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Stability of Strict Equilibria in Best Experienced Payoff Dynamics: Simple Formulas and Applications*

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Dedicated to the memory of Bill Sandholm

Abstract

We consider a family of population game dynamics known as Best Experienced Payoff Dynamics. Under these dynamics, when agents are given the opportunity to revise their strategy, they test some of their possible strategies a fixed number of times. Crucially, each strategy is tested against a new randomly drawn set of opponents. The revising agent then chooses the strategy whose total payoff was highest in the test, breaking ties according to a given tie-breaking rule. Strict Nash equilibria are rest points of these dynamics, but need not be stable. We provide some simple formulas and algorithms to determine the stability or instability of strict Nash equilibria. *JEL* classification numbers: C72, C73.

Keywords: Best Experienced Payoff; Procedural rationality; Payoff-sampling dynamics; Stability

1. Introduction

Most dynamics in Evolutionary Game Theory can be neatly seen as a combination of a *population game* and a *revision protocol* (Sandholm, 2010). The *population game* assigns

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to each population state a vector of payoffs, *one* for each strategy in the population. The *revision protocol* specifies how agents, using the payoff assigned to each strategy, update their current strategy. A crucial assumption embedded in this framework is that, at any population state, there is *one single* payoff assigned to each strategy. In population games where agents are matched to play a symmetric normal form game, the payoff assigned to each strategy is often the expected payoff the agent will obtain when using that strategy. But how can agents know this expected payoff? Unless there is complete matching, agents somehow know the exact population state, or agents are explicitly communicated the precise expected payoff for each strategy, it seems unrealistic to assume that they will all share exactly the same expectations for any given strategy. From this point of view, it is noteworthy that many evolutionary dynamics from the economics literature are informationally demanding in one important respect: they require agents to be fully informed about the population's current aggregate behavior. This assumption seems rather strong in the large-population contexts to which evolutionary models are most naturally applied.

In many situations, it seems more natural to assume that agents acquire information by interacting with only a *sample* of the population, rather than assuming that they have access to accurate statistics of the whole population. There are two distinct lines of research that follow this approach while keeping the assumption that agents respond optimally to the information they have.

The first line assumes that agents take samples of the actions being played in the population, and they use these samples to make inferences about the distribution of actions in the whole population, and to best respond to the estimates thus formed. This is the approach followed by Sandholm (2001), Kosfeld et al. (2002), Osborne and Rubinstein (2003), Kreindler and Young (2013), Oyama et al. (2015), Heller and Mohlin (2018), Salant and Cherry (2020), and Sawa and Wu (2021). Under this approach, note that agents must be aware of the population game they are playing, so they can best reply to their point estimates of the population distribution of actions.[1]

A second approach –significantly less demanding on agents' informational and computational skills– was pioneered by Osborne and Rubinstein (1998) and Sethi (2000). Here, revising agents try out a subset of the available strategies by playing them against randomly drawn counterparts, and then choose the strategy that performed best in the test. Crucially, each game is played against new randomly drawn counterparts, so sub-optimal strategies may be selected in the test if they happened to be lucky in the random sampling

[1] The dynamics induced by this protocol have been termed *sampling best response* dynamics (see e.g. Oyama et al. (2015)) and *action-sampling* dynamics (see e.g. Sethi (2021); Arigapudi et al. (2021, 2022)).

of co-players. In this approach, note that agents do not even need to know that they are playing a game. Agents who follow this revision protocol have been called *procedurally rational agents* (Osborne and Rubinstein, 1998), and the evolutionary dynamics they produce are the so-called *payoff-sampling* dynamics (Sethi, 2021; Arigapudi et al., 2021, 2022) or, more generally, *Best Experienced Payoff* (BEP) dynamics (Sandholm et al., 2019).[2] These dynamics are the main object of study in this paper.

The procedurally rational agents described before, and their associated BEP dynamics and equilibria, have been used in a variety of applications including consumer choice procedures and product pricing strategies (Spiegler, 2006a), markets with asymmetric information (Spiegler, 2006b), trust and delegation of control (Rowthorn and Sethi, 2008), the Traveler's Dilemma (Berkemer, 2008), market entry (Chmura and Güth, 2011), ultimatum bargaining (Miękisz and Ramsza, 2013), use of common-pool resources (Cárdenas et al., 2015), contributions to public goods (Mantilla et al., 2020), the Centipede game (Sandholm et al., 2019; Izquierdo and Izquierdo, 2021), the Prisoner's Dilemma (Arigapudi et al., 2021), and coordination problems (Izquierdo et al., in press). Sethi (2021) studies the equilibria of these processes in symmetric, finitely repeated games, with several applications.

Under BEP dynamics, strict Nash equilibria of a game correspond to states that are rest points, but they may not be stable. Sandholm et al. (2020), building on Sethi's (2000) pioneering work, provide several sufficient conditions for instability and for asymptotic stability of strict equilibria under BEP dynamics. Arigapudi et al. (2021) refine one of the most general sufficient stability conditions in Sandholm et al. (2020), providing a tighter one. While many of the stability and instability conditions in Sandholm et al. (2020) are really simple and can be immediately checked from the payoffs of the game, the most general stability condition (Theorem 2 II in Arigapudi et al. (2021)), and the most general instability condition (Proposition 5.4 in Sandholm et al. (2020)) are –if taken at face value– actually difficult to check, as they state a condition over all sets in a certain power set, or require finding a subset of strategies that satisfies some condition. Here we show that these general stability and instability conditions can be checked by conducting a simple analysis, whose complexity is equivalent to carrying out an iterated elimination of dominated strategies, and which admits a simple interpretation. We also provide some tighter tests for specific BEP dynamics.

The rest of the paper is structured as follows. Section 2 contains a short introduction to Best Experienced Payoff processes and their dynamics. In Section 3 we summarize previous results on stability of strict equilibria, indicating also the new contributions in

[2] The term *payoff-sampling* dynamics is used when revising agents test *all* their available actions. Sandholm et al. (2019) generalized payoff-sampling dynamics, allowing revising agents to consider *subsets* of their available actions. This generalization led to the so-called family of *Best Experienced Payoff* (BEP) dynamics.

this paper. Section 4 presents the new stability tests and formulas, Section 5 shows an application of our results to tacit coordination games, and in Section 6 we state some conclusions. The proofs, and some additional information, have been grouped in an appendix. All figures in this paper can be easily replicated with open-source freely available software which also performs exact computations of rest points and exact linearization analyses (*EvoDyn-3s* (Izquierdo et al., 2018) for figures 1-4 and BEP-TCG (Izquierdo and Izquierdo, 2022) for figures 7-8).

2. Best experienced payoff protocols and dynamics

For notational simplicity, we keep our presentation to p-player symmetric games played in one population, but all our results can be easily extended to asymmetric games played in p populations. Following Sandholm et al. (2020), we consider a unit-mass population of agents who are matched to play a symmetric p-player normal form game $G = \{S, U\}$. This game is defined by a strategy set $S = \{1, \ldots, n\}$, and a payoff function $U \colon S^p \to \mathbb{R}$, where $U(i; j_1, \ldots, j_{p-1})$ represents the payoff obtained by a strategy i player whose opponents play strategies $j_1, \ldots, j_{p-1}$. Our symmetry assumption requires that the value of U not depend on the ordering of the last $p-1$ arguments. When $p = 2$, we sometimes write U_{ij} instead of $U(i; j)$.

Aggregate behavior in the population is described by a *population state* x in the simplex $X = \{x \in \mathbb{R}^n_+ \colon \sum_{i \in S} x_i = 1\}$, with x_i representing the fraction of agents in the population using strategy $i \in S$. The standard basis vector $e_i \in X$ represents the pure (monomorphic) state at which all agents play strategy i.

We consider Best Experienced Payoff (BEP) protocols defined by a triple (τ, κ, β). Under BEP protocols, agents occasionally revise their current strategy by conducting tests of alternative strategies.

The first parameter, namely the *test-set rule* τ, indicates how the set of strategies to be tested is chosen. Specifically, here we consider the test-set rule τ^α, under which the revising agent, when considering whether to change his current strategy, will also test other $\alpha - 1$ randomly selected strategies in S (besides testing his current strategy). Naturally, $\alpha \in \mathbb{N}$ and $1 < \alpha \leq n$. If all the strategies in S are tested, i.e. if $\alpha = n$, then we have the test-all rule, denoted by τ^{all}.

The second parameter, called the *number of trials* $\kappa \in \mathbb{N}$, specifies the number of times that each strategy will be played in the test. Thus, each strategy in the test set will be played by the revising agent over κ matches, with each match requiring a new independent sampling of $p-1$ co-players.

The last parameter in the BEP protocol, namely the *tie-breaking rule* β, indicates the rule used to decide which strategy is selected when the best result (i.e. the greatest total payoff) in the tests is obtained by more than one strategy. We will omit the last parameter when our results are independent of the tie-breaking rule. Otherwise, we will focus on two tie-breaking rules. The uniform-if-tie rule, β^{unif}, selects any of the strategies that obtain the best total payoff in the tests, each of these strategies with equal probability. This is the rule that has been considered in almost all cases in the literature. The stick-if-tie rule, β^{stick}, chooses to keep using the current strategy if it obtains the best total payoff in the tests, and, otherwise, it breaks ties by random uniform selection among the strategies that obtained the best total payoff.

Well-known results of Benaïm and Weibull (2003) show that the behavior of a large but finite population following the procedure above is closely approximated by the solution of the associated *mean dynamic*, a differential equation which describes the expected motion of the population from each state. This mean dynamic for BEP processes is (Sethi, 2000):

$$\dot{x}_i = w_i(x) - x_i \tag{1}$$

where $w_i(x)$ is the probability with which strategy i is selected by a revising agent, i.e., the probability that it is tested, it obtains the best total payoff, and, if there are ties, it is selected by the tie-breaking rule. The calculation of the term $w_i(x)$, i.e. the mean dynamic, for BEP$(\tau^{\alpha}, \kappa, \beta)$ processes, was formalized by Sandholm et al. (2020).

3. Stability and instability under BEP dynamics. Antecedents and contribution

3.1 Background on stability and linear stability

Consider a C^1 differential equation $\dot{x} = V(x)$ defined on X whose forward solutions $(x(t))_{t\geq 0}$ do not leave X. State x^* is a *rest point* or equilibrium of the dynamics if $V(x^*) = 0$, so that the unique solution starting from x^* is stationary.

A rest point x^* is *Lyapunov stable* if for every neighborhood O of x^*, there exists a neighborhood O' of x^* such that every forward solution that starts in $O' \cap X$ is contained in O. If x^* is not Lyapunov stable it is *unstable*.

A rest point x^* is *attracting* if there is a neighborhood O of x^* such that all solutions that start in $O \cap X$ converge to x^*. If a rest point x^* is Lyapunov stable and attracting, it is *asymptotically stable*. In this case, the maximal (relatively) open[3] set of states in X from which solutions converge to x^* is called the *basin* of attraction of x^*. If the basin of attraction of x^* contains int(X), we call x^* *almost globally asymptotically stable*; if it is X itself, we call x^* *globally asymptotically stable*.

By the definition of the derivative, the value of V in a (relative) neighborhood $O \cap X$ of a rest point x^* can be approximated via

$$V(x) = \mathbf{0} + DV(x^*)(x - x^*) + o(|x - x^*|)$$

where $DV(x^*)$ is the Jacobian matrix of V (more precisely, the Jacobian of a C^1 extension of V to R^n such that the first-order partial derivatives of the component functions of the extension are defined at x^*) evaluated at state x^*. The stability of x^* can be analyzed by considering the eigenvalues of $DV(x^*)$ corresponding to those eigenvectors lying in the tangent space $TX = \{z \in \mathbb{R}^n : \sum_i z_i = 0\}$. If all such eigenvalues have negative real parts, then x^* is linearly stable. If any of those eigenvalues has positive real part, then x^* is linearly unstable. A linearly stable rest point is asymptotically stable, and solutions starting near the rest point converge to it at an exponential rate (Perko, 2001; Sandholm, 2010).

3.2 Linear stability analysis of strict Nash equilibria under BEP dynamics

Focusing now on the BEP dynamics (1), consider a strict strategy s in a symmetric p-player game, i.e., a strategy s such that the strategy profile $(s, s, ..., s)$ is a strict Nash equilibrium of the game. Following Osborne and Rubinstein's (1998) pioneering study of rest points of the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic, and Sethi's (2000) stability analysis of the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, Sandholm et al. (2020) show that the linear stability analysis of a strict Nash equilibrium state e_s – a monomorphic state where all players use the same strict strategy s – under any BEP(τ, κ, β) dynamic, can be reduced to the analysis of an $n \times n$ matrix $V^{\kappa,s} = (v_{ij}^{\kappa,s})$ of total payoffs $v_{ij}^{\kappa,s}$, defined by

$$v_{ij}^{\kappa,s} = (\kappa - 1)U(i; s, s, ..., s) + U(i; j, s, ..., s)$$

To simplify the notation, we will drop the superindex s when it is clear that we are refering to a specific equilibrium strategy s, in which case we will use V^κ and v_{ij}^κ. The Jacobian of the dynamics at the equilibrium e_s can be calculated from the terms in V^κ. The term v_{ij}^κ is the total payoff to strategy i when, over its κ trials, it meets exclusively players using the strict Nash strategy s, except in one trial, where exactly one of the $(p - 1)$ co-players

[3] A set is relatively open in X if it is the intersection of X with an open set in $\mathbb{R}^n$.

uses strategy j. The reason why these are the only relevant payoffs for a linear stability analysis is that, in the proximity of the strict equilibrium, where $x_s = 1 - \epsilon$, the probability of any random sample of $\alpha\,\kappa\,(p-1)$ co-players with more than one co-player choosing a strategy other than s is $O(\epsilon^2)$.

