

A maximally superintegrable deformation of the N -dimensional quantum Kepler–Coulomb system

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Abstract. The N -dimensional quantum Hamiltonian

$$\hat{H} = -\frac{\hbar^2|\mathbf{q}|}{2(\eta + |\mathbf{q}|)}\nabla^2 - \frac{k}{\eta + |\mathbf{q}|}$$

is shown to be exactly solvable for any real positive value of the parameter η . Algebraically, this Hamiltonian system can be regarded as a new maximally superintegrable η -deformation of the N -dimensional Kepler–Coulomb Hamiltonian while, from a geometric viewpoint, this superintegrable Hamiltonian can be interpreted as a system on an N -dimensional Riemannian space with nonconstant curvature. The eigenvalues and eigenfunctions of the model are explicitly obtained, and the spectrum presents a hydrogen-like shape for positive values of the deformation parameter η and of the coupling constant k .

1. Introduction

Let us consider the N -dimensional (ND) classical Hamiltonian given by

$$\mathcal{H}_\eta(\mathbf{q}, \mathbf{p}) = \mathcal{T}_\eta(\mathbf{q}, \mathbf{p}) + \mathcal{U}_\eta(\mathbf{q}) = \frac{|\mathbf{q}|\mathbf{p}^2}{2(\eta + |\mathbf{q}|)} - \frac{k}{\eta + |\mathbf{q}|}, \quad (1)$$

where k and η are real parameters, $\mathbf{q} = (q_1, \dots, q_N)$, $\mathbf{p} = (p_1, \dots, p_N) \in \mathbb{R}^N$ are conjugate coordinates and momenta, and $\mathbf{q}^2 \equiv |\mathbf{q}|^2 = \sum_{i=1}^N q_i^2$. We recall that \mathcal{H}_η has been proven to be a maximally superintegrable Hamiltonian by making use of symmetry techniques [1]. This means that \mathcal{H}_η is endowed with the maximum possible number of $(2N - 1)$ functionally independent constants of motion (including \mathcal{H}_η itself).

Explicitly, $(2N - 3)$ of such integrals are provided by the radial symmetry of the system, namely,

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j \leq m} (q_i p_j - q_j p_i)^2, \quad \mathcal{C}_{(m)} = \sum_{N-m < i < j \leq N} (q_i p_j - q_j p_i)^2, \quad m = 2, \dots, N; \quad (2)$$



such that $\mathcal{C}^{(N)} = \mathcal{C}_{(N)} \equiv \mathbf{L}^2$ is the square of the total angular momentum. Furthermore, \mathcal{H}_η is endowed with an ND Laplace–Runge–Lenz vector \mathbf{R} . This means there exist N additional constants of motion coming from the components of \mathbf{R} , which are given by

$$\mathcal{R}_i = \sum_{j=1}^N p_j (q_j p_i - q_i p_j) + \frac{q_i}{|\mathbf{q}|} (\eta \mathcal{H}_\eta + k), \quad i = 1, \dots, N. \quad (3)$$

The squared modulus of \mathbf{R} is radially symmetric, and turns out to be expressible in terms of \mathcal{H}_η and \mathbf{L}^2 :

$$\mathbf{R}^2 = \sum_{i=1}^N \mathcal{R}_i^2 = 2\mathbf{L}^2 \mathcal{H}_\eta + (\eta \mathcal{H}_\eta + k)^2.$$

We remark that each of the three sets $\{\mathcal{H}_\eta, \mathcal{C}^{(m)}\}$, $\{\mathcal{H}_\eta, \mathcal{C}_{(m)}\}$ ($m = 2, \dots, N$) and $\{\mathcal{R}_i\}$ ($i = 1, \dots, N$) is formed by N functionally independent functions in involution, and the set $\{\mathcal{H}_\eta, \mathcal{C}^{(m)}, \mathcal{C}_{(m)}, \mathcal{R}_i\}$ for $m = 2, \dots, N$ with a fixed i provides the set of $(2N - 1)$ functionally independent functions. As it was shown in [1], the set of constants of the motion \mathcal{R}_i can be obtained explicitly by applying a Stäckel transform [2, 3] to the nondeformed Kepler–Coulomb (KC) Hamiltonian.

