share. Agents get what they demand as long as the sum of the two demands is no more than one hundred per cent of the pie. Otherwise, they get nothing (see the payoff matrix used in the model in Table 1).

Agents are endowed with a memory (of size m) in which they store the portion of the pie demanded by their opponents in the last m rounds. In order to make a decision, in AEY's model, an individual chooses the best reply that maximizes the expected payoff considering their past experiences. In our model we consider a simpler decision rule, which dictates that individuals choose the best reply against the most frequent demand in their memory (ties are resolved randomly without bias). This last rule is cognitively less demanding than AEY's and, naturally, it induces different results than those obtained with the original decision rule [24]. The response is assumed "noisy" in the sense that agents may make mistakes in their decisions (or simply experiment from time to time) with small probability. Hence, with probability $(1 - \varepsilon)$ an individual chooses the best reply and with probability ε she chooses one of the three possible demands at random (low, medium, or high with the same probability). Afterwards, agents are paired up again with other agents (chosen at random) and the bargaining process continues.

The influence of some parameters of the model (such as the number of agents, the memory size, the payoff matrix, or the decision rule) has been thoroughly analysed in [24], but that study only considered the situation where every player could interact with any other player (i.e., a complete interaction network).

In a later extension of the model [25], agents were located on a regular square lattice in such a way that one agent could play only with any of her eight surrounding neighbours (Moore neighbourhood). The influence of this topology on the outcome of the Nash Demand

Table 1: Payoff matrix of the Nash demand game.

	H	M	L
H	(0,0)	(0,0)	**(70,30)**
M	(0,0)	**(50,50)**	(50,30)
L	**(30,70)**	(30,50)	(30,30)

Game has been analysed. Results show that the mesoscopic properties of the interaction networks of players with the same tag have an important influence on the diffusion of the emergent norms. The regular square lattice was a first attempt to adapt the AEY's model to more realistic scenarios of human interaction. Nevertheless, regular interaction is still far from most real-life patterns of relation.

The model that we present in this paper is an extension of AEY's model [23] where the agents (In this paper the term "agent"—coming from the ABM literature—and the term "node"—coming from the network theory—could be applied indistinctly. However we prefer to use the term "agent" in the description of the agents' behavior and in the interaction network properties.) are located on a Watts and Strogatz network [30]. Some real human social networks have been proved [26] to be highly clustered (which, roughly speaking, means that your friends' friends are likely to be also your friends) and to have a short average geodesic distance (which means that one can travel from any agent to any other in a small number of hops, the so-called *small-world* effect) (In small-world networks, average path length scales with the logarithm of the number of agents N in the network.). These two properties do not occur simultaneously in regular or in random networks. On the one hand, structured regular networks (such as lattices) are known to be highly clustered but they have long average geodesic distances. On the other hand, random graphs, in which every possible edge occurs independently with a fixed probability, have a short average geodesic distance but they are not highly clustered. Watts and Strogatz [30] designed a mechanism that gives a family of networks which can combine both properties at the same time (In a range of values for the parameter "probability of rewiring."): small-world effect and high clustering.

In the model implemented in this paper we use the Watts-Strogatz algorithm [30] to create networks with different values of the probability of rewiring β. This parameter smoothly interpolates between extreme cases of a regular ring lattice and a random network, traversing "small-world" networks along the way (see Figure 1). As the probability of rewiring increases, the network becomes less regular (and thus less clustered) and the appearance of long-distance links reduces the average geodesic distance.

In our model, the network is created at the beginning of each run and remains fixed thereafter. At each time period, all the agents are selected in a random order to play the Nash Demand Game with one of their (randomly selected) neighbours. It is important to note that each time period consists of N matches, and consequently it is probable that an agent plays more than once in each period.

In the subsequent experiments we will show how the probability of rewiring (and thus the properties of the resulting network) affects the regimes that can be reached in the AEY's game.

Notice that, unlike in previous works [24, 25] and to focus on topological effects, we have not considered the fact that the agents could have tags (i.e., distinguishable labels, such as the colour of the agent, which other agents can identify and condition their decisions on them). In [23, 24] the consideration of tags led to two different games (intra- and intertype

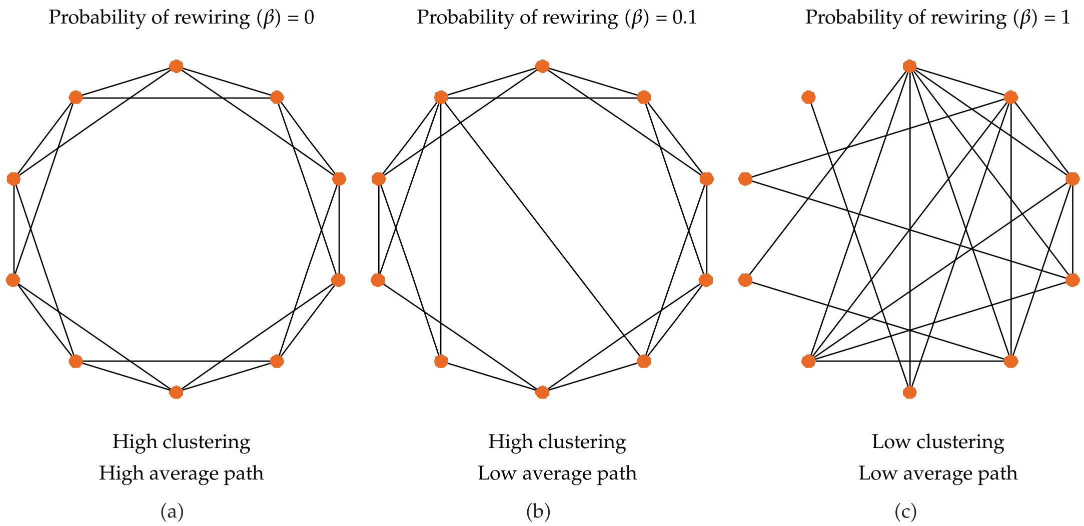


Figure 1: Network structure for several values of the probability of rewiring. N (number of nodes) = 10, k (average degree) = 4.