Thus, when α strategies are tested, the relevant sampling events –those whose probability is $O(1)$ or $O(\epsilon)$, but is not $O(\epsilon^2)$– are:

i) Those in which all the $\alpha\,\kappa\,(p-1)$ randomly sampled co-players use strategy s. In this case, a test of strategy s provides the total payoff v^{κ}_{ss} and a test of strategy $i \neq s$ provides the total payoff v^{κ}_{is}. Since s is a strict Nash strategy, $v^{\kappa}_{ss} > v^{\kappa}_{is}$, so, if strategy s is in the test set, then it will be selected.

ii) Those in which all but one of the sampled co-players use strategy s and exactly one co-player (the "deviating co-player") uses strategy $j \neq s$. Assuming all strategies are tested:

- If, in a battery of tests (for which $n\,\kappa\,(p-1)$ co-players are sampled), the single deviating co-player using strategy j is met when testing strategy s, the total payoffs in the battery of tests are v^{κ}_{sj} (when testing s) and $\{v^{\kappa}_{is}\}_{i \in S \smallsetminus \{s\}}$ (when testing the other strategies). Defining $S_2 \equiv \operatorname{argmax}_{i \neq s} v^{\kappa}_{is} = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$, we have that either the selected strategy belongs to S_2, or the selected strategy is s, depending on the comparison of v^{κ}_{sj} and $v^{\kappa}_{ts} \equiv \max_{i \neq s} v^{\kappa}_{is}$. In case of equality, the tie-breaking rule would apply.
- If the deviating co-player is met when testing strategy $i \neq s$, the total payoffs are v^{κ}_{ss}, v^{κ}_{ij} and $\{v^{\kappa}_{ks}\}_{k \in S \smallsetminus \{s,i\}}$. Since every element in $\{v^{\kappa}_{ks}\}_{k \in S \smallsetminus \{s,i\}}$ is less than v^{κ}_{ss}, the selected strategy is either s or i, depending on the comparison of v^{κ}_{ss} and v^{κ}_{ij}. In case of equality, the tie-breaking rule would apply.

To analyze the stability of a strict equilibrium state e_s, Sandholm et al. (2020) consider a change of variables that takes e_s to the origin $\mathbf{0}$ (by eliminating the coordinate x_s, given that $\sum_{i=1}^{n} x_i = 1$) and show that the Jacobian of the dynamics at the origin is $DW(\mathbf{0}) = DW^{+}(\mathbf{0}) - I_{(n-1)}$, where $DW^{+}(\mathbf{0})$ is a matrix of non-negative terms that can be easily calculated from the terms in V^{κ}, following the previous discussion.

3.3 Instability results

A series of instability results (i.e. sufficient conditions for instability) can be derived from the analysis of V^{κ} by considering that the Perron-Frobenius eigenvalue of $DW^{+}(\mathbf{0})$ is at least as large as the Perron-Frobenius eigenvalue of any principal submatrix of $DW^{+}(\mathbf{0})$, which is in turn bounded from below by the minimum sum of the elements in each of its columns (or rows). If the Perron-Frobenius eigenvalue of $DW^{+}(\mathbf{0})$ is greater than 1, then $DW(\mathbf{0})$ has a real positive eigenvalue[4] and, consequently, e_s is unstable. A general condition that guarantees instability following this approach is provided by Proposition 5.4 (ii) in Sandholm et al. (2020), which states that e_s is linearly unstable under any BEP($\tau^{\alpha}, \kappa, \beta$) dynamic if, for some nonempty $J \subseteq S \setminus \{s\}$,

$$(2) \qquad (p-1)\kappa \frac{\alpha-1}{n-1}\left(\sum_{i \in J} \mathbf{1}[v^{\kappa}_{ij} > v^{\kappa}_{ss}] + \mathbf{1}[S_2 \subseteq J]\,\mathbf{1}[v^{\kappa}_{sj} < v^{\kappa}_{ts}]\right) > 1 \text{ for all } j \in J,$$

where $\mathbf{1}[\cdot]$ denotes a Boolean function that takes the value 1 if the condition in the brackets is met, and the value 0 otherwise. Under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics (i.e., for $\alpha = n$) and given a subset of strategies $J \subseteq S \setminus \{s\}$, this result considers a tight bound on the column sums of the submatrix of $DW^{+}(\mathbf{0})$ corresponding to the strategies in J,[5] and it is, up to our knowledge, the most general available result that guarantees instability under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics (for any tie-breaking rule) with either $\kappa > 1$ or $p > 2$. And the result applies to BEP($\tau^{\alpha}, \kappa, \beta$) dynamics as well.

3.4 Stability results

A series of stability results (i.e. sufficient conditions for stability) can also be derived from the analysis of V^{κ} by considering that, if $DW^{+}(\mathbf{0})$ is a triangular matrix, its eigenvalues are its diagonal elements. If the eigenvalues λ_i of $DW^{+}(\mathbf{0})$ are all less than one, then the eigenvalues of $DW(\mathbf{0})$, which are $\lambda'_i = \lambda_i - 1$, are all negative and, consequently, e_s is stable. This can be used to show, for instance, that, under any BEP($\tau^{\alpha}, \kappa, \beta$) dynamics, any strict equilibrium state is asymptotically stable if the number of trials is larger than a certain threshold (Sandholm et al., 2020, Corollary 5.8).

Under BEP($\tau^{\text{all}}, \kappa$) dynamics, the most general condition that guarantees that the Jacobian of $DW^{+}(\mathbf{0})$ can be arranged as a triangular matrix whose diagonal elements are 0 is the existence of an ordering of the strategies in S such that, for all $i, j \neq s$ with $i \geq j$ we have: $v^{\kappa}_{ss} > v^{\kappa}_{ij}$ and, if $i \in S_2$, $v^{\kappa}_{sj} > v^{\kappa}_{is}$. This is a refinement of Proposition 5.9 in Sandholm et al. (2020) that can be shown to be equivalent to the sufficient condition for asymptotic stability in Theorem 2 (II) in Arigapudi et al. (2021).

Arigapudi et al. (2021) focus on the BEP($\tau^{\text{all}}, \kappa$) dynamic and on a family of games that

[4] If λ is an eigenvalue of $DW^{+}(\mathbf{0})$, then $(\lambda - 1)$ is an eigenvalue of $DW(\mathbf{0}) = DW^{+}(\mathbf{0}) - I$.
[5] See note at the beginning of appendix A.2.

satisfy a specific genericity requirement, which here we term κ-*generic* games (Arigapudi et al., 2021, Definition 4). They show that their sufficient condition for asymptotic stability of strict Nash equilibria is both sufficient and necessary in κ-generic games with either more than two players ($p > 2$) or more than one test of each strategy ($\kappa > 1$). However, their stability condition is difficult to check if followed literally, since it involves testing a requirement on each and every set in the power set of $S \setminus \{s\}$. The requirement of having a κ-*generic* game can also be too stringent in practical cases, as it may not be satisfied even by two-player games with generic payoff matrices. As an illustration, none of the more than 20 numeric examples in Osborne and Rubinstein (1998), Sethi (2000), Sandholm et al. (2019, 2020), Sethi (2021) and Arigapudi et al. (2021) are κ-generic.

3.5 Contribution

In this paper we:

i) Show that the general sufficient condition for instability of strict equilibria indicated above (Sandholm et al., 2020, Proposition 5.4 (ii)), which applies under any BEP(τ^{α}, κ) dynamics, can be checked using a simple algorithm. The complexity of this algorithm is equivalent to performing an iterated elimination of dominated strategies.

ii) Show that a similarly simple algorithm can be used to check the general sufficient condition for asymptotic stability of strict equilibria under BEP($\tau^{\text{all}}, \kappa$) dynamics indicated in Section 3.4,[6] i.e., the most general condition that guarantees, under any tie-breaking rule, a triangular Jacobian $DW(\mathbf{0})$ with all diagonal values (eigenvalues) equal to -1. We also provide a tighter stability test under the specific tie-breaking rule β^{stick}, a rule that favors stability under BEP($\tau^{\text{all}}, \kappa$) dynamics.

iii) Discuss conditions under which the sufficient condition for asymptotic stability in ii) is also necessary for stability, for different BEP($\tau^{\text{all}}, \kappa$) dynamics. This extends the results of Arigapudi et al. (2021) by removing the constraint that the game be κ-*generic*.

iv) Apply our results to explore the predictive power of BEP dynamics in tacit coordination games. In these games, most game theoretical models do not correspond well with experimental evidence.

[6]As indicated before, this is equivalent to the sufficient condition for asymptotic stability in Arigapudi et al. (2021), Theorem 2, II.

4. Stability and instability tests

4.1 s-stabilizing and potentially s-stabilizing strategies

In this section, we define s-stabilizing and potentially s-stabilizing strategies in subsets $J \subseteq S \setminus \{s\}$. Informally, a s-stabilizing strategy in J is a strategy that, under a BEP($\tau^{\text{all}}, \kappa$) dynamic, does not contribute to the growth of the fraction of players using the strategies in J, when the population state is close to the strict equilibrium state e_s. In contrast, if a strategy is not potentially s-stabilizing in J, it is associated to at least some minimum contribution to the growth of the fraction of players using the strategies in J, when the population state is close to the strict equilibrium state e_s, under any BEP(τ^{α}, κ) dynamic.

Definition (s-stabilizing and potentially s-stabilizing strategies)**.** Let s be a strategy such that the strategy profile $(s, s, ..., s)$ is a strict Nash equilibrium of the game. Let S_2 be the set of strategies that obtain the second-best payoff, v^{κ}_{ts}, when playing against s-players, i.e., $S_2 \equiv \text{argmax}_{i \neq s} v^{\kappa}_{is} = \text{argmax}_{i \neq s} U(i; s, s, ..., s)$, and $v^{\kappa}_{ts} \equiv \max_{i \neq s} v^{\kappa}_{is}$. Let J be a non-empty set $J \subseteq S \setminus \{s\}$. A strategy $j \in J$ is s-stabilizing in J, for a number of trials κ, if

- $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ for all $i \in J$, and
- If $S_2 \cap J \neq \emptyset$, then $v^{\kappa}_{sj} > v^{\kappa}_{ts}$.

A strategy $j \in J$ is potentially s-stabilizing in J, for a number of trials κ, if

- $v^{\kappa}_{ij} \leq v^{\kappa}_{ss}$ for all $i \in J$, and
- If $S_2 \subseteq J$, then $v^{\kappa}_{sj} \geq v^{\kappa}_{ts}$. □

Clearly, every s-stabilizing strategy in J is potentially s-stabilizing in J. To understand the previous conditions, consider a test of each strategy by a revising agent who, when sampling the required $n\,\kappa\,(p-1)$ co-players, meets just once a deviating co-player not using strategy s, but using strategy $j \in J$ instead. The condition $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ guarantees that, if the deviating j-player is met when testing strategy $i \in J$, the total payoff v^{κ}_{ij} to strategy i is less than the total payoff v^{κ}_{ss} to strategy s, so strategy s is selected. Similarly, the condition $((S_2 \cap J \neq \emptyset) \Rightarrow v^{\kappa}_{sj} > v^{\kappa}_{ts})$ guarantees that, if the deviating j-player is met when testing strategy s (in which case the maximum of the payoffs obtained by all the strategies is either v^{κ}_{sj} or v^{κ}_{ts}), no strategy $i \in J$ is selected. Intuitively, in a neighborhood of e_s, if j is s-stabilizing in J then we could say that j does not help any other strategy in J (including itself) to destabilize e_s.

With relation to the analysis of the Jacobian of the BEP($\tau^{\text{all}}, \kappa$) dynamics at e_s, if a strategy j is s-stabilizing in a subset of strategies J, then j has a null contribution (on the column corresponding to j) to the principal submatrix of $DW^{+}(\mathbf{0})$ associated to J. And if, starting from $J_1 = S \setminus \{s\}$, a process of iterative elimination of s-stabilizing strategies (see appendix A.1) eliminates all strategies in $S \setminus \{s\}$, then $DW^{+}(\mathbf{0})$ can be arranged (by reordering the strategies) as a triangular matrix with a zero diagonal (so the eigenvalues of $DW^{+}(\mathbf{0})$ are 0, and the eigenvalues of $DW(\mathbf{0})$ are -1), proving asymptotic stability of e_s.

Note that if, for a number of trials κ_0 and some subset of strategies J, a strategy $j \in J$ is s-stabilizing in J, then j is s-stabilizing in J for any $\kappa > \kappa_0$.

In contrast, if a strategy j is not potentially s-stabilizing in some subset of strategies J (such that $j \in J$), then the (positive or destabilizing) contribution of j to the principal submatrix of the Jacobian of the dynamics corresponding to the strategies in J, in the column corresponding to j, is guaranteed to be above a certain threshold value, under any BEP(τ^{α}, κ) dynamic. If there is some subset of strategies J such that every strategy $j \in J$ is not potentially s-stabilizing in J, the fact that the sum of the terms in every column of the principal submatrix of $DW(\mathbf{0})$ associated to J is above a threshold value can be used to obtain a lower bound for the Perron–Frobenius eigenvalue of $DW(\mathbf{0})$, and to guarantee instability of the equilibrium.

Note that if, for a number of trials κ_0 and some subset of strategies J, a strategy $j \in J$ is not potentially s-stabilizing in J, then j is not potentially s-stabilizing in J for any $\kappa < \kappa_0$.

4.2 Instability under BEP(τ^{α}, κ) dynamics

Our first proposition shows that a tight sufficient test for instability of strict equilibria under any BEP(τ^{α}, κ) dynamics can be carried out by analyzing the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$. Although the process of iterated elimination may be considered evident, a formal description can be found in appendix A.1. All the proofs have been relegated to appendix A.2.

Note that if a strategy $j \in J \subseteq S \setminus \{s\}$ is potentially s-stabilizing in J, then j is also potentially s-stabilizing in any subset of J containing j. As a consequence, the order in which potentially s-stabilizing strategies are iteratively eliminated does not alter the final set of surviving strategies.