This maximally superintegrable Hamiltonian is obviously endowed with an $\mathfrak{so}(N)$ Lie–Poisson symmetry, since it can be constructed on an ND spherically symmetric space. In particular, we can consider the $N(N - 1)/2$ generators of rotations $J_{ij} = q_i p_j - q_j p_i$ with $i < j$ and $i, j = 1, \dots, N$ which span the $\mathfrak{so}(N)$ Lie–Poisson algebra with Poisson brackets given by

$$\{J_{ij}, J_{ik}\} = J_{jk}, \quad \{J_{ij}, J_{jk}\} = -J_{ik}, \quad \{J_{ik}, J_{jk}\} = J_{ij}, \quad i < j < k.$$

Hence the $(2N - 3)$ angular momentum integrals $\mathcal{C}^{(m)}$ and $\mathcal{C}_{(m)}$ (2) turn out to be the quadratic Casimirs of some rotation subalgebras $\mathfrak{so}(m) \subset \mathfrak{so}(N)$. Moreover we observe that the ‘ η -deformation’ of the Laplace–Runge–Lenz vector \mathcal{R}_i (3) closes the same Poisson algebra as its nondeformed counterpart, namely:

$$\{J_{ij}, \mathcal{R}_k\} = \delta_{ik} \mathcal{R}_j - \delta_{jk} \mathcal{R}_i, \quad \{\mathcal{R}_i, \mathcal{R}_j\} = -2\mathcal{H}_\eta J_{ij}.$$

Therefore, since

$$\{J_{ij}, \mathcal{H}_\eta\} = \{\mathcal{R}_i, \mathcal{H}_\eta\} = 0, \quad \forall i, j,$$

the Hamiltonian \mathcal{H}_η behaves as a ‘constant’ with respect to the $N(N + 1)/2$ ‘generators’ $\{J_{ij}, \mathcal{R}_i\}$. This fact makes possible to identify the classical integrals of the motion with the generators of an $\mathfrak{so}(N + 1)$ algebra similarly to what happens with the usual ND Euclidean KC system (see [4] and references therein).

We stress that maximally superintegrable Hamiltonians in N dimensions are quite scarce, even on the Euclidean space. The two representative examples of this class of systems are the KC system and the isotropic harmonic oscillator, for which all bounded trajectories are periodic (Bertrand’s Theorem). In this respect, we recall that a maximally superintegrable ‘deformation/generalization’ of the ND isotropic oscillator was firstly presented in [5] and its quantum counterpart was constructed and solved in [6, 7]. In fact, on the same footing of such a maximally superintegrable oscillator system, the system \mathcal{H}_η (1) can be regarded as a genuine (maximally superintegrable) η -deformation of the ND usual KC system, since the limit $\eta \rightarrow 0$ of \mathcal{H}_η (1) yields

$$\mathcal{H}_0 = \frac{1}{2} \mathbf{p}^2 - \frac{k}{|\mathbf{q}|}.$$

Moreover, from a geometric perspective the kinetic energy term \mathcal{T}_η can be interpreted as the one generating the geodesic motion of a particle with unit mass on a conformally flat space $\mathcal{M}^N = (\mathbb{R}^N, g)$, which is the complete Riemannian manifold with metric

$$ds^2 = \left(1 + \frac{\eta}{|\mathbf{q}|}\right) d\mathbf{q}^2 \tag{4}$$

and nonconstant scalar curvature given by

$$R = \eta(N - 1) \frac{4(N - 3)r + 3(N - 2)\eta}{4r(\eta + r)^3},$$

where we have introduced the radial coordinate $r = |\mathbf{q}|$. Here it is straightforward to check that the limit $\eta \rightarrow 0$ of (4) returns the flat Euclidean metric $ds^2 = d\mathbf{q}^2$ with $R = 0$. In fact, we stress that \mathcal{H}_η can be naturally related to the Taub-NUT system [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] since \mathcal{M}^N can be regarded as the (Riemannian) ND Taub-NUT space [20].

The aim of this contribution is to anticipate the main results concerning a maximally superintegrable quantization of the classical Hamiltonian (1), since a complete study of this new exactly solvable quantum system will be given elsewhere [21]. In the next Section, the properties of the classical system, including its effective potential, are presented. In Section 3 the corresponding quantum Hamiltonian is constructed by imposing the existence of the quantum analog of the full set of $(2N - 1)$ classical integrals of the motion (2) and (3). Finally, the explicit solution of the spectral problem is sketched in Section 4.