games). In [25], as the agents were placed on a regular grid, each type of game gave rise to a different network (intra- and intertype networks). In this work, however, the agents are placed on a small-world network. If we had considered tags in this network and had split it into two independent networks (as in [25]), the resulting networks would not necessarily have small-world properties.

3. General Analysis

Before doing a computational exploration of the agent-based model, it is particularly interesting to conduct a previous analysis using the framework of Markov Chains [39], to get useful insights about the expected dynamics and behaviour of the model. In terms of Markovian properties, the system is a time-homogeneous Markov Chain. Considering that the interaction network is fixed and known, the state of the system is completely described by a set of N vectors $\{X_i(t)\}_{i=1}^{N} = \{L_i(t)/m, M_i(t)/m, H_i(t)/m\}_{i=1}^{N}$ of the relative frequencies of opponents' demands for each agent i, where $L_i(t)$, $M_i(t)$ and $H_i(t)$ denote the number of times that agent i's opponents demanded L, M, and H, respectively, in the m most recent interactions just before time t.

As previously explained, an interaction between two agents is modelled as a Nash demand game [21] with three discrete strategies or decisions $\{H, M, L\}$—the corresponding matrix payoff is represented in Table 1. For the one-shot game, there are three pure-strategy Nash equilibriums, one equitable (M, M), and two other (symmetric) inequitable (H, L) and (L, H). These states play a role of focal points that drive the bargaining evolution and explain the asymptotic dynamics of the system.

3.1. The Unperturbed Model

The system dynamics are determined by the presence or absence of errors in agents' decisions. In the absence of errors, that is, *the unperturbed model*, the system has absorbing states in which sooner or later it will be trapped (if we run the model for long enough). These absorbing states are directly related with the pure-strategy Nash equilibriums just mentioned before. Obviously, the interaction network conditions the probabilities of these states to be reached. Assuming global interactions, that is, every agent can play with everybody without any restriction, there is only one absorbing state corresponding to the equitable strategy. This happens when everyone in the population expects the others will demand M, and consequently everyone demands M, so the system ends up reaching an absorbing state, called *equity (EQ) state*. This state is equitable because all agents get equal payoffs and is also efficient (in Pareto sense) because no agent can be made better off without making another agent worse off.

When we assume local interactions, that is, an agent can only play with her neighbours in the interaction network, besides the EQ state, there are two other absorbing states corresponding to the inequitable strategies (In networks with more than one component, there can be more types of absorbing states. In these cases, because each component is independent of the others, the absorbing state of the system is defined as the combination of the absorbing states reached by each component.). This happens when there are two separated groups of agents, in terms of the network, in which the individuals of one of the groups expect the others will demand L and hence they will demand H; and at the same time, the individuals of the other group will expect and demand the complementary decisions. In these cases, the system reaches an absorbing state, which is efficient but not equitable in the payoffs obtained by each agent. Considering the interaction network, these inequitable absorbing states can only happen if the network is bipartite, that is, the network can be divided into two independent subnetworks such that the agents of one of them are only linked to agents of the other, and vice versa. In general, these states are rather improbable due to this topological necessity. For instance (see Figure 2), whenever the interaction network has triplets of agents or any odd cycles, the coordination in the strategies (H, L) is not stable since there is at least one pair of agents with incentives to change their current states. Note in the examples that the expected evolution is a series of continuous changes in agents' strategies between H and L. This unstable pattern (which is directly related with a fractious regime that will be defined below) can persist for very long, until the only absorbing state (i.e., the EQ state) is reached.

3.2. *The Perturbed Model*

When errors are possible in the agents' decisions, (Errors refer to the noisy response explained in the model's section.), that is, *the perturbed model*, the system becomes ergodic, regardless of the interaction network. In this case, there is a unique limiting distribution—and consequently independent of the initial conditions—over the state space which determines the probability of finding the system in each of its states in the long run. This limiting distribution can be estimated by sampling just one simulation run for a sufficiently long time, and computing the fraction of the time that the system spends in each state, that is, the occupancy distribution [39]. For a finite population and global interaction this limiting distribution concentrates on the EQ state, which is the only stochastically stable state [22].