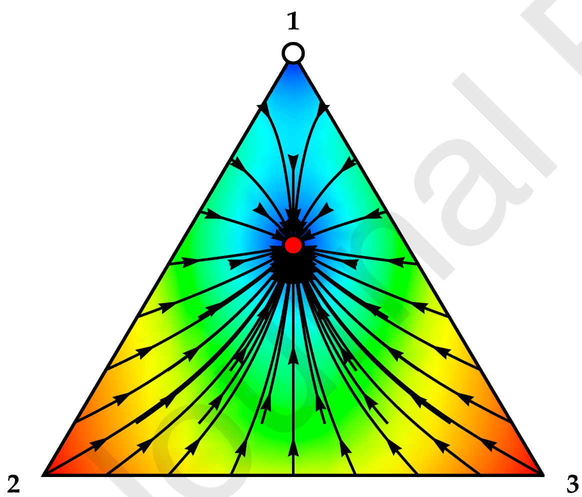
Proposition 4.1. *Let e_s be a strict equilibrium. If for a number of trials $\kappa_0 > \frac{n-1}{(p-1)(\alpha-1)}$ some strategy survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is unstable under any BEP(τ^{α}, κ) for any κ satisfying $\frac{n-1}{(p-1)(\alpha-1)} < \kappa \leq \kappa_0$.*

Corollary 4.2. *Let e_s be a strict equilibrium. If for a number of trials κ_0 some strategy survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is unstable under any BEP($\tau^{\text{all}}, \kappa$) for any κ with $1 < \kappa \leq \kappa_0$, and, if $p > 2$, for any $\kappa \leq \kappa_0$.*

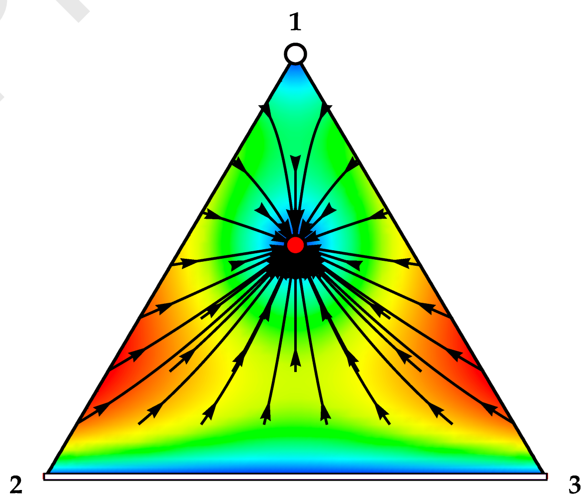
Example 4.1. Consider the game with payoff matrix

$$U_{ij} = V^{\kappa=1} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \text{ which leads to } V^{\kappa=2} = \begin{pmatrix} 6 & 3 & 3 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}.$$

Corollary 4.2 shows that the equilibrium state e_1 is unstable under BEP($\tau^{\text{all}}, \kappa = 2$) dynamics. This can be proved by noting that, for $\kappa = 2$, strategies 2 and 3 survive the iterated elimination of potentially 1-stabilizing strategies, since none of them is potentially 1-stabilizing in $J = S \setminus \{s\} = \{2, 3\}$. This is so because, for $s = 1$ and $j \in J$, we have that $S_2 = \{2, 3\} \subseteq J$ but $v_{1j}^{\kappa=2} = 3 < 4 = v_{t1}^{\kappa=2}$. However, for $\kappa = 2$, this game satisfies the necessary conditions for asymptotic stability in Theorem 2 in Arigapudi et al. (2021), which are not sufficient in this case, since the game is not κ-generic. Thus, Corollary 4.2 (and Proposition 4.1, more generally) can be used to prove the instability of strict equilibria on which Theorem 2 in Arigapudi et al. (2021) remains silent.



(i) tie-breaking rule β^{unif}



(ii) tie-breaking rule β^{stick}

Figure 1: BEP($\tau^{\text{all}}, 2, \beta$) dynamics in the game of Example 4.1 for two tie-breaking rules: β^{unif} (left) and β^{stick} (right).

Figure 1 shows the BEP($\tau^{\text{all}}, 2, \beta$) dynamics in the game of Example 4.1 for two tie-breaking rules: β^{unif} (left) and β^{stick} (right).[7] As proved above for any tie-breaking rule, it

[7]In the figures, colors represent speed of motion: red is fastest, blue is slowest. Isolated rest points are

can be seen that state e_1 is unstable under both dynamics. ◆

4.3 Asymptotic stability under BEP($\tau^{\text{all}}, \kappa$) dynamics

Proposition 4.3. *Let e_s be a strict equilibrium. If for a number of trials κ_0 no strategy survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is asymptotically stable under any BEP($\tau^{\text{all}}, \kappa$) with $\kappa \geq \kappa_0$.*

As before, note that if a strategy $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J, then j is also s-stabilizing in any subset of J containing j. As a consequence, the order in which s-stabilizing strategies are iteratively eliminated does not alter the final set of surviving strategies. For a fixed κ, the stability condition in Proposition 4.3 can be shown to be equivalent to the stability condition in Arigapudi et al. (2021) [Theorem 2, II],[8] so the former can be seen as a quick and easy way of checking the latter. In terms of the complexity of checking these conditions according to their formulation, Proposition 4.3 involves checking the existence of s-stabilizing strategies in at most $n-1$ subsets of S, while a direct check of the stability condition in Arigapudi et al. (2021) involves checking an existence condition in 2^{n-1} subsets of S (in all the subsets of $S \setminus \{s\}$). If, for instance, the number of strategies is $n = 11$, the difference would be checking 10 subsets using Proposition 4.3 versus checking $2^{10} = 1024$ subsets otherwise.

Example 4.2. Consider the coordination game with payoff matrix

$$U_{ij} = \begin{pmatrix} U_{11} & 0 & 0 & \dots & 0 \\ 0 & U_{22} & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & U_{(n-1)(n-1)} & 0 \\ 0 & 0 & \dots & 0 & U_{nn} \end{pmatrix}, \tag{3}$$

with $U_{ss} > 0$ for all $s \in S$, so all strategies are strict Nash strategies. In this game, for $i, j \in S \setminus \{s\}$ and $i \neq j$, we have $v_{ss}^{\kappa,s} = \kappa U_{ss}$, $v_{sj}^{\kappa,s} = (\kappa-1)U_{ss}$, $v_{ii}^{\kappa,s} = U_{ii}$ and $v_{ij}^{\kappa,s} = 0$. Therefore, strategy $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if the following two conditions are satisfied:

represented with circles: red if the rest point is asymptotically stable, and white if it is unstable. Connected components of rest points are represented with lines: purple if Lyapunov stable, and white if unstable.

[8] It is not difficult to show that the sufficient condition for stability in Arigapudi et al. (2021) can be equivalently formulated in terms of iterated elimination of strategies that are not weakly supported (according to their definition) by any other strategy. This is so because if a strategy j is not weakly supported by any strategy in a set J that includes j, then j is not weakly supported by any strategy in any subset of J that includes j.

- $v_{ij}^{\kappa} < v_{ss}^{\kappa}$ for all $i \in J$ $\quad\Leftrightarrow U_{jj} < \kappa U_{ss} \quad\Leftrightarrow \kappa > \dfrac{U_{jj}}{U_{ss}}.$
- If $S_2 \cap J \neq \varnothing$ then $v_{sj}^{\kappa} > v_{ts}^{\kappa}$ $\quad\Leftrightarrow (\kappa - 1)U_{ss} > 0 \quad\Leftrightarrow \kappa > 1.$

Thus, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Similarly, it is easy to check that strategy $j \in J \subseteq S \setminus \{s\}$ is potentially s-stabilizing in J if and only if $\kappa \geq \frac{U_{jj}}{U_{ss}}$.

Now, let $U_{\max} = \max_{i \in S} U_{ii}$ be the highest possible payoff and let $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$ be the set of strategies that obtain the highest possible payoff in the game when playing against themselves.

Applying Proposition 4.3, we can deduce that, for any strict strategy $s \in S$, state e_s is asymptotically stable under any BEP($\tau^{\text{all}}, \kappa$) for every $\kappa > \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that all strategies are s-stabilizing, so no strategy survives the iterated elimination of s-stabilizing strategies. In particular, if $s \in S^{\max}$, e_s is asymptotically stable for every $\kappa > \frac{U_{\max}}{U_{\max}} = 1$.

Applying Corollary 4.2, we can deduce that if $s \notin S^{\max}$, state e_s is unstable under any BEP($\tau^{\text{all}}, \kappa$) for every $1 < \kappa < \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that any strategy $i \in S^{\max}$ is not potentially s-stabilizing in any subset that contains it, so it survives the iterated elimination of potentially s-stabilizing strategies in $S \setminus \{s\}$.

So, to sum up, in coordination game (3) with $U_{ss} > 0$ for all $s \in S$, under any BEP($\tau^{\text{all}}, \kappa$) with $\kappa > 1$, e_s is asymptotically stable for $\kappa > \frac{U_{\max}}{U_{ss}}$ and e_s is unstable for $1 < \kappa < \frac{U_{\max}}{U_{ss}}$.

The stability of e_s in the remaining cases, i.e. for $\kappa = 1$ and for $\kappa = \frac{U_{\max}}{U_{ss}}$ (if $\frac{U_{\max}}{U_{ss}} \in \mathbb{N}$), depends on the tie-breaking rule.

Figure 2 illustrates these results by showing the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in the coordination game (3) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 2$, e_1 is unstable (since $1 < \kappa < \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$), e_2 is asymptotically stable (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{2} = 1.5$), and e_3 is asymptotically stable (since $s = 3 \in S^{\max}$ and $\kappa > 1$). For $\kappa \geq 4$, e_1 becomes asymptotically stable too (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$). ◆

4.4 Stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics

In this section we study whether the lack of fulfillment of the sufficient condition for asymptotic stability in Proposition 4.3 can guarantee instability. Arigapudi et al. (2021) show that, for BEP($\tau^{\text{all}}, \kappa$) dynamics with either $\kappa > 1$ or $p > 2$, a sufficient stability condition that is equivalent to Proposition 4.3, is both sufficient and necessary in κ-generic games. However, the requirement of being κ-generic can be quite restrictive in practice, as pointed out in Section 3.

Here we remove the genericity condition and focus on BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in any

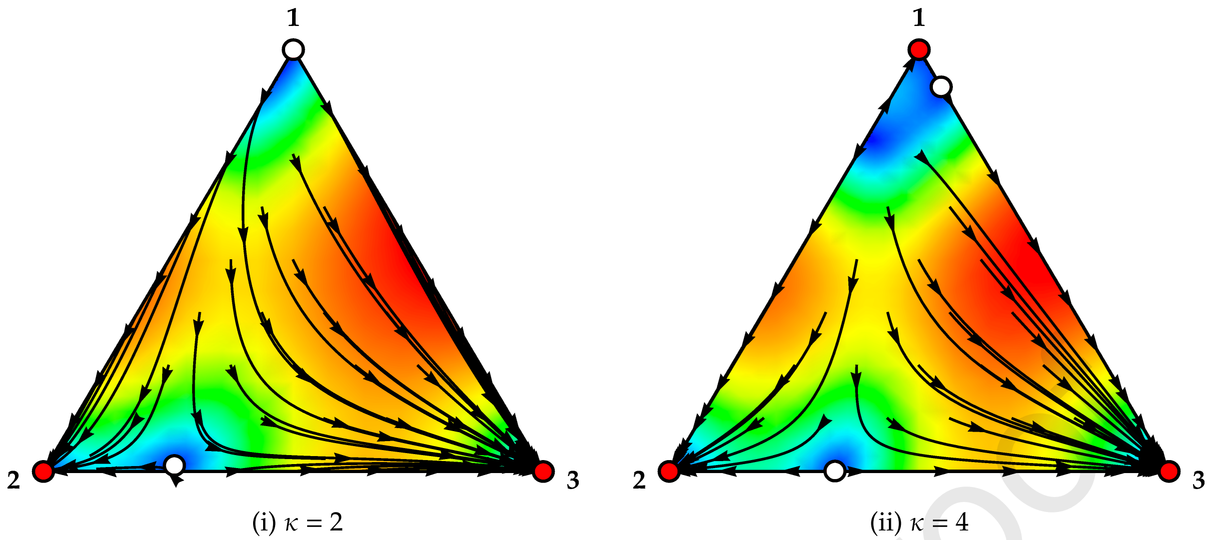


Figure 2: Coordination game (3) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, for $\kappa = 2$ (left) and $\kappa = 4$ (right).

game, given that this is the family of BEP dynamics considered in most previous studies in the literature. In the next section, we will also consider BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, as this alternative tie-breaking rule can be regarded as more natural in many cases.

For BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, we show that the sufficient condition for asymptotic stability in Proposition 4.3 is also necessary for stability for any $\kappa > n$; more tightly, for any $\kappa > \frac{|S_2|+1}{p-1}$. If the second-best payoff when playing against s-players is obtained by a single strategy (i.e., if $|S_2| = 1$),[9] then this property holds for any $\kappa > 2$ (for any κ, if $p > 3$).

Our next result (i.e. Proposition 4.4) is also relevant because it shows that under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, beyond some small values of κ, and as κ grows, we will find either permanent asymptotic stability or a single transition from instability to permanent asymptotic stability. A transition from stability to instability can only happen within the small values of κ indicated in the proposition.

Proposition 4.4. *Let e_s be a strict equilibrium and let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$. If for a number of trials κ_0 no strategy survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is asymptotically stable under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics for any $\kappa \geq \kappa_0$. Otherwise:*

- *State e_s is unstable under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics for any κ satisfying $\frac{|S_2|+1}{p-1} < \kappa \leq \kappa_0$, and also for any $\kappa > \frac{2}{p-1}$ satisfying $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} < \kappa \leq \kappa_0$.*

[9]Note that the condition $|S_2| = 1$ is much weaker than the condition that a game has to satisfy in order to be κ-generic.