The asymptotic behaviour is not very useful if we want to apply the model to real situations in which the "long run" is a very vague concept. For that reason, it is interesting to pay attention to the transient dynamics too, following the guidelines proposed by Axtell et al. [23]. It can be shown by computer simulation that starting from random initial conditions

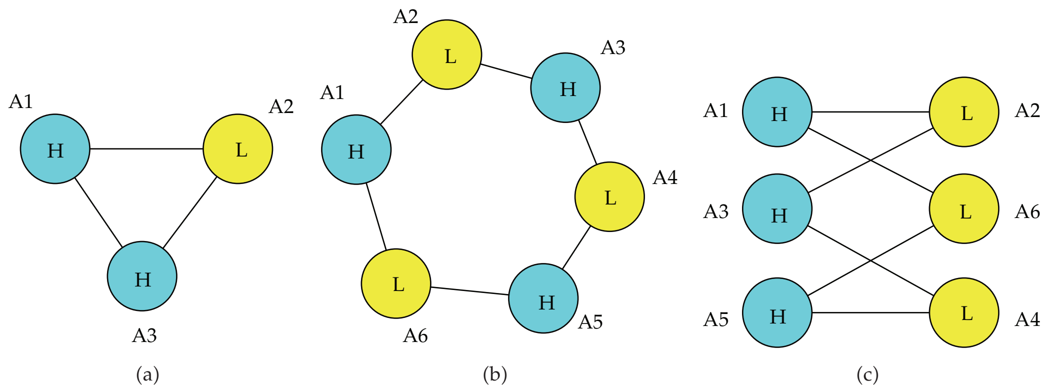


Figure 2: An inequitable state in a triplet (a), and in an even cycle of 6 agents (b) with its corresponding bipartite representation (c). Agents who demand high (H) are depicted in light blue, while those ones who demand low (L) are in light yellow. Note that for inequitable states to be absorbing, the interaction network has to be bipartite (second and third figures), that is, it should not have odd cycles.

the system quickly settles in one of two relevant regimes where it spends a considerable fraction of the time (It is also possible that the system reaches a persistent inequitable regime, where a set of agents persistently demands H against another set of agents that consequently demands L. Notwithstanding the topological conditions to reach this regime is very unlikely as explained in the analysis of the unperturbed model.). One of these persistent regimes, which we call *EQ regime*, is characterized by the EQ state and its surroundings in the state space. In the other, called *fractious (FR) regime* (We use the concept proposed originally by Axtell et al. [23], although there are other names in the literature, such as "fluctuating agents" [40, 41], used to refer the same concept, that is, agents that intermittently change their strategy.), the agents alternate their demands between H and L. Axtell et al. [23] demonstrated that the transition time from this fractious regime to the stochastically stable state can be enormously long—the system presents broken ergodicity [42]—; in fact this time grows exponentially with the number of agents and their memory length.

4. The Equity Norm

All the states and persistent regimes defined in the previous section correspond to a set of Markovian states of the system. However, in order to complete the analysis and discussion of the model, we also need to characterize some of the individual states in which an agent can be from the point of view of the agent's behaviour, that is, which of the three possible decisions $\{L, M, H\}$ an agent will take in the next interaction. We say that an agent follows the *equity norm* whenever she demands M in the next interaction (ignoring the effect of error on decisions). Obviously, this type of behaviour is directly related with the corresponding persistent regimes where the system can settle in the transient period: when all agents follow the equity norm the system reaches the EQ regime (which is equivalent to say that the population follows the equity norm).

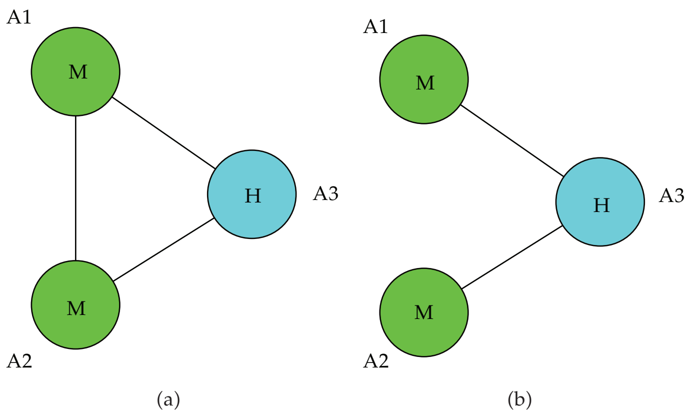


Figure 3: The effect of a mutation within a triplet (a), and on a nonclustered triad (b), when agents are initially coordinated in the equity norm and the mutant changes her demand from M (light green) to H (light blue).

4.1. Clustering Effect on the Equity Norm

It is well known that many real social networks show a significant propensity to form groups or clusters of agents more densely interconnected among them than what could be expected by pure randomness [26, 30, 43]. A typical statistic of this property is the clustering coefficient of a network C [30], measured (4.1) as the average of the clustering coefficients C_i of the agents, that is, the proportion of links between her neighbours (triplets) divided by the number of links that could exist between all of them, which depend directly on the number of connections (degree k_i) of the agents

$$C = \frac{1}{N}\sum_{i}^{N} C_i = \frac{1}{N}\sum_{i}^{N}\frac{2}{k_i(k_i-1)}\#\text{ triplets}_i. \tag{4.1}$$

This measure of network transitivity estimates the probability that two neighbours of any agent have a link between them too, and consequently that they all form a triplet. We have seen that the existence of triplets hinders the stability of the inequitable regime, and, in the case of a state close to them it is very probable that the system falls in the trap of the fractious regime. Besides, it is also interesting to understand the effect of clustering on the equity norm. A simple analysis of the two idealized cases showed in Figure 3 should give us some insights about how clustering influences the stability of this state (in terms of persistence in a finite time period).