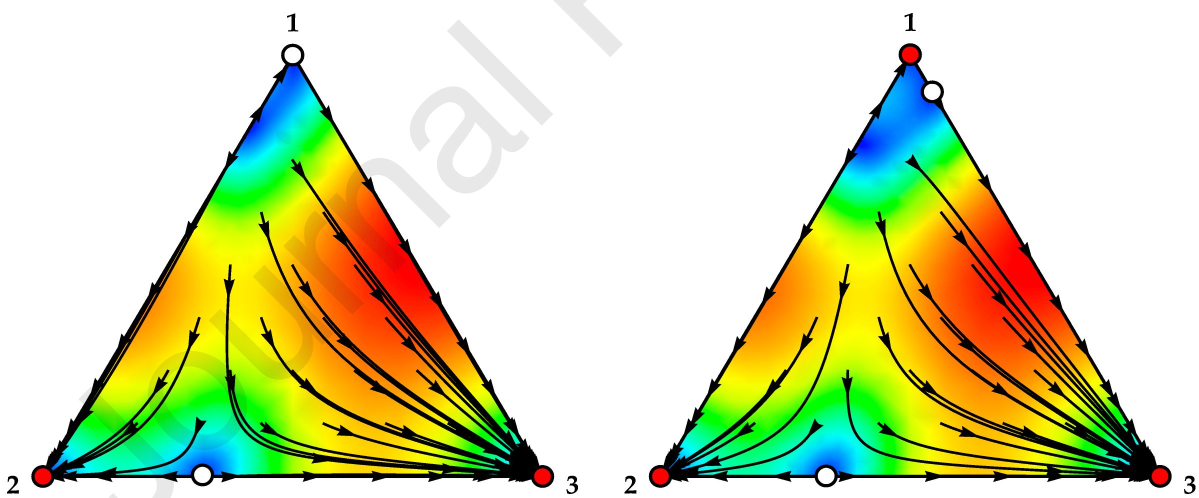
- *State e_s is unstable under BEP($\tau^\alpha, \kappa, \beta^{\text{unif}}$) dynamics for any κ satisfying $\frac{n-1}{\alpha-1}\frac{\min(|S_2|+1,\alpha)}{p-1} < \kappa \leq \kappa_0$.*

Example 4.3. Consider the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics on the coordination game with payoff matrix (3), with $U_{ss} > 0$ for all $s \in S$. Recall that $U_{\max} = \max_{i\in S} U_{ii}$ and $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$.

In addition to what we inferred in Example 4.2 for any BEP($\tau^{\text{all}}, \kappa$) dynamic, applying Proposition 4.4 we can address the stability of $e_{s\notin S^{\max}}$ for $\kappa = \frac{U_{\max}}{U_{ss}}$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$). We could not do this using Corollary 4.2 because this stability depends on the tie-breaking rule. Here we deduce that if $s \notin S^{\max}$, state e_s is unstable under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) for any $\kappa \leq \frac{U_{\max}}{U_{ss}}$, assuming $\kappa > 2$.

In Example 4.2 we showed that, in this game, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Thus, if $s \notin S^{\max}$ and $\kappa \leq \frac{U_{\max}}{U_{ss}}$, any strategy $i \in S^{\max}$ survives the iterated elimination of s-stabilizing strategies in $S \setminus \{s\}$ so, applying Proposition 4.4 and noting that $\frac{v^1_{ss} - \min_{j\in S\setminus\{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} = \frac{U_{ss}-0}{U_{ss}-0} = 1$ and $p = 2$, we can state that e_s is unstable for any $\kappa \leq \frac{U_{\max}}{U_{ss}}$, assuming $\kappa > \frac{2}{p-1} = 2$.

Figure 2 shows the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in the coordination game (3) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 3$, e_1 is unstable (since $\kappa \leq \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$), while for $\kappa \geq 4$, e_1 is asymptotically stable (since $\kappa > \frac{U_{\max}}{U_{ss}} = \frac{3}{1} = 3$). ◆



(i) $\kappa = 3$ (ii) $\kappa = 4$

Figure 3: Coordination game (3) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, for $\kappa = 3$ (left) and $\kappa = 4$ (right).

4.5 Stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics

For BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, here we provide an improved sufficient condition for asymptotic stability, tighter than Proposition 4.3, and prove that this sufficient condition for asymptotic stability is also necessary for stability for any $\kappa > \frac{|S_2|}{p-1}$. If the second-best payoff when playing against s-players is obtained by a single strategy (i.e., $|S_2| = 1$), then this condition holds for any $\kappa > 1$ (for any κ, if $p > 2$). To show this, first we need to define weakly s-stabilizing strategies.

Definition (Weakly s-stabilizing strategies)**.** We say that a strategy $j \in J$ is weakly s-stabilizing in J, for a number of trials κ, if

- $v_{ij}^{\kappa} \leq v_{ss}^{\kappa}$ for all $i \in J$, and
- If $S_2 \cap J \neq \emptyset$, then $v_{sj}^{\kappa} \geq v_{ts}^{\kappa}$. □

Any s-stabilizing strategy in J is weakly s-stabilizing in J, so if the iterated elimination of s-stabilizing strategies in $S_2 \setminus \{s\}$ eliminates all strategies (proving stability under BEP($\tau^{\text{all}}, \kappa$) dynamics), so does the iterated elimination of weakly s-stabilizing strategies. The second process, however, can prove stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics in additional cases. We illustrate this fact in Example 4.4 (also, compare Figure 3(i) vs Figure 4(ii)).

Proposition 4.5. *Let e_s be a strict equilibrium and let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$. If for a number of trials κ_0 no strategy survives the iterated elimination of weakly s-stabilizing strategies in $S \setminus \{s\}$, then state e_s is asymptotically stable under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) for any $\kappa \geq \kappa_0$. Otherwise, it is unstable for any κ with $\frac{|S_2|}{p-1} < \kappa \leq \kappa_0$, and also for any $\kappa > \frac{1}{p-1}$ with $\frac{v_{ss}^1 - \min_{j \in S \setminus \{s\}} v_{sj}^1}{v_{ss}^1 - v_{ts}^1} < \kappa \leq \kappa_0$.*

Note that, if $|S_2| = 1$, then the condition $\kappa > \frac{|S_2|}{p-1}$ holds for any $\kappa > 1$ (for any κ, if $p > 2$).

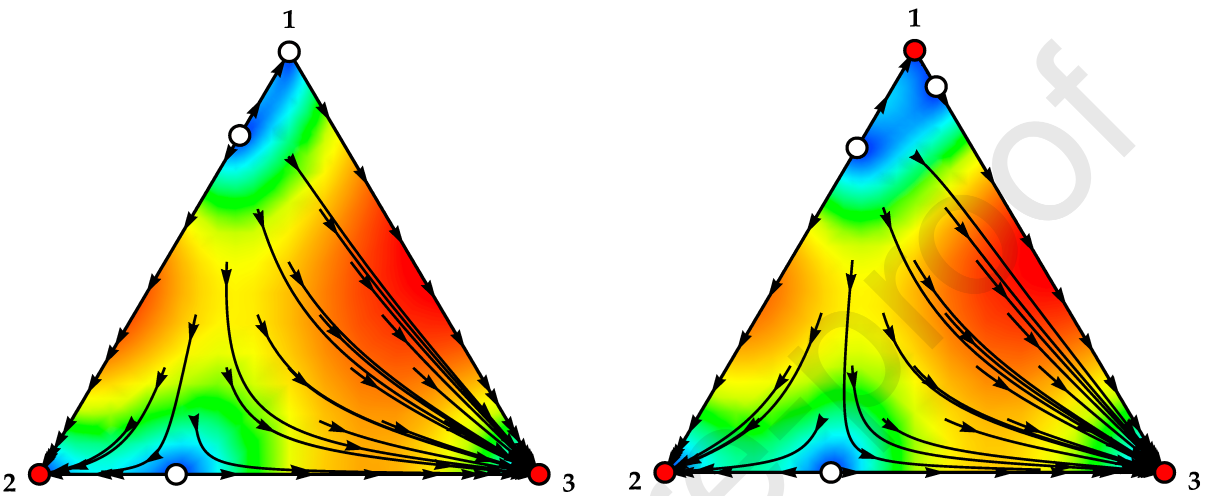
Example 4.4. Consider the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic on the coordination game with payoff matrix (3), with $U_{ss} > 0$ for all $s \in S$. Recall that $U_{\max} = \max_{i \in S} U_{ii}$ and $S^{\max} = \{i \in S \mid U_{ii} = U_{\max}\}$.

In Example 4.2 we showed that, in this game, $j \in J \subseteq S \setminus \{s\}$ is s-stabilizing in J if and only if $\kappa > \max\left(\frac{U_{jj}}{U_{ss}}, 1\right)$. Following the same reasoning, it is easy to check that $j \in J \subseteq S \setminus \{s\}$ is weakly s-stabilizing in J if and only if $\kappa \geq \frac{U_{jj}}{U_{ss}}$.

Applying Proposition 4.5, we can then deduce that state e_s is asymptotically stable for every $\kappa \geq \frac{U_{\max}}{U_{ss}}$, since this condition guarantees that all strategies are weakly s-stabilizing, so no strategy survives the iterated elimination of weakly s-stabilizing strategies. In particular, if $s \in S^{\max}$, e_s is asymptotically stable for every $\kappa \geq \frac{U_{\max}}{U_{\max}} = 1$. If $s \notin S^{\max}$

and $\kappa < \frac{U_{max}}{U_{ss}}$, any strategy $i \in S^{max}$ is not weakly s-stabilizing, so it survives the iterated elimination of weakly s-stabilizing strategies in $S \setminus \{s\}$. Therefore, noting that $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} = \frac{U_{ss}-0}{U_{ss}-0} = 1$ and $p = 2$, we can state that e_s is unstable for any κ such that $\frac{1}{p-1} = 1 < \kappa < \frac{U_{max}}{U_{ss}}$.

To sum up, under BEP($\tau^{all}, \kappa, \beta^{stick}$), e_s is asymptotically stable if $\kappa \geq \frac{U_{max}}{U_{ss}}$, and unstable if $1 < \kappa < \frac{U_{max}}{U_{ss}}$. In particular, if $s \in S^{max}$, then e_s is asymptotically stable for every κ.



(i) $\kappa = 2$ (ii) $\kappa = 3$

Figure 4: Coordination game (3) with $n = 3$ strategies, where $U_{ii} = i$, under BEP($\tau^{all}, \kappa, \beta^{stick}$) dynamics, for $\kappa = 2$ (left) and $\kappa = 3$ (right).

Figure 4 shows the BEP($\tau^{all}, \kappa, \beta^{stick}$) dynamics in the coordination game (3) with $n = 3$ strategies and $U_{ii} = i$. For $\kappa = 2$, e_1 is unstable (since $1 < \kappa < \frac{U_{max}}{U_{ss}} = \frac{3}{1} = 3$), e_2 is asymptotically stable (since $\kappa \geq \frac{U_{max}}{U_{ss}} = \frac{3}{2} = 1.5$), and e_3 is asymptotically stable (since $s = 3 \in S^{max}$). For $\kappa \geq 3$, e_1 is also asymptotically stable (since $\kappa \geq \frac{U_{max}}{U_{ss}} = \frac{3}{1} = 3$). ◆

5. Application: tacit coordination games

5.1 Introduction

In this section, we apply our results to the tacit coordination games studied by Van Huyck et al. (1990).[10] These games formalize a symmetric situation where a group of individuals must decide how much effort to put into a common project. If everyone works

[10] We thank an anonymous reviewer for suggesting the application of our stability results under BEP dynamics to these coordination games, which combine an interesting structure with available experimental evidence, and illustrate a nice feature of BEP dynamics.

equally hard, the return obtained by each of the individuals per unit of effort exceeds its cost. Thus, the more effort they collectively put, the greater the profit (i.e. payoff) they will obtain. However, the output of the project depends solely on the minimum effort made by any of the individuals; thus, if any one works less than the rest, the extra effort put by the others goes to waste.

Formally, tacit coordination games are symmetric p-player games with strategy space $S = \{1, ..., n\}$ (denoting the player's effort or contribution) and payoff function

$$U(i; j_1, \ldots, j_{p-1}) = a \min(i, j_1, \ldots, j_{p-1}) - b\, i,$$

where $a > b \geq 0$ are two parameters controlling the return and the cost of effort units, respectively.

Note that every homogeneous pure strategy profile $(i, i, ..., i)$, in which all the p players choose the same strategy i, is a Nash equilibrium, and these equilibria are strictly Pareto ranked, with their rank preference growing with i. However, at any given situation, selecting the lowest strategy chosen by the rest of the players is always a best reply. This means that, at any equilibrium $(i, ..., i)$, if any player deviates to a lower strategy $j < i$, then, following suit and changing to strategy j is always a best response. This creates a tension that can induce players to lower their strategy or "effort" as soon as any other player does –or as soon as they *believe* that any other player *may* do it.

Table 1 represents the payoff function for the 3-strategy case ($n = 3$). The row headings on the payoff matrices in Table 1 indicate the strategy chosen by the player that receives the payoff. The column headings indicate the minimum value of the strategies chosen by the other $(p - 1)$ players.

Table 1: Payoff matrices for a p-player tacit coordination game with three strategies ($n = 3$). Left: general case. Middle: $a = 2$ and $b = 1$. Right: $a = 1$ and $b = 0$.

	min of others' strategies		
	1	2	3
1	$a - b$	$a - b$	$a - b$
2	$a - 2b$	$2a - 2b$	$2a - 2b$
3	$a - 3b$	$2a - 3b$	$3a - 3b$

	min of others' st.		
	1	2	3
1	1	1	1
2	0	2	2
3	−1	1	3

	min of others' st.		
	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

For $b > 0$, the unique best reply to any (partial) pure strategy profile $(j_1, \ldots, j_{p-1})$ used by the other players is the minimum of their contributions, i.e. $\min(j_1, \ldots, j_{p-1})$. The monomorphic states e_i, with $i \in \{1, ..., n\}$, are consequently the only pure-strategy Nash equilibrium states of the game, and they are all strict. Strategy 1 (the maxmin strategy) is called the secure strategy, while strategy n is called the efficient strategy because, if adopted by everyone, it corresponds to the efficient equilibrium profile $(n, ..., n)$. For $b = 0$, the efficient strategy n is weakly dominant (see Table 1, right matrix) and e_n is the only strict Nash state.