The first case represents a triplet of agents initially following the equity norm, when one of them changes (mutates) her demand from M to H. The topology of agents' interactions weakens the state of the mutant, who has incentives to change her strategy back because all her expected opponents demand M. On the contrary, the other two agents do not have incentives to change their current strategy M because it is successful in half of their expected interactions. In the second case, there is the same triplet but without the link between the two agents who demand M. Now, these last agents have incentives to change their current strategy M, making the equity norm less robust against random mutations.

Obviously, the analysis is not so trivial if the network is bigger and much more complex, but the intuition, inferred from these simple examples, is that the equity norm is much more robust against random mutations when agents are clustered than when they are not. Consequently, we should expect that the evolution of the bargaining (under the hypothesis of the model proposed in this work) tends to reach the EQ regime more frequently in networks with higher clustering. The design of experiments and the computer simulations described in the next section aim to confirm this intuition.

4.2. The Diffusion Process of the Equity Norm

The purpose of this section is to describe how the equity norm emerges and spreads across the population in finite time (transient dynamics). In simple and abstract terms, the dynamic process evolves as follows: the population starts from a randomly initialized state; these random initial conditions make it likely that, initially, one or more agents adopt the equity norm and coordinate with each other in small groups that reinforce the norm; if this coordination process occurs quickly, and some of these equity nuclei are able to reach a critical size (which depends on the particular properties of the network they are embedded in), then they will be able to expand their limits and grow, making the equity norm spread across the whole population. Unlike other diffusion phenomena already studied in the literature [38, 44–46], the diffusion process in our model is more difficult to follow, since the adoption mechanism of the norm depends on a learning decision rule and a stochastic response. To overcome this obstacle, we propose an abstraction that captures the essence of the process and allows us to understand the effect of the network structure on the system dynamics more clearly.

In order to do so, we initially define a new unit of analysis called *equity nucleus*, that is, a connected component of the subgraph of agents that follow the equity norm. At any time, there could be none, one, or more equity nuclei; and we will measure their sizes and their clustering coefficients with the purpose of correlating these properties with the posterior evolution of the nuclei: they may grow until they finally invade the population or they may decrease until their disappearance in the transient period.

Second, we need to determine a metric to measure the change in an equity nucleus after a complete interaction at each time period t, that is, every agent plays the Nash demand game with one of her neighbours randomly selected. To that end, we define two new concepts: the inner border and the outer border of a nucleus. Given any equity nucleus in the population (see Figure 4), its *inner border* is the set of agents in the nucleus who have one or more neighbours out of it, that is, neighbours who will not play M in the next interaction; its *outer border* is the set of agents not belonging to the nucleus who have one or more neighbours within it.

Note that any change in a nucleus must involve one of these two borders. A nucleus can grow by adding new members of the outer border who adopt the equity norm. Similarly, a nucleus can decrease as a consequence of losing members of the inner border who leave the norm. Obviously, the real nuclei dynamics might be a little different, since in each time period t there are N individual interactions that can modify these borders in different ways. For example, one interaction could make a border grow and the next interaction could make it decrease. Despite that, this approach is accurate enough for understanding the effect of the network structure.

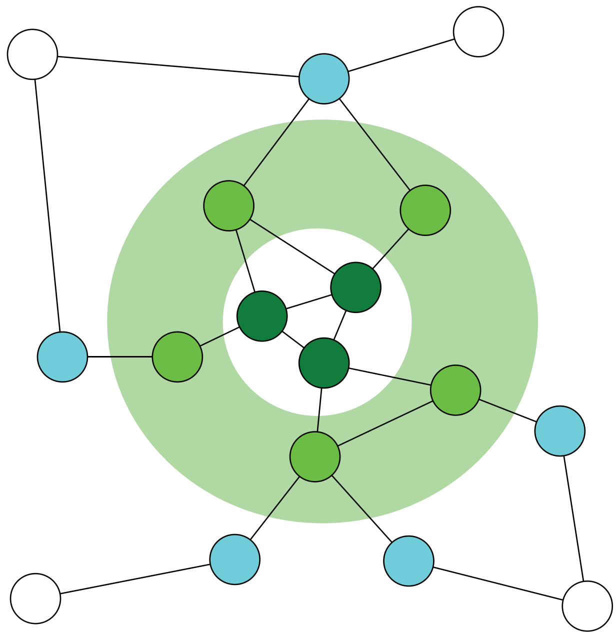


Figure 4: An equity nucleus (dark and light green agents). The inner border is made up of the (light green) agents belonging to the nucleus who have one or more neighbours out of it. The outer border consists of the (light blue) agents not belonging to the nucleus who have one or more neighbours within it.

Finally, we set a procedure to compute all these properties over a simulation run. Before a complete interaction at time period t, we identify all equity nuclei $j \in \{1, \ldots, K\}$ and their borders and compute and collect their sizes and clustering $\{S_j(t), C_j(t)\}_{j=1}^{K}$. After all agents have played the game, we compute the changes in the inner and outer borders of every nucleus. We will use the set of pairs, $\{S_j(t), C_j(t)\}_{j=1}^{K}$ and $\{S_j(t+1), C_j(t+1)\}_{j=1}^{K}$, to infer some conclusions of the expected development of an equity nucleus depending on its size and clustering.