All symmetric strict Nash equilibria satisfy most equilibrium refinements and correspond to evolutionarily stable states, according to the standard definition of evolutionary stability (Weibull, 1995).[11] However, experimental evidence clearly shows that human subjects do discriminate between different strict equilibria in these games. Van Huyck et al. (1990) present and discuss neat experimental evidence on these games with $n = 7$ strategies, repeatedly played within (fixed) groups of different sizes. Their most striking findings are summarized below:[12]

- Games with $b > 0$. The behavior of human subjects in these games clearly depends on the number of players. When the game is played in very small groups (i.e., $p = 2$ players), there is a clear tendency to choose the efficient strategy.[13] In contrast, in groups with several players ($p \approx 15$), the distribution of strategies is initially diverse, and then the vast majority of players approach the lowest effort (i.e. the secure strategy 1) fairly quickly –in ten periods or less–, even when the experiment is repeated with the same group of co-players: "most people appear to consider the highest effort a good bet in small groups, but not in large groups" (Crawford, 1991). Note that this clear pattern of discrimination between strict Nash equilibria, dependent on the number of players and against the payoff-dominance criterion in the case of large groups, cannot be explained along the lines of traditional game theory (Crawford, 1991).

- Games with $b = 0$. In between two rounds of repeatedly playing the stage game with $b > 0$ within a large group (at both of which nearly all subjects ended up choosing the lowest effort; even faster and more sharply in the second round), Van Huyck et al. (1990) put the same groups to play one round of the game with $b = 0$ for five periods. In stark contrast with the results for $b > 0$, in this intermediate round with $b = 0$, nearly all players chose the highest effort in virtually all periods. This suggests that the consistent results obtained with $b > 0$ (both before and after playing the repeated game with $b = 0$) were not due to players' misunderstanding of the incentives structure, but to strategic uncertainty, i.e. players' uncertainty about how

[11] Crawford (1991) provides a detailed analysis of these games and shows that the only equilibrium state that satisfies a finite-population definition of evolutionary stability is the secure state e_1.

[12] We refer to each play of the stage game as one *period*, and we use the term *round* for several consecutive periods.

[13] Van Huyck et al. (1990) also present results on setups where players were randomly paired after every period. In that case, they did not find any stable pattern of behavior.

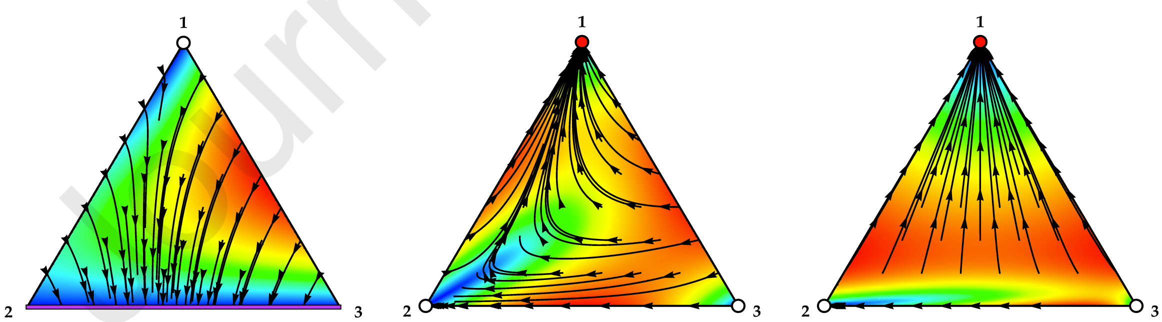
the other players may respond to the multiplicity of strict Nash equilibria (Crawford, 1991).

5.2 Results

Without aiming to provide an explanation for the regularities found by Van Huyck et al. (1990) – we refer the reader to Crawford's (1991) insightful analysis for a discussion of possible explanations –, our goal in this section is to explore whether BEP dynamics can capture the discrimination between different strict Nash equilibria shown by humans in tacit coordination games, and its dependence on the number of players p. We include the most relevant results of this analysis below (proofs are included in the appendix).

The stability analysis of strict equilibria states under BEP dynamics is more interesting for low values of the number of trials κ, since for sufficiently large values of κ, every strict equilibrium is asymptotically stable.

a) Games with $b > 0$. The stability of the different strict Nash states is highly dependent on the number of players p. For two players, the efficient state e_n (maximum contribution) is Lyapunov stable under any BEP($\tau^{\text{all}}, 1$), while (assuming that the number of strategies is greater than $1 + \frac{a}{a-b}$) the secure state e_1 (minimum contribution) is unstable (see Figure 5(i) for the three-strategy case). By contrast, for more than two players, the efficient state e_n is unstable (under any BEP($\tau^{\text{all}}, 1$) dynamic[14]), while the secure state e_1 is asymptotically stable (under every BEP($\tau^{\text{all}}, \kappa$) dynamics). Figure 5 illustrates these results for a game with $n = 3$ strategies under BEP($\tau^{\text{all}}, 1, \beta^{\text{stick}}$) dynamics.



(i) $p = 2$ (ii) $p = 3$ (iii) $p = 15$

Figure 5: Tacit coordination game with $n = 3$ strategies and $a = 2b > 0$ (see Table 1) under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{stick}}$) dynamics, for number of players $p = 2$ (left), $p = 3$ (middle) and $p = 15$ (right).

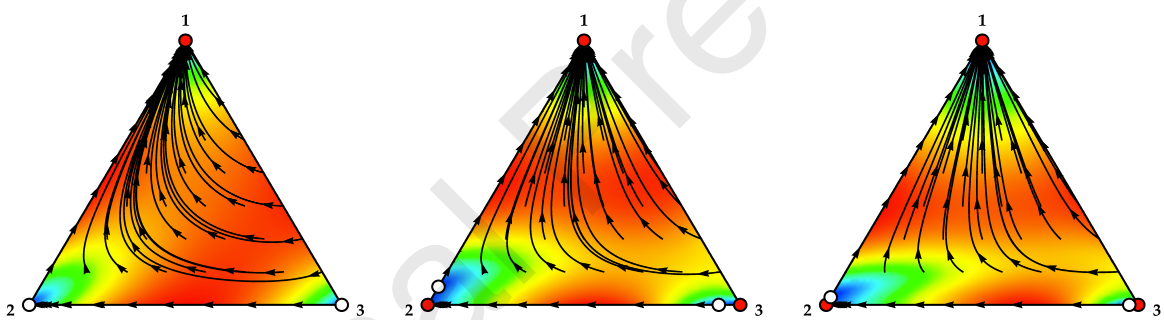
[14] Proposition 4.1 also shows that the efficient state e_n is unstable for every BEP($\tau^{\alpha}, 1$) dynamics if $p > n$.

The fact that increasing the number of players favors the instability of e_n and the stability of e_1 is in full accordance with experimental evidence.

Let us now analyze the stability of the intermediate strict Nash states $e_2, ..., e_{n-1}$ for $p > 2$. We find three cases:

i) $a < 2b$. In this case, every intermediate state $e_2, ..., e_{n-1}$ is unstable under BEP($\tau^{\text{all}}, \kappa < \frac{a}{a-b}$) dynamics, and also under BEP($\tau^{\alpha}, \kappa < \frac{a}{a-b}$) dynamics if $p > n$, leaving e_1 as the only asymptotically stable strict Nash state (this includes the cases $\kappa = 1$ and $\kappa = 2$, given that $\frac{a}{a-b} > 2$). In turn, every intermediate state $e_2, ..., e_{n-1}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > \frac{a}{a-b}$) dynamics. The stability of the intermediate states in the borderline case $\kappa = \frac{a}{a-b}$ depends on the tie-breaking rule (instability under β^{unif}, stability under β^{stick}).

Figure 6 illustrates these results for a game with $n = 3$ strategies under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics. Note that, in the cases where the intermediate state is stable (i.e. $\kappa > 2$), its basin of attraction is rather small compared with the basin of attraction of the secure state e_1.

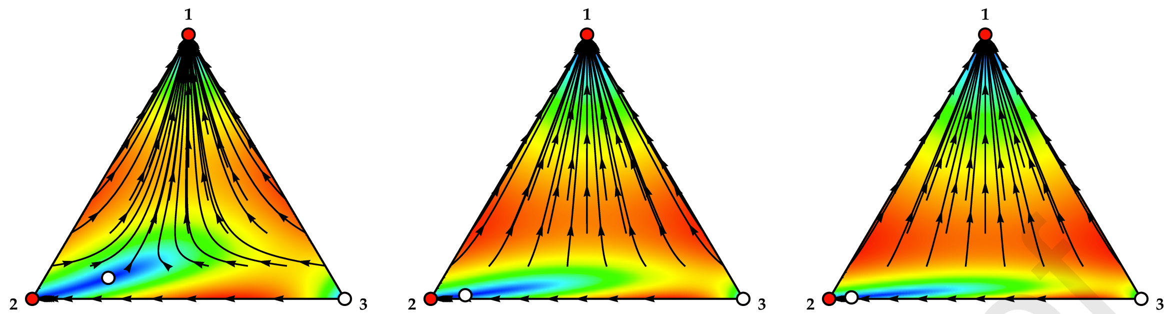


(i) $\kappa = 1$ (ii) $\kappa = 3$ (iii) $\kappa = 5$

Figure 6: Tacit coordination game with $n = 3$ strategies, $p = 3$ players, $a = 3$, and $b = 2$ (see Table 1) under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics, for number of trials $\kappa = 1$ (left), $\kappa = 3$ (middle) and $\kappa = 5$ (right).

ii) $a = 2b$. In this case, every intermediate state $e_2, ..., e_{n-1}$ is unstable under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{stick}}$) dynamics for $p > 3$, leaving e_1 as the only asymptotically stable strict Nash state. This is also the case under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{unif}}$) dynamics with $p > 4$, and under BEP($\tau^{\alpha}, \kappa = 1, \beta^{\text{unif}}$) dynamics with $p \geq 2n$. For $\kappa > 2$, every intermediate state is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > 2$) dynamics. The case $\kappa = 2$ depends on the tie-breaking rule (stability under β^{stick}, instability under β^{unif}).

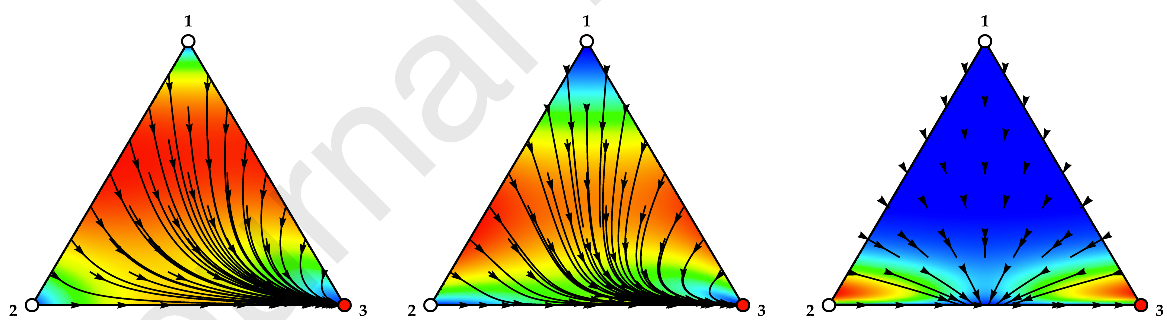
iii) $a > 2b$. In this case, for $p > 2$, every intermediate state $e_2, ..., e_{n-1}$ is asymptotically stable under BEP($\tau^{\text{all}}, \kappa$) dynamics. Note, however, that the basin of attraction of these intermediate states is again rather small compared with the basin of attraction of the secure state e_1, especially if p is large (see Figure 7).



(i) $p = 5$ (ii) $p = 10$ (iii) $p = 15$

Figure 7: Tacit coordination game with $n = 3$ strategies and $a > 2b > 0$ (see Table 1) under BEP($\tau^{\text{all}}, \kappa = 1$) dynamics (for any tie-breaker), for number of players $p = 5$ (left), $p = 10$ (middle) and $p = 15$ (right).

b) Games with $b = 0$. In this case, strategy n is weakly dominant and the efficient state e_n is the only strict Nash equilibrium state. For $p = 2$ players, this efficient state is almost globally asymptotically stable under both BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) and BEP($\tau^{\text{all}}, 1, \beta^{\text{stick}}$). Besides, for any number of players, e_n is asymptotically stable under BEP($\tau^{\text{all}}, \kappa > 1$) dynamics, and also under the BEP($\tau^{\text{all}}, 1, \beta^{\text{stick}}$) dynamic.[15] Figure 8 illustrates these results for a game with $n = 3$ strategies under the BEP($\tau^{\text{all}}, 1, \beta^{\text{stick}}$) dynamic.



(i) $p = 2$ (ii) $p = 3$ (iii) $p = 15$

Figure 8: Tacit coordination game with $n = 3$ strategies and $b = 0$ (see Table 1) under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{stick}}$) dynamics, for number of players $p = 2$ (left), $p = 3$ (middle) and $p = 15$ (right).

Thus, in general terms (but also with a few exceptions –e.g. see footnote 15), BEP($\tau^{\text{all}}, \kappa$) dynamics with a low number of trials κ seem to exhibit regularities similar to those

[15]However, Proposition 4.4 shows that e_n is unstable under the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic for $p > 3$, and under BEP($\tau^{\alpha}, 1, \beta^{\text{unif}}$) dynamics for $p \geq 2n$.

observed in the experimental evidence for tacit coordination games, i.e.: (i) for $b > 0$, clear discrimination between strict Nash states, selecting the secure state e_1 in large groups but not in games with two players, and (ii) for $b = 0$, a clear tendency to select the weakly dominant strategy.