5. Results

5.1. Design of Experiments

In the ABM model proposed, agents are embedded in a small-world interaction network (SWN from now). We have chosen the small-world algorithm by Watts and Strogatz [30] to model the interaction network because it provides a useful framework to study the clustering effect, besides other properties of the network, using only one parameter. Then, the rewiring probability is going to be the main control parameter, which will govern the network creation and its properties. The design of experiments aims to show how this kind of network family affects the system dynamics.

The parameterization of all scenarios studied in this paper corresponds to a model of $N = 100$ agents randomly distributed in a particular instance of the SWN for a fixed rewiring probability, and an average degree equal to 8 (for beta = 0 and degree = 8, the properties of the resulting network are close to the ones of the regular square lattice used in previous research [25]). Each agent is endowed with a memory of length $m = 10$, randomly initialized at the beginning of a simulation run. At a time period t, each agent (selected in a random order) randomly selects one of her neighbours to interact. Both agents decide the best reply against the most frequent demand in their memory. However, with a small probability $\varepsilon = 0.01$ an agent decides randomly between the three possible demands $\{L, M, H\}$. We have sampled 100 simulation runs during $T_f = 2000$ time periods for each combination of parameters. This time is enough for the system to reach a persistent regime (either EQ or FR).

During a simulation run, we say that an agent follows the equity norm strongly whenever she has at least $(1 - \varepsilon) \times m$ instances of M in her memory. Similarly, the system reaches the *FR regime* whenever every agent has at least a combination of $(1-\varepsilon) \times m$ instances of both L and H (Note that the memory vector has a finite number of instances, so we approximate $(1 - \varepsilon) \times m$ to the lower integer and $\varepsilon \times m$ to the higher integer.). Finally, in order to identify an *equity nucleus*, and its inner and outer borders, we apply the equity norm definition and consider that an agent belongs to an equity nucleus if the mode of her memory is M, which is enough to guarantee that the agent will demand M in the next interaction in absence of errors.

5.2. Frequencies of Transient Regimes

As explained before, in the transient dynamics of the system, simulations often reach one of two expected regimes: the EQ regime or the FR regime. The first one corresponds to the emergence of the equity norm, while the second represents a confusing and disordered state in agents' decisions that prevents any coordination in the bargaining. Now, the first question that arouses our interest is to understand how small-world networks condition the emergence of these regimes. To determine this influence we have computed the frequencies of both regimes when the rewiring probability β of the network varies. Figure 5 presents the frequency of the EQ regime reached by a set of simulations at the end of the runs (Note that since the system reaches one of the two transient regimes, the rest of the cases correspond to simulations which ended at the FR regime.).

The first inference that can be made from the results is that the emergence of the EQ regime depends significantly on the rewiring probability, and more concretely on the structure of the interaction network. In the case of regular ring lattices ($\beta = 0$) characterized by the highest values of clustering and path length, the population follows the norm in (almost) all cases. As randomness increases ($\beta > 0$) the networks show lower values of clustering and average path length, and the frequency of the norm decays with them, particularly with the clustering coefficient, being finally quite close to 50% in the extreme case of pure random networks ($\beta = 1$). Moreover, in the well-known small-world range ($0.05 < \beta < 0.1$) (The small-world range is sensible to variations in the rest of the parameters, the size of the network and the average degree of the agents.), characterized by high values of clustering and low values of average path length, the frequency of the norm is nearly the same as in regular ring lattices. Then, it seems that the average path length does not explain the dynamics of the norm in the bargaining model (at least for small networks), which makes sense because the distance between agents, that is, the minimum number of links between two agents, does not seem to play any role in how agents take decisions. This contrasts with the clustering coefficient, which does reflect the characteristics of the neighbourhoods, and is consequently related with the agents' interactions and the bargaining evolution.

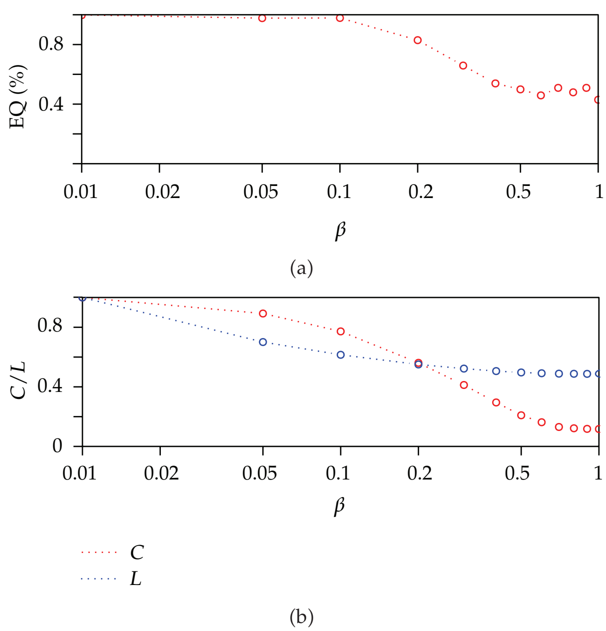


Figure 5: Above, the frequency of the EQ regime reached at the end of the simulations when the rewiring probability β (represented in logarithmic scale on the abscissa axis) varies. Below, the average of the clustering coefficient C and the average path length L of the interaction networks of the simulations (both statistics are normalized dividing each value by the corresponding ones of the regular ring lattice ($\beta = 0$)).

5.3. Equity Norm Diffusion

In this section we characterize the diffusion process of the equity norm. We apply our particular approach based on observing the emergence and evolution of clusters of agents playing the norm-equity nuclei. We also try to correlate the dynamics of these nuclei with their network properties and estimate their expected change. We will see how the rewiring probability of the small-world networks conditions significantly not only the probabilities of the emergence of successful equity nuclei, but also their growing speed over the population.

The diffusion of the equity norm is quite similar to the movement of a wave of adopters in a population embedded in a social network. By randomness, one or more small groups of linked agents start to follow the equity norm (equity nuclei), and depending on their internal structure and the structure of the network that surrounds them, they have greater or lower probabilities of growing successfully by incorporating new members which modify the properties of the nuclei. Overall, if an equity nucleus reaches a critical size with particular properties, it will invade the population, but these properties will depend highly on the parameters of the interaction network.

We have analysed the observed dynamics of the equity nuclei by means of a gradient map obtained through computer simulation data (this procedure has also been used in [47]). The statistical procedure to make this sort of graph is described as follows: we start from the matrix of change $\{S_j(t), C_j(t), S_j(t+1), C_j(t+1)\}_{j=1}^{K}$ that collects the size (S) and the clustering (C) of all equity nuclei before and after a game round, and which have been computed following the way described in previous sections. Each row element can be interpreted as a vector of change of a nucleus in the size-clustering space. Consequently, the matrix of change of a particular parameterization of the model collects a set of vectors of change of equity nuclei that happened in the corresponding simulation runs. We divide the size-clustering space in a regular square lattice. For each cell we compute the vectorial sum of all vectors with initial points included in the cell and represent the resulting vector as an arrow of normalized magnitude (we are only interested in the direction of the expected movement). We additionally colour each square according to the probability that a nucleus of given S and C increases in size, computed as the relative frequency of occurrence in the simulated data. The gradient maps for different rewiring probabilities are shown in Figure 6.

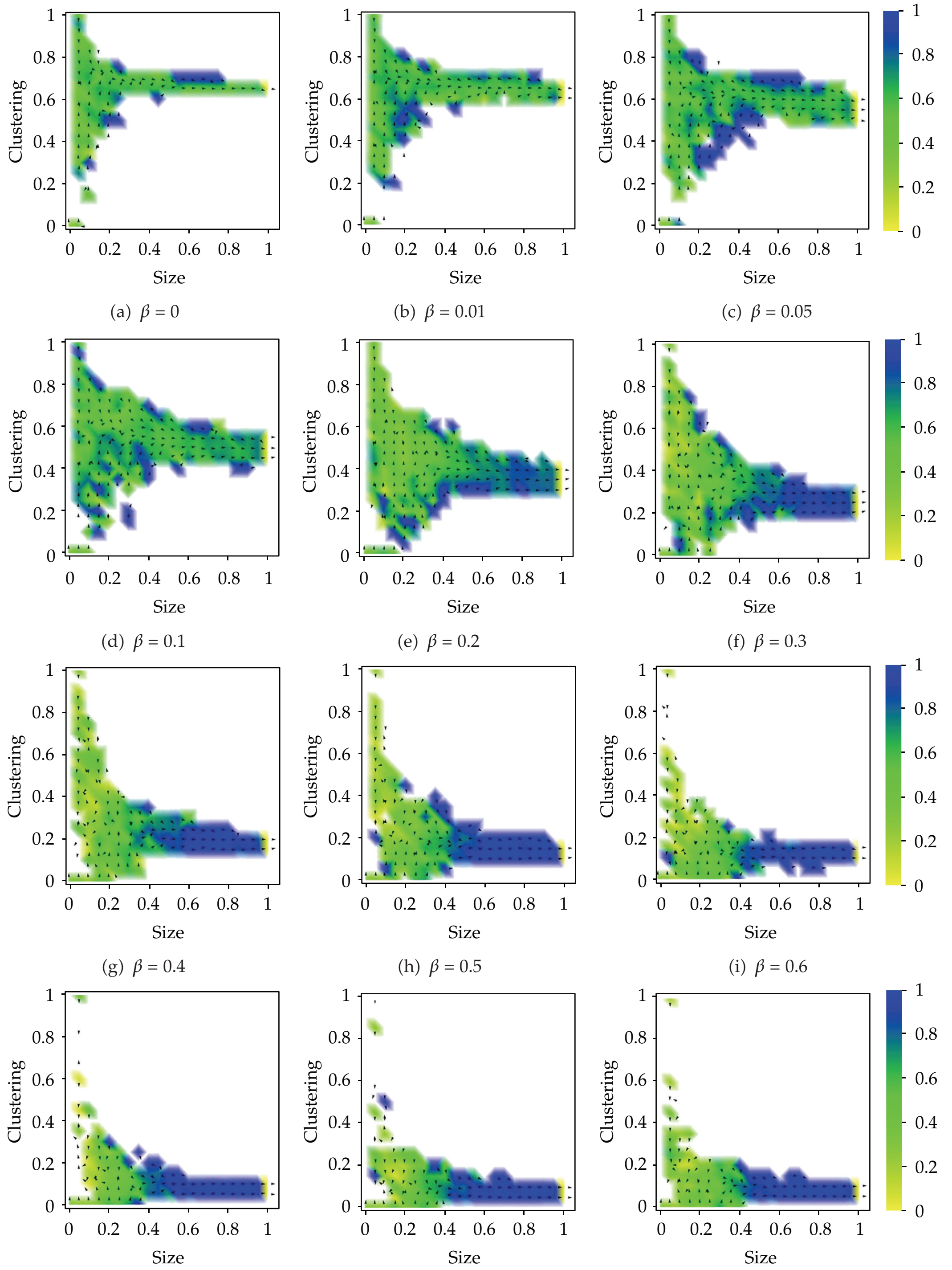
In most cases of Figure 6 we see two different movement regions: a first one corresponding to a developing stage, in which the equity nuclei emerge and grow slowly, and a second one corresponding to a spreading stage, in which the consolidated nuclei grow fast and invade finally the whole population.