5.3 Discussion

Strict Nash states are stable under most deterministic evolutionary dynamics (Sandholm, 2014), such as all monotone imitative dynamics (e.g. the replicator dynamics (Taylor and Jonker, 1978)), all sign-preserving excess payoff dynamics (e.g. the BNN dynamic (Brown and von Neumann, 1950)), and all pairwise comparison dynamics (e.g. the Smith (1984) dynamic). The intuition is that, in a small neighborhood of a strict Nash state e_s, the strict Nash strategy s is the unique best reply to every population state in terms of expected payoffs.

However, consider a population state in which most players use the strict Nash strategy s and a small fraction ϵ of players use strategy $j \neq s$. Under random matching, the probability that an s-player happens to be in a p-player group in which there is at least one j-co-player is approximately ϵ times the number $(p-1)$ of co-players (considering a first-order approximation). This means that, given a fixed fraction of j-players in a population, the larger the number of players p in a game, the larger the probability of finding at least one co-player using strategy j. If agents revise their strategies based on the performance that those strategies provide when tested in specific groups of p (randomly drawn) players –instead of looking at the expected payoff of each strategy in the population–, then the number of players p can have a large influence on the population dynamics.

Experimental results in tacit coordination games constitute a clear example of the practical relevance of the number of players in the stability of different strict Nash equilibria, and BEP dynamics in tacit coordination games also illustrate this effect. Focusing on tacit coordination games with $a < 2b$ and BEP($\tau^{\text{all}}, \kappa = 1$) dynamics, suppose that most players (a fraction $1-\epsilon$) use strategy $s > 1$ and a small fraction ϵ of players use strategy $j < s$. Most of the revising s-players who, when testing strategy s, happen to meet a j-co-player in their group, will then adopt some strategy lower than s (under test-all, they will likely adopt strategy $s-1$, because, when testing strategies against a group of s-players, $s-1$ is the second-best reply, after s). As discussed before, the number of such revising agents is roughly proportional to the number of co-players $(p-1)$.[16] To be specific, they will be approximately $(1-\epsilon)\,\epsilon\,(p-1)$. In turn, most of the ϵ j-players will adopt strategy s when

[16]In contrast, note that under best-response dynamics (in terms of expected payoff), no s-player would change strategy for sufficiently low ϵ.

revising. Thus, if $(1-\epsilon)\,\epsilon\,(p-1) > \epsilon$, i.e. if $p > \frac{2-\epsilon}{1-\epsilon}$, state e_s is unstable. This is the intuition why, for a sufficiently large number of co-players ($p > 2$ under τ^{all}; $p > n$ under τ^{α}), the efficient and the intermediate strict Nash states become unstable.[17]

Note that BEP dynamics capture the effect that the number of players p can have in the stability of a strict Nash state in tacit coordination games, via the probability of meeting a deviating j-co-player in a group of p players, which is an increasing function of p. This increasing probability of meeting a deviating j-co-player is also likely to be an important factor to explain the effect of the number of players in the experimental results, as pointed out by Van Huyck et al. (1990, p. 236). In any case, it is important to emphasize that many experimental designs in the literature do not readily fit in the evolutionary framework we have assumed here, and one would expect additional factors to be at play in those experimental studies (see Crawford (1991)).

6. Conclusions

Strict Nash equilibria correspond to rest points under Best Experienced Payoff dynamics, but these rest points may be unstable. In this paper we provide a simple test, with a simple interpretation, that guarantees asymptotic stability under BEP($\tau^{\text{all}}, \kappa$) dynamics. We also provide a related simple test that guarantees instability of strict equilibria under the more general family of BEP(τ^{α}, κ) dynamics.

Focusing on BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics, which is the family of BEP dynamics prevalent in the literature, and for values of the number of trials κ above a small threshold value $\kappa_1 \leq n$, our stability test proves either asymptotic stability or, otherwise, instability. We also show that, for $\kappa > n$ and as κ increases, any strict equilibrium is either always asymptotically stable or there is a single transition from instability to asymptotic stability, within a bounded range of values of κ[18]. Similar results are obtained for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic, for which we present an even tighter asymptotic stability test.

Finally, in order to illustrate our results and to explore the predictive power of BEP dynamics, we have conducted a detailed analysis of the stability of strict equilibria in tacit coordination games. In these games, experimental evidence is at odds with the predictions of most game theoretical analyses.

[17] Similar arguments can be applied for the other sampling dynamics, i.e. *sampling best response* dynamics or *action-sampling* dynamics, under which strict Nash states can also be unstable (Sandholm, 2001).

[18] Sandholm et al. (2020) provide bounds on the values of κ that can correspond to instability.

A. Appendix

A.1 Iterated elimination of strategies

Definition (Survivors of iterated elimination of strategies satisfying condition $\mathbb{C}$ in a finite set Ω)**.** Let $J^0 \equiv \Omega$ and define J^m recursively by

$$J^m = \{i \in J^{m-1} \mid i \text{ does not satisfy condition } \mathbb{C} \text{ in } J^{m-1}\}.$$

The (potentially empty) set $J^{|\Omega|}$ is the set of strategies that survive iterated elimination of strategies satisfying condition $\mathbb{C}$ in set Ω. An algorithm for this procedure is described in Algorithm 1.

Algorithm 1 Iterated elimination of strategies satisfying condition $\mathbb{C}$ in set Ω

$J \leftarrow \Omega$
while $\exists j \in J \mid j$ satisfies condition $\mathbb{C}$ in J **do**
    $J \leftarrow J \setminus \{j \in J \mid j$ satisfies condition $\mathbb{C}$ in $J\}$
end while ▹ J at the end is the set of all surviving strategies after iterated elimination

A.2 Proofs

Note. Bound on the Perron-Frobenius eigenvalue of $DW^+(\mathbf{0})$ *under BEP dynamics, based on the columns of the principal submatrices of* $DW^+(\mathbf{0})$.

Under BEP($\tau^{\text{all}}, \kappa$) dynamics, the inflow (positive) terms in column j of $DW^+(\mathbf{0})$ are associated to the terms $\mathbf{1}[v_{ij}^{\kappa} > v_{ss}^{\kappa}]$, $\mathbf{1}[v_{ts}^{\kappa} > v_{sj}^{\kappa}]$, $\mathbf{1}[v_{ij}^{\kappa} = v_{ss}^{\kappa}]$ or $\mathbf{1}[v_{ts}^{\kappa} = v_{sj}^{\kappa}]$, when the corresponding cases in the brackets hold, i.e., when the indicator function takes the value 1. The inflow associated to the last two terms, $\mathbf{1}[v_{ij}^{\kappa} = v_{ss}^{\kappa}]$ and $\mathbf{1}[v_{ts}^{\kappa} = v_{sj}^{\kappa}]$, is 0 under tie-breaking rules that always select the agent's current strategy if it is among the optimal tested strategies (such as β^{stick}). In this case, the less favorable for the instability of s, the inflow (positive) terms in column j of $DW^+(\mathbf{0})$ are $(p-1)\kappa\,\mathbf{1}[v_{ij}^{\kappa} > v_{ss}^{\kappa}]$, at position $DW_{ij}^+(\mathbf{0})$, plus a total inflow of $(p-1)\kappa\,\mathbf{1}[v_{ts}^{\kappa} > v_{sj}^{\kappa}]$ distributed (according to the tie-breaking rule) among the rows of $DW^+(\mathbf{0})$ corresponding to the strategies in S_2. Consequently, given a subset $J \subseteq S \setminus \{s\}$ and considering its associated principal submatrix $DW_J^+(\mathbf{0})$, corresponding to the strategies in J, the largest value that we can guarantee (for every tie-breaking rule) for the sum of the terms in the column of $DW_J^+(\mathbf{0})$ corresponding to strategy j is $(p-1)\kappa \sum_{i \in J} \mathbf{1}[v_{ij}^{\kappa} > v_{ss}^{\kappa}]$, plus, if $S_2 \subseteq J$, $(p-1)\kappa\,\mathbf{1}[v_{ts}^{\kappa} > v_{sj}^{\kappa}]$. Considering τ^{α}, for $\kappa > \frac{n-1}{(p-1)(\alpha-1)}$ (with τ^{all}, either $p > 2$ or $k > 2$ are enough to satisfy this condition) it can be shown, following arguments similar to the proof of fact 2 in Arigapudi et al. (2021), that proposition 5.4 (ii) in Sandholm et al. (2020), which is based on the bound discussed here (by columns), is more general than proposition 5.4 (i), which is based on a bound by rows that considers only the terms $\mathbf{1}[v^{\kappa}_{ij} > v^{\kappa}_{ss}]$. □

Proof of Proposition 4.1. Considering $\kappa = \kappa_0$, if the iterated elimination of potentially s-stabilizing strategies does not eliminate all strategies in $S \setminus \{s\}$, then there is some non-empty set $J \subseteq S \setminus \{s\}$ which does not contain any potentially s-stabilizing strategies. This implies that for every $j \in J$, either $\exists i \in J$ such that $v^{\kappa}_{ij} > v^{\kappa}_{ss}$ or ($S_2 \subseteq J$ and $v^{\kappa}_{sj} < v^{\kappa}_{ts}$). With these conditions, proposition 5.4 of Sandholm et al. (2020) guarantees instability of the strict equilibrium if $\kappa > \frac{n-1}{(p-1)(\alpha-1)}$. The extension to $\kappa < \kappa_0$ comes from the fact that if a strategy is not potentially s-stabilizing in J for a number of trials κ_0, then it is not potentially s-stabilizing in J for any $\kappa < \kappa_0$. □

Proof of Proposition 4.3. Following Sandholm et al. (2020), consider a change of variables for the population state $(x_1, x_2, ..., x_n)$ that sends the equilibrium e_s to the origin $\mathbf{0}$, by eliminating the coordinate x_s while keeping the labeling of the other coordinates. In this system, consider the Jacobian of the dynamics at the equilibrium, $DW(\mathbf{0})$. Let $DW_J(\mathbf{0})$ be the square submatrix of $DW(\mathbf{0})$ whose rows and columns correspond to the strategies in J. If j is s-stabilizing in J for $\kappa = \kappa_0$, then the column of $DW_J(\mathbf{0})$ corresponding to strategy j is made up (see Sandholm et al. (2020)) by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Let $(j_1, j_2, ..., j_{n-1})$ be an ordering of the $(n-1)$ strategies in $S \setminus \{s\}$ that iteratively eliminates s-stabilizing strategies. Then the column of $DW(\mathbf{0})$ corresponding to strategy j_1 is made up by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Considering the cofactor expansion of the determinant of the Jacobian along the column corresponding to j_1, and denoting by $DW_{-\{j_1\}}(\mathbf{0})$ the submatrix of $DW(\mathbf{0})$ obtained by eliminating the column and row corresponding to j_1, we have that $|DW(\mathbf{0})| = (-1)\,|DW_{-\{j_1\}}(\mathbf{0})|$. Now, the column of $DW_{-\{j_1\}}(\mathbf{0})$ corresponding to strategy j_2 is made up by zeros in all non-diagonal positions, with a value -1 at the diagonal position. Proceeding secuentially with the other strategies we obtain $|DW(\mathbf{0})| = (-1)\,|DW_{-\{j_1\}}(\mathbf{0})| = (-1)^2\,|DW_{-\{j_1,j_2\}}(\mathbf{0})| = ... = (-1)^{n-1}$, i.e., all the eigenvalues of the Jacobian have negative real parts, which implies asymptotic stability of the equilibrium. The result for $\kappa \geq \kappa_0$ follows from the fact that if a strategy is s-stabilizing in J for a number of trials κ_0, then it is s-stabilizing in J for any $\kappa > \kappa_0$. □

Proof of Proposition 4.4. The stability part comes from Proposition 4.3. For the instability part, first consider $\kappa = \kappa_0$. If the iterated elimination of s-stabilizing strategies does not eliminate all strategies in $S \setminus \{s\}$, then there is some non-empty set $J \subseteq S \setminus \{s\}$ which does not contain any s-stabilizing strategies. This means that for every $j \in J$, either $\exists i \in J$ such that $v^\kappa_{ij} \geq v^\kappa_{ss}$ or ($S_2 \cap J \neq \emptyset$ and $v^\kappa_{sj} \leq v^\kappa_{ts}$). Considering this and Lemma A.1 below, which is a direct adaptation of proposition 5.4 in Sandholm et al. (2020) for the BEP($\tau^\alpha, \kappa, \beta^{\text{unif}}$) dynamics, we have that the minimum possible value of the left hand side on Equation (4) is $(p-1)\kappa\frac{1}{|S_2|+1}$, so the condition $\kappa > \frac{|S_2|+1}{p-1}$ guarantees instability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics. If $\frac{v^1_{ss} - \min_{j \in S \setminus \{s\}} v^1_{sj}}{v^1_{ss} - v^1_{ts}} < \kappa$, then $v^\kappa_{sj} > v^\kappa_{ts}$ for all $j \neq s$ and the minimum possible value indicated before is $(p-1)\kappa\frac{1}{2}$, so the condition $\kappa > \frac{2}{p-1}$ guarantees instability. The adaptation of these results to BEP($\tau^\alpha, \kappa, \beta^{\text{unif}}$) dynamics is immediate considering Equation (5). The extension to $\kappa < \kappa_0$ comes from the fact that if a strategy is not s-stabilizing in J for a number of trials κ_0, then it is not s-stabilizing in J for any $\kappa < \kappa_0$.

Lemma A.1. *Let e_s be a strict equilibrium, let $S_2 = \operatorname{argmax}_{i \neq s} U(i; s, s, ..., s)$, and let $t \in S_2$. Under any BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic, state e_s is linearly unstable if, for some nonempty $J \subseteq S \setminus \{s\}$, the following condition holds for all $j \in J$:*

$$(4)\qquad (p-1)\kappa\left(\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \tfrac{1}{2}\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} = v^\kappa_{ss}]\right) + (p-1)\kappa\,|S_2 \cap J|\left(\tfrac{1}{|S_2|}\mathbf{1}[v^\kappa_{sj} < v^\kappa_{ts}] + \tfrac{1}{|S_2|+1}\mathbf{1}[v^\kappa_{sj} = v^\kappa_{ts}]\right) > 1$$

And under any BEP($\tau^\alpha, \kappa, \beta^{\text{unif}}$) dynamic, letting $b = \min(|S_2|, \alpha - 1)$, state e_s is linearly unstable if, for some nonempty $J \subseteq S \setminus \{s\}$, the following condition holds for all $j \in J$:

$$(5)\qquad (p-1)\kappa\,\frac{\alpha-1}{n-1}\left(\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \tfrac{1}{2}\sum_{i \in J} \mathbf{1}[v^\kappa_{ij} = v^\kappa_{ss}]\right) + (p-1)\kappa\,\frac{\alpha-1}{n-1}\,|S_2 \cap J|\left(\tfrac{1}{b}\mathbf{1}[v^\kappa_{sj} < v^\kappa_{ts}] + \tfrac{1}{b+1}\mathbf{1}[v^\kappa_{sj} = v^\kappa_{ts}]\right) > 1$$

□

Proof of Proposition 4.5. The stability part comes from adapting the proof of Proposition 4.3 to the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic, considering that the Jacobian $DW(\mathbf{0})$ for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic has components (Sandholm et al., 2020):

$$DW_{ij}(\mathbf{0}) = \begin{cases} (p-1)\kappa\;\mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] - \mathbf{1}[j = i] & \text{if } i \notin S_2, \\ (p-1)\kappa\left(\mathbf{1}[v^\kappa_{ij} > v^\kappa_{ss}] + \frac{1}{|S_2|}\mathbf{1}[v^\kappa_{is} > v^\kappa_{sj}]\right) - \mathbf{1}[j = i] & \text{if } i \in S_2. \end{cases}$$

For the instability part follow the steps in the proof of Proposition 4.4, noting that if a non-empty set $J \subseteq S \setminus \{s\}$ does not contain any weakly s-stabilizing strategies, then, for every $j \in J$, either $\exists i \in J$ such that $v^{\kappa}_{ij} > v^{\kappa}_{ss}$ or ($S_2 \cap J \neq \varnothing$ and $v^{\kappa}_{sj} < v^{\kappa}_{ts}$). Note also that the equivalent of Equation (2) for the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamic is

$$(p-1)\kappa \left(\sum_{i \in J} \mathbf{1}[v^{\kappa}_{ij} > v^{\kappa}_{ss}] + |S_2 \cap J| \left(\tfrac{1}{|S_2|} \mathbf{1}[v^{\kappa}_{sj} < v^{\kappa}_{ts}] \right) \right) > 1$$

□

Proofs of statements in Section 5.2 (Results on Tacit Coordination Games). For the analysis of the stability of the strict Nash states of a p-player tacit coordination game under BEP dynamics, we calculate the values $v^{\kappa,s}_{ij}$, which in this case are:

$$\begin{aligned} v^{\kappa,s}_{ij} &= (\kappa-1)U(i; s, s, ..., s) + U(i; j, s, ..., s) \\ &= (\kappa-1)\, a\, \min(i,s) - \kappa\, b\, i + a \begin{cases} \min(i,j) & \text{if } p = 2 \\ \min(i,j,s) & \text{if } p > 2 \end{cases}. \end{aligned}$$

The $n \times n$ matrices $V^{\kappa=1,s}$ for $p = 2$ and for $p > 2$ are shown in tables 2 and 3 respectively. Note that $V^{\kappa=1,s}$ for $p = 2$ (Table 2) is the payoff matrix U_{ij}. Matrices $V^{\kappa,s}$ for $\kappa > 1$ can be easily calculated from the corresponding matrix $V^{\kappa=1,s}$ by adding, to every column in $V^{\kappa=1,s}$, column s of $V^{\kappa=1,s}$ times $(\kappa - 1)$.

Table 2: Matrix $V^{\kappa=1,s}$ for tacit coordination games with $p = 2$ players. This is also the payoff matrix of the game if $p = 2$.

	1	2	3	...	n
1	$a-b$	$a-b$	$a-b$	...	$a-b$
2	$a-2b$	$2a-2b$	$2a-2b$	...	$2a-2b$
3	$a-3b$	$2a-3b$	$3a-3b$	...	$3a-3b$
⋮	⋮	⋮	⋮	⋱	⋮
n	$a-nb$	$2a-nb$	$3a-nb$	...	$na-nb$

a) Games with $b > 0$.

Results about the stability of the efficient state e_n and of the secure state e_1.

– For $p = 2$, the efficient state e_n is Lyapunov stable under any BEP($\tau^{\text{all}}, 1$).

Proof. Direct application of Proposition 5.11(i) in Sandholm et al. (2020), noting that $U_{nn} > U_{ij}$ for all $i, j \neq n$ (see Table 2). □

Table 3: Matrix $V^{\kappa=1,s}$ for tacit coordination games with more than two players (i.e. $p > 2$).

	1	2	…	$s-1$	s	$s+1$	…	n
1	$a-b$	$a-b$	…	$a-b$	$a-b$	$a-b$	…	$a-b$
2	$a-2b$	$2a-2b$	…	$2a-2b$	$2a-2b$	$2a-2b$	…	$2a-2b$
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮
$s-1$	$a-(s-1)b$	$2a-(s-1)b$	…	$(s-1)(a-b)$	$(s-1)(a-b)$	$(s-1)(a-b)$	…	$(s-1)(a-b)$
s	$a-sb$	$2a-sb$	…	$(s-1)a-sb$	$s(a-b)$	$s(a-b)$	…	$s(a-b)$
$s+1$	$a-(s+1)b$	$2a-(s+1)b$	…	$(s-1)a-(s+1)b$	$sa-(s+1)b$	$sa-(s+1)b$	…	$sa-(s+1)b$
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮
n	$a-nb$	$2a-nb$	…	$(s-1)a-nb$	$sa-nb$	$sa-nb$	…	$sa-nb$

– For $p = 2$ and $n > 1 + \frac{a}{a-b}$, the secure state e_1 is unstable under any BEP($\tau^{\text{all}}, 1$).

  Proof. Direct application of Proposition 5.4(i) in Sandholm et al. (2020), considering the subset of strategies $J = \{n, n-1\}$ and noting that, if $n > 1 + \frac{a}{a-b}$, then $U_{11} < U_{ij}$ for $i, j \in J$ (see Table 2). □

– Proposition 4.1 shows that the efficient state e_n is unstable under any BEP($\tau^{\text{all}}, 1$) dynamic if $p > 2$, and under every BEP($\tau^{\alpha}, 1$) dynamics if $p > n$.

  Proof. Table 3 shows matrix $V^{\kappa=1,s}$ for $p > 2$. For $s = n$ we have $S_2 = \{n-1\}$ and $v^{1,s}_{s(n-1)} = (n-1)\,a - n\,b < (n-1)(a-b) = v^{1,s}_{ts}$. Consequently, strategy $(n-1)$ is not potentially n-stabilizing in any set that contains it, and Proposition 4.1 shows that e_n is unstable under the BEP($\tau^{\text{all}}, 1$) dynamic for $p > 2$, and under every BEP($\tau^{\alpha}, 1$) dynamics for $p > n$. □

– Proposition 4.3 shows that the secure state e_1 is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa$) dynamics if $p > 2$.

  Proof. Table 3 shows matrix $V^{\kappa=1,s}$ for $p > 2$. For $s = 1$ and $\kappa = 1$, the condition $v^{\kappa=1}_{ij} < v^{\kappa=1}_{ss} = s\,(a-b)$, which implies satisfaction of part of the conditions for s-stabilizing strategies, holds for all $i, j \neq s$. Looking at matrix $V^{\kappa=1,s=1}$, we have $S_2 = \{2\}$ and $v^{1,s}_{sj} = a - b > a - 2b = v^{1,s}_{ts}$. Consequently, all strategies are 1-stabilizing in $S \setminus \{1\}$ for $\kappa = 1$, and we can apply Proposition 4.3 to state that the secure state e_1 is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa$). □

Results about the stability of the intermediate strict Nash states $e_2, ..., e_{n-1}$.

i) $a < 2b$. Proposition 4.1 shows that every intermediate state $e_2, ..., e_{n-1}$ is unstable under BEP($\tau^{\text{all}}, \kappa < \frac{a}{a-b}$) dynamics for $p > 2$, and under every BEP($\tau^{\alpha}, \kappa < \frac{a}{a-b}$) dynamics for $p > n$. Conversely, if $\kappa > \frac{a}{a-b}$, then Proposition 4.3 shows that every intermediate state is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > \frac{a}{a-b}$) dynamics for $p > 2$. The case $\kappa = \frac{a}{a-b}$ depends on the tie-breaking rule. For $p > 2$, Proposition 4.4 shows that every intermediate state is unstable under BEP($\tau^{\text{all}}, \kappa = \frac{a}{a-b}, \beta^{\text{unif}}$), and Proposition 4.5 shows that every intermediate state is stable under BEP($\tau^{\text{all}}, \kappa = \frac{a}{a-b}, \beta^{\text{stick}}$).

Proof. Table 3 shows matrix $V^{\kappa=1,s}$ for $p > 2$. For $s \in \{2, ..., n-1\}$, the condition $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$, which implies satisfaction of part of the conditions for s-stabilizing strategies, holds for all $i, j \neq s$. Given that $a < 2b$, we have $S_2 = \{s-1\}$. We can now compute $v^{\kappa,s}_{s(s-1)} = \kappa s(a-b) - a$, and $v^{\kappa,s}_{ts} = v^{\kappa,s}_{(s-1)s} = \kappa(s-1)(a-b)$. Therefore, the condition $v^{\kappa,s}_{s(s-1)} \geq v^{\kappa,s}_{ts}$ holds if and only if $\kappa \geq \frac{a}{a-b}$.

Thus, if $\kappa < \frac{a}{a-b}$, then $v^{\kappa,s}_{s(s-1)} < v^{\kappa,s}_{ts}$, so strategy $(s-1)$ is not potentially s-stabilizing in any set that contains it, and Proposition 4.1 shows that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is unstable under BEP($\tau^{\text{all}}, \kappa < \frac{a}{a-b}$) dynamics for $p > 2$, and under every BEP($\tau^{\alpha}, \kappa < \frac{a}{a-b}$) dynamics for $p > n$.

Conversely, if $\kappa > \frac{a}{a-b}$, then $v^{\kappa,s}_{s(s-1)} > v^{\kappa,s}_{ts}$, so strategy $(s-1)$ is s-stabilizing in $S \setminus \{s\}$. After eliminating strategy $(s-1)$, all the other strategies are s-stabilizing in $J = S \setminus \{s, s-1\}$, since $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$ for all $i, j \neq s$ and $S_2 \cap J = \varnothing$. Thus, no strategy survives the iterated elimination of s-stabilizing strategies, and we can apply Proposition 4.3 to state that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > \frac{a}{a-b}$).

The case $\kappa = \frac{a}{a-b} > 2$ depends on the tie-breaking rule.[19] For $p > 2$, Proposition 4.4 can be applied to prove that every intermediate state is unstable under BEP($\tau^{\text{all}}, \kappa = \frac{a}{a-b}, \beta^{\text{unif}}$),[20] and Proposition 4.5 can be applied to prove that every intermediate state is asymptotically stable under BEP($\tau^{\text{all}}, \kappa = \frac{a}{a-b}, \beta^{\text{stick}}$).[21] □

ii) $a = 2b$. Proposition 4.5 shows that every intermediate state $e_2, ..., e_{n-1}$ is unstable under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{stick}}$) dynamics for $p > 3$. Proposition 4.4 shows that they are also unstable under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{unif}}$) dynamics for $p > 4$ and under every BEP($\tau^{\alpha}, \kappa = 1, \beta^{\text{unif}}$) dynamics for $p \geq 2n$.

Conversely, if $\kappa > 2$, then Proposition 4.3 shows that every intermediate state is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > 2$) dynamics for $p > 2$.

[19] Given that $a < 2b$, we have $\frac{a}{a-b} > 2$.

[20] In this case, strategy $(s-1)$ is not s-stabilizing in any set that contains it and $\frac{|S_2|+1}{p-1} = \frac{2}{p-1} < 2$ if $p > 2$.

[21] In this case, strategy $(s-1)$ is weakly s-stabilizing in $S \setminus \{s\}$. After eliminating strategy $(s-1)$, all the other strategies are weakly s-stabilizing in $J = S \setminus \{s, s-1\}$, since $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$ for all $i, j \neq s$ and $S_2 \cap J = \varnothing$. Thus, no strategy survives the iterated elimination of weakly s-stabilizing strategies.

The borderline case $\kappa = 2$ depends on the tie-breaking rule. For $p > 2$, Proposition 4.4 shows that every intermediate state is unstable under BEP($\tau^{\text{all}}, \kappa = 2, \beta^{\text{unif}}$), and Proposition 4.5 shows that every intermediate state is stable under BEP($\tau^{\text{all}}, \kappa = 2, \beta^{\text{stick}}$).

Proof. Table 3 shows matrix $V^{\kappa=1,s}$ for $p > 2$. For $s \in \{2, ..., n-1\}$, the condition $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$, which implies satisfaction of part of the conditions for s-stabilizing strategies, holds for all $i, j \neq s$. Given that $a = 2b$, we have $S_2 = \{s-1, s+1\}$, and $v^{\kappa,s}_{ts} = v^{\kappa,s}_{(s-1)s} = v^{\kappa,s}_{(s+1)s} = \kappa\,(s-1)(a-b)$. We also have $v^{\kappa,s}_{s(s-1)} = \kappa\, s\,(a-b) - a$ and $v^{\kappa,s}_{s(s+1)} = \kappa\, s\,(a-b)$. Thus, $v^{\kappa,s}_{s(s-1)} > v^{\kappa,s}_{ts}$ if and only if $\kappa > \frac{a}{a-b} = 2$; and it is always the case that $v^{\kappa,s}_{s(s+1)} > v^{\kappa,s}_{ts}$.