For regular ring lattice ($\beta = 0$) and small rewiring probabilities ($\beta \leq 0.05$), the difference between these two stages is not quite clear, and although there is a region in which nuclei emerge, grow, and decay, nuclei do not need to reach a big critical size to consolidate an unstoppable growth. Remember that for all these cases the equity norm is always established, what is explained by the high level of clustering of these networks. Taking into account the regular structure of the interaction network it is not surprising that the spreading of the equity norm in quasiregular ring lattices is very homogeneous and slow: the norm supported by very clustered nuclei advances invading also very clustered subnetworks, and this slows down significantly the diffusion speed reflected in the gradient map by probabilities of growing close to 0.5.

On the other hand, for greater rewiring probabilities ($\beta \geq 0.2$) these two regions are much easier to observe in the gradient map (a green region versus a blue region). In these cases the frequency of the establishment of the equity norm decays with the rewiring probability (see Figure 5), being finally close to 0.5 in pure random networks ($\beta = 1$). The developing stage in which small nuclei grow and decay is represented by a more extensive area (green region), and the critical size necessary to start an unstoppable growth is bigger. Unlike regular ring lattices, once a consolidated nucleus emerges and the equity norm spreads over the whole population (blue region), the diffusion speed is much faster because the norm has to invade less clustered subnetworks.

We can summarize these inferences into the next statements: locally structured networks—in the sense of having more clustering—promote the emergence of the equity norm, while less locally structured networks facilitate the appearance of disordered or fractious states (according to the data of Figure 5); nevertheless, the clustering of the network can slow down the diffusion of the equity norm making more difficult the process of adoption (according to the data of Figure 6). For example, in the case of quasiregular ring lattices, an equity nucleus that invades the whole population always emerges, sooner or later, although the clustering of the network slows down the convergence to the norm. On the contrary, in more random networks, the probability of this event decreases with lower clustering values, although if an equity nucleus succeeds, the speed of the convergence to the norm is much faster.

Figure 7 shows the speed of the diffusion of the equity norm, that is, the minimum time necessary for the whole population to converge to the norm. In accordance with previous results, the times of convergence are significantly higher for quasiregular ring lattices ($\beta \leq 0.1$) than for more random networks ($\beta \geq 0.2$), as a consequence of the higher resistance to adopt the norm that clustered groups of non-equity agents show in the convergence



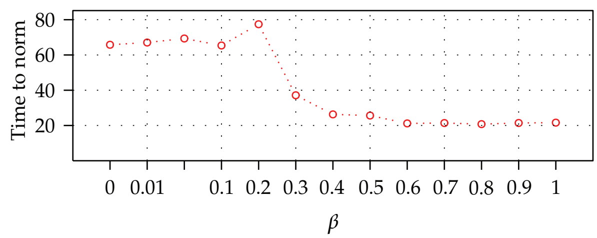
(a) $\beta = 0$ (b) $\beta = 0.01$ (c) $\beta = 0.05$

(d) $\beta = 0.1$ (e) $\beta = 0.2$ (f) $\beta = 0.3$

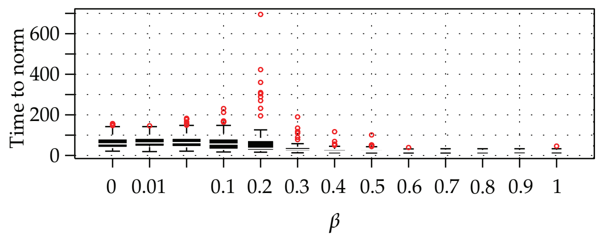
(g) $\beta = 0.4$ (h) $\beta = 0.5$ (i) $\beta = 0.6$

(j) $\beta = 0.7$ (k) $\beta = 0.9$ (l) $\beta = 1$

Figure 6: Gradient maps of the observed dynamics of equity nuclei for different values of the rewiring probability. Each arrow represents the direction of the change in the size-clustering space, while the colour of the cells is the probability of growing in size, which can be interpreted as a measure of the speed in nuclei growth. When there is no simulation data for a particular combination of size and clustering, the corresponding square cell of the map is coloured in white.