Therefore, if $\kappa > \frac{a}{a-b} = 2$, then $v^{\kappa,s}_{s(s-1)} > v^{\kappa,s}_{ts}$, so strategies $(s-1)$ and $(s+1)$ are both s-stabilizing in $S \setminus \{s\}$. After eliminating both of them, all the other strategies are s-stabilizing in $J = S \setminus \{s, s-1, s+1\}$, since $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$ for all $i, j \neq s$ and $S_2 \cap J = \varnothing$. Thus, no strategy survives the iterated elimination of s-stabilizing strategies, and we can apply Proposition 4.3 to state that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > \frac{a}{a-b} = 2$) if $p > 2$.

The case where $\kappa \leq \frac{a}{a-b} = 2$ depends on the tie-breaker. Let us start with β^{unif}. If $\kappa \leq \frac{a}{a-b} = 2$, then $v^{\kappa,s}_{s(s-1)} \leq v^{\kappa,s}_{ts}$, so strategy $(s-1)$ is not s-stabilizing in any set that contains it. Thus, Proposition 4.4 can be applied to prove that every intermediate state is unstable under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{unif}}$) if $p > 4$, under BEP($\tau^{\text{all}}, \kappa = 2, \beta^{\text{unif}}$) if $p > 2$, and under every BEP($\tau^{\alpha}, \kappa = 1, \beta^{\text{unif}}$) dynamics for $p \geq 2n$.[22]

Let us now focus on β^{stick}. If $\kappa = \frac{a}{a-b} = 2$, then $v^{\kappa,s}_{s(s-1)} = v^{\kappa,s}_{ts}$, so strategies $(s-1)$ and $(s+1)$ are both weakly s-stabilizing in $S \setminus \{s\}$. After eliminating both of them, all the other strategies are weakly s-stabilizing in $J = S \setminus \{s, s-1, s+1\}$, since $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$ for all $i, j \neq s$ and $S_2 \cap J = \varnothing$. Thus, no strategy survives the iterated elimination of weakly s-stabilizing strategies, and we can apply Proposition 4.5 to state that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa = 2, \beta^{\text{stick}}$).

If $\kappa = \frac{a}{a-b} < 2$, then $v^{\kappa,s}_{s(s-1)} < v^{\kappa,s}_{ts}$, so strategy $(s-1)$ is not weakly s-stabilizing in any set that contains it. In this case, Proposition 4.5 shows that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is unstable under BEP($\tau^{\text{all}}, \kappa = 1, \beta^{\text{stick}}$) dynamics for

[22] Note that $\frac{|S_2|+1}{p-1} = \frac{3}{p-1} < 1$ if $p > 4$; $\frac{|S_2|+1}{p-1} = \frac{3}{p-1} < 2$ if $p > 2$; and $\frac{n-1}{\alpha-1} \frac{\min(|S_2|+1, \alpha)}{p-1} < 1$ if $p \geq 2n$.

$p > 3$.[23] □

iii) $a > 2b$. Proposition 4.3 shows that every intermediate state $e_2, ..., e_{n-1}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa$) dynamics for $p > 2$.

Proof. Table 3 shows matrix $V^{\kappa=1,s}$ for $p > 2$. For $s \in \{2, ..., n-1\}$, the condition $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$, which implies satisfaction of part of the conditions for s-stabilizing strategies, holds for all $i, j \neq s$. Given that $a > 2b$, we have $S_2 = \{s+1\}$. We can now compute $v^{\kappa,s}_{s(s+1)} = \kappa\, s\,(a-b)$, and $v^{\kappa,s}_{ts} = v^{\kappa,s}_{(s+1)s} = \kappa\,(s\,(a-b)-b)$. Therefore, the condition $v^{\kappa,s}_{s(s+1)} > v^{\kappa,s}_{ts}$ always holds. Thus, strategy $(s+1)$ is s-stabilizing in $S \setminus \{s\}$. After eliminating strategy $(s+1)$, all the other strategies are s-stabilizing in $J = S \setminus \{s, s+1\}$, since $v^{\kappa}_{ij} < v^{\kappa}_{ss} = \kappa\, s\,(a-b)$ for all $i, j \neq s$ and $S_2 \cap J = \emptyset$. Thus, no strategy survives the iterated elimination of s-stabilizing strategies, and we can apply Proposition 4.3 to state that every intermediate state $e_s \in \{e_2, ..., e_{n-1}\}$ is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa$). □

b) Games with $b = 0$.

– For $p = 2$ players, the efficient state e_n is almost globally asymptotically stable under both BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) and BEP($\tau^{\text{all}}, 1, \beta^{\text{stick}}$).

Proof. Direct application of Proposition 5.11(ii) in Sandholm et al. (2020), noting that if $s = n$, for all $i, j \neq s$, we have $U_{sj} = a\, j \geq \min(U_{ij}, U_{is}) = \min(a\ \min(i,j), i\, a) = a\ \min(i,j)$ (see Table 2). □

– For $p \geq 2$, Proposition 4.3 shows that e_n is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > 1$) dynamics, and Proposition 4.5 shows that e_n is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) dynamics.

Proof. For $s = n$ and $i, j \neq s$, we have $v^{\kappa}_{ij} = (\kappa-1)\, i\, a + a\ \min(i,j)$, and $v^{\kappa}_{ss} = (\kappa-1)\, n\, a + a\, n = \kappa\, n\, a$. Thus, $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ for all $i, j \neq s$, which implies satisfaction of part of the conditions for s-stabilizing strategies and for weakly s-stabilizing strategies.

Looking at matrix $V^{\kappa=1,s=n}$, we have $S_2 = \{n-1\}$. We can now compute $v^{\kappa,s}_{s(n-1)} = v^{\kappa,s}_{n(n-1)} = (\kappa-1)\, a\, n + a\,(n-1)$, and $v^{\kappa,s}_{ts} = v^{\kappa,s}_{(n-1)n} = (\kappa-1)\, a\,(n-1) + a\,(n-1)$. Therefore, condition $v^{\kappa,s}_{s(n-1)} > v^{\kappa,s}_{ts}$ holds if $\kappa > 1$, and condition $v^{\kappa,s}_{s(n-1)} \geq v^{\kappa,s}_{ts}$ holds if $\kappa \geq 1$. Thus, strategy $(s+1)$ is s-stabilizing in $S \setminus \{s\}$ if $\kappa > 1$ and weakly

[23] Note that $\frac{|S_2|}{p-1} = \frac{2}{p-1} < 1$ if $p > 3$.

s-stabilizing in $S \setminus \{s\}$ for $\kappa = 1$. After eliminating strategy $(n-1)$, all the other strategies are s-stabilizing in $J = S \setminus \{s, n-1\}$ for any κ, since $v^{\kappa}_{ij} < v^{\kappa}_{ss}$ for all $i, j \neq s$ and $S_2 \cap J = \emptyset$.

Thus, if $\kappa > 1$, no strategy survives the iterated elimination of s-stabilizing strategies, and we can apply Proposition 4.3 to state that e_n is asymptotically stable is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa > 1$).

Similarly, if $\kappa = 1$, no strategy survives the iterated elimination of weakly s-stabilizing strategies, and we can apply Proposition 4.5 to state that e_n is asymptotically stable under every BEP($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$). □

– Proposition 4.4 shows that, under BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, e_n is unstable for every number of players $p > 3$, and under every BEP($\tau^{\alpha}, 1, \beta^{\text{unif}}$) dynamics for $p \geq 2n$.

Proof. Looking at matrix $V^{\kappa=1,s=n}$, we have $S_2 = \{n-1\}$. We can now compute $v^{\kappa=1,s}_{s(n-1)} = v^{1,n}_{n(n-1)} = a\,(n-1)$, and $v^{\kappa=1,s}_{ts} = v^{1,n}_{(n-1)n} = a\,(n-1)$. Therefore, $v^{\kappa,s}_{s(n-1)} = v^{\kappa,s}_{ts}$. Consequently, strategy $(n-1)$ is not n-stabilizing in any set that contains it, and Proposition 4.4 shows that e_n is unstable under the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic for $p > 3$, and under every BEP($\tau^{\alpha}, 1, \beta^{\text{unif}}$) dynamics for $p \geq 2n$.[24] □

□

References

Arigapudi, S., Heller, Y., and Milchtaich, I. (2021). Instability of defection in the prisoner's dilemma under best experienced payoff dynamics. *Journal of Economic Theory*, 197:105174.

Arigapudi, S., Heller, Y., and Schreiber, A. (2022). Sampling dynamics and stable mixing in hawk-dove games. Working paper available at `https://arxiv.org/abs/2107.08423`.

Benaïm, M. and Weibull, J. W. (2003). Deterministic approximation of stochastic evolution in games. *Econometrica*, 71:873–903.

Berkemer, R. (2008). Disputable advantage of experience in the traveler's dilemma. Unpublished manuscript, Technical University of Denmark. Abstract in *International Conference on Economic Science with Heterogeneous Interacting Agents, Warsaw, 2008*.

[24]Note that $\frac{|S_2|}{p-1} = \frac{2}{p-1} < 1$ if $p > 3$; and $\frac{n-1}{\alpha-1}\frac{\min(|S_2|+1,\alpha)}{p-1} = \frac{n-1}{\alpha-1}\frac{2}{p-1} < 1$ if $p \geq 2n$.

Brown, G. W. and von Neumann, J. (1950). Solutions of games by differential equations. In Kuhn, H. W. and Tucker, A. W., editors, *Contributions to the Theory of Games I*, volume 24 of *Annals of Mathematics Studies*, pages 73–79. Princeton University Press, Princeton.

Cárdenas, J. C., Mantilla, C., and Sethi, R. (2015). Stable sampling equilibrium in common pool resource games. *Games*, 6:299–317.

Chmura, T. and Güth, W. (2011). The minority of three-game: An experimental and theoretical analysis. *Games*, 2:333–354.

Crawford, V. P. (1991). An "evolutionary" interpretation of Van Huyck, Battalio, and Beil's experimental results on coordination. *Games and Economic Behavior*, 3(1):25–59.

Heller, Y. and Mohlin, E. (2018). Social learning and the shadow of the past. *Journal of Economic Theory*, 177:426–460.

Izquierdo, L. R. and Izquierdo, S. S. (2022). BEP-TCG. Software available at `https://doi.org/10.5281/zenodo.6975573`.

Izquierdo, L. R., Izquierdo, S. S., and Rodriguez, J. (2022). Fast and scalable global convergence in single-optimum decentralized coordination problems. *IEEE Transactions on Control of Network Systems*.

Izquierdo, L. R., Izquierdo, S. S., and Sandholm, W. H. (2018). EvoDyn-3s: A Mathematica computable document to analyze evolutionary dynamics in 3-strategy games. *SoftwareX*, 7:226–233.

Izquierdo, S. S. and Izquierdo, L. R. (2021). "Test two, choose the better" leads to high cooperation in the centipede game. *Journal of Dynamics and Games*.

Kosfeld, M., Droste, E., and Voorneveld, M. (2002). A myopic adjustment process leading to best reply matching. *Journal of Economic Theory*, 40:270–298.

Kreindler, G. E. and Young, H. P. (2013). Fast convergence in evolutionary equilibrium selection. *Games and Economic Behavior*, 80:39–67.

Mantilla, C., Sethi, R., and Cárdenas, J. C. (2020). Efficiency and stability of sampling equilibrium in public goods games. *Journal of Public Economic Theory*, 22(2):355–370.

Miękisz, J. and Ramsza, M. (2013). Sampling dynamics of a symmetric ultimatum game. *Dynamic Games and Applications*, 3:374–386.

Osborne, M. J. and Rubinstein, A. (1998). Games with procedurally rational players. *American Economic Review*, 88:834–847.

Osborne, M. J. and Rubinstein, A. (2003). Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior*, 45(2):434–441. Special Issue in Honor of Robert W. Rosenthal.

Oyama, D., Sandholm, W. H., and Tercieux, O. (2015). Sampling best response dynamics and deterministic equilibrium selection. *Theoretical Economics*, 10:243–281.

Perko, L. (2001). *Differential Equations and Dynamical Systems*. Springer, New York, third edition.

Rowthorn, R. and Sethi, R. (2008). Procedural rationality and equilibrium trust. *The Economic Journal*, 118:889–905.

Salant, Y. and Cherry, J. (2020). Statistical inference in games. *Econometrica*, 88(4):1725–1752.

Sandholm, W. H. (2001). Almost global convergence to p-dominant equilibrium. *International Journal of Game Theory*, 30:107–116.

Sandholm, W. H. (2010). *Population Games and Evolutionary Dynamics*. MIT Press, Cambridge.

Sandholm, W. H. (2014). Local stability of strict equilibria under evolutionary game dynamics. *Journal of Dynamics and Games*, 1(3):485–495.

Sandholm, W. H., Izquierdo, S. S., and Izquierdo, L. R. (2019). Best experienced payoff dynamics and cooperation in the centipede game. *Theoretical Economics*, 14(4):1347–1385.

Sandholm, W. H., Izquierdo, S. S., and Izquierdo, L. R. (2020). Stability for best experienced payoff dynamics. *Journal of Economic Theory*, 185:104957.

Sawa, R. and Wu, J. (2021). Statistical inference in evolutionary dynamics. Available at http://dx.doi.org/10.2139/ssrn.3767635.

Sethi, R. (2000). Stability of equilibria in games with procedurally rational players. *Games and Economic Behavior*, 32:85–104.

Sethi, R. (2021). Stable sampling in repeated games. *Journal of Economic Theory*, 197:105343.

Smith, M. J. (1984). The stability of a dynamic model of traffic assignment—an application of a method of Lyapunov. *Transportation Science*, 18:245–252.

Spiegler, R. (2006a). Competition over agents with boundedly rational expectations. *Theoretical Economics*, 1(2):207–231.

Spiegler, R. (2006b). The Market for Quacks. *The Review of Economic Studies*, 73(4):1113–1131.

Taylor, P. D. and Jonker, L. (1978). Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences*, 40:145–156.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *The American Economic Review*, 80(1):234–248.

Weibull, J. W. (1995). *Evolutionary Game Theory*. MIT Press, Cambridge.