(a)



(b)

Figure 7: Above, the average of the time of convergence to the equity norm, and below the corresponding boxplot, when the rewiring probability β varies. Note that the results are computed with the simulation runs that ended in the equity norm a number that decreases with the rewiring probability (see Figure 5). The range $[0.1 < \beta < 0.2]$ separates two system behaviours: a first one characterized by high clustering networks in which the system always reaches the EQ regime in the simulation time, and a second one characterized by significantly lower values of clustering in which the system alternates between the EQ regime and the FR regime but in cases when the system reaches the EQ regime it takes it lower times of convergence.

process. Note that the results represented graphically in Figure 7 correspond to the times of convergence of all simulation runs that reached the EQ regime; this percentage adds up to 100% in the case of quasiregular ring lattices, but decreases with the rewiring probability from 80% ($\beta = 0.2$) to quite less than 50% ($\beta = 1$). There seems to be a sort of phase change in the range $[0.1 < \beta < 0.2]$, coincident with a significant drop of the clustering levels of the network (see Figure 5), that explains the significant dispersion in the times of convergence for $\beta = 0.2$. For rewiring probabilities greater than this value, we can infer that the time of convergence to the equity norm decreases with β, reaching the lowest value in pure random networks ($\beta = 1$).

Finally, we have extended the computing analysis of the bargaining model by running other simulations in order to check the sensitivity of the results to changes in other parameters, mainly in the size of the population. Figure 8 shows the frequency of the EQ regime for different population sizes. Overall, these results are not qualitative different from the previous ones obtained with a population of 100 agents. Nevertheless, it is interesting to observe that the growth of the size of the network seems to promote the establishment of the equity norm in even more random realizations. For example, for $\beta = 0.2$ and a number of agents greater than 400, the equity norm is reached in almost 100% of the simulation runs, in contrast to the 80% reached by populations of 100 agents.

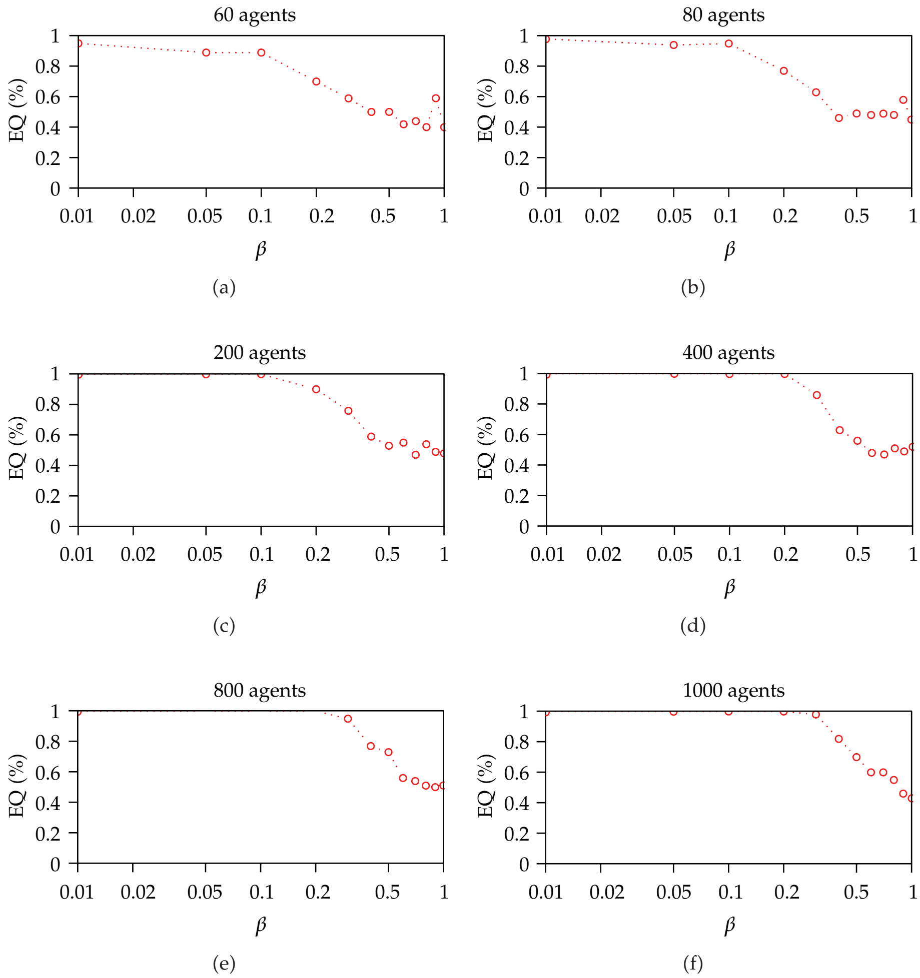


Figure 8: The frequency of the EQ regime for different sizes of the interaction network. The qualitative results do not differ from those ones analysed in the previous sections for a population of 100 agents. Regular ring lattices and networks with low rewiring probability support the dominance of the equity norm, while more random networks contribute to the emergence of fractious states.

6. Conclusions

In this work we have addressed the effect of topologies of interaction ranging from regular ring lattices to random networks, including small-world networks on the Nash demand game in a finite population of agents. Our analysis shows that locally structured networks—in the sense of having more clustering—promote the emergence of the equity norm, while less locally structured networks facilitate the appearance of disordered or fractious states. At the same time, results indicate that the clustering of the network can slow down the diffusion of the equity norm making more difficult the process of adoption. For example, in the case of quasiregular ring lattices, an equity nucleus that invades the whole population always emerges, sooner or later, although the clustering of the network slows down the convergence to the norm. On the contrary, in more random networks, the probability of this event decreases with lower clustering values; although if an equity nucleus succeeds, the speed of the convergence to the norm is much faster. Our findings seem robust to the size of population and corroborate the influence of some properties of the interaction structure in learning and evolutionary games.

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