2. The classical Hamiltonian and its effective potential

Firstly, it should be remarked that, quite surprisingly, the potential \mathcal{U}_η (1) with $\eta \neq 0$ can be considered as the ND generalization of an ‘intrinsic’ oscillator on the curved space \mathcal{M}^N . This statement comes from the approach introduced in [20], that generalizes the Bertrand’s Theorem [22] to 3D conformally flat Riemannian spaces, thus providing a more general notion of KC and harmonic oscillator potentials [23, 24, 25]. To be self-contained, let us briefly recall these ideas by considering a 3D spherically symmetric space \mathcal{M}^3 with coordinates (q_1, q_2, q_3) and equipped with a metric

$$g_{ij} = f(|\mathbf{q}|)^2 \delta_{ij},$$

such that $f(|\mathbf{q}|) = f(r)$ is the conformal factor. Then the corresponding Laplace–Beltrami operator on \mathcal{M}^3 is given by

$$\Delta_{\mathcal{M}^3} = \sum_{i,j=1}^3 \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \sqrt{g} g^{ij} \frac{\partial}{\partial q_j},$$

where g^{ij} is the inverse of the metric tensor g_{ij} , and g is the determinant of g_{ij} . The radial symmetric Green function $U(|\mathbf{q}|) = U(r)$ on \mathcal{M}^3 (up to multiplicative and additive constants) is defined as the positive nonconstant solution to the equation

$$\Delta_{\mathcal{M}^3} U(r) = 0 \quad \text{on} \quad \mathcal{M}^3 \setminus \{\mathbf{0}\},$$

namely,

$$U(r) = \int^r \frac{dr'}{r'^2 f(r')}. \tag{5}$$

The prescription for the ND case [5] is to keep the very same definitions given in [24, 25, 26, 27, 28, 29] for the KC and oscillator potentials on \mathcal{M}^3 . In particular, the *intrinsic KC potential* on the ND space \mathcal{M}^N will be defined by

$$\mathcal{U}_{\text{KC}}(r) := AU(r) + B, \tag{6}$$

while the *intrinsic oscillator potential* is defined to be proportional to the inverse square of the KC potential

$$\mathcal{U}_O(r) := \frac{C}{U^2(r)} + D, \tag{7}$$

where A, B, C and D are real constants.

Hence according to the above definitions and by considering the Taub-NUT metric (4), with conformal factor

$$f(r) = \sqrt{1 + \frac{\eta}{r}},$$

it is straightforward to obtain that the corresponding intrinsic potentials read

$$\mathcal{U}_{KC}(r) = A\sqrt{1 + \frac{\eta}{r}} + B, \quad \mathcal{U}_O(r) = C\frac{r}{r + \eta} + D.$$

Therefore, whenever $\eta \neq 0$, we find that the potential (1) corresponds to \mathcal{U}_O provided that $C = k/\eta$ and $D = -C$:

$$\mathcal{U}_\eta(r) = \frac{k}{\eta} \left(\frac{r}{r + \eta} - 1 \right) = -\frac{k}{\eta + r},$$

which shows that \mathcal{U}_η can be interpreted as an intrinsic oscillator on \mathcal{M}^N with metric (4).

In order to understand the dynamical properties of the system (such as the existence of bounded states and any other critical features) we shall make its radial symmetry manifest by introducing the hyperspherical coordinates r, θ_j ($j = 1, \dots, N - 1$):

$$q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k, \quad 1 \leq j < N, \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k. \tag{8}$$

Their corresponding canonical momenta p_r, p_{θ_j} can be straightforwardly computed, and in these hyperspherical variables the ND Hamiltonian (1) takes the following 1D radial form

$$\mathcal{H}_\eta(r, p_r) = \mathcal{T}_\eta(r, p_r) + \mathcal{U}_\eta(r) = \frac{r}{2(\eta + r)} \left(p_r^2 + \frac{\mathbf{L}^2}{r^2} \right) - \frac{k}{\eta + r}, \tag{9}$$

where the total angular momentum is given by

$$\mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k}.$$

The contribution to the dynamics of both of the non-flat metric and the potential can be better understood if we consider a new set of canonical variables Q, P given by

$$Q(r) = \sqrt{r(\eta + r)} + \eta \log(\sqrt{r} + \sqrt{r + \eta}),$$

$$P(r, p_r) = \sqrt{\frac{r}{\eta + r}} p_r.$$

In terms of these new variables the Hamiltonian (9) is written as

$$\mathcal{H}_\eta(Q, P) = \frac{1}{2}P^2 + \frac{\mathbf{L}^2}{2r(Q)(\eta + r(Q))} + \mathcal{U}(r(Q)) \equiv \frac{1}{2}P^2 + \mathcal{U}_{\text{eff}}(Q),$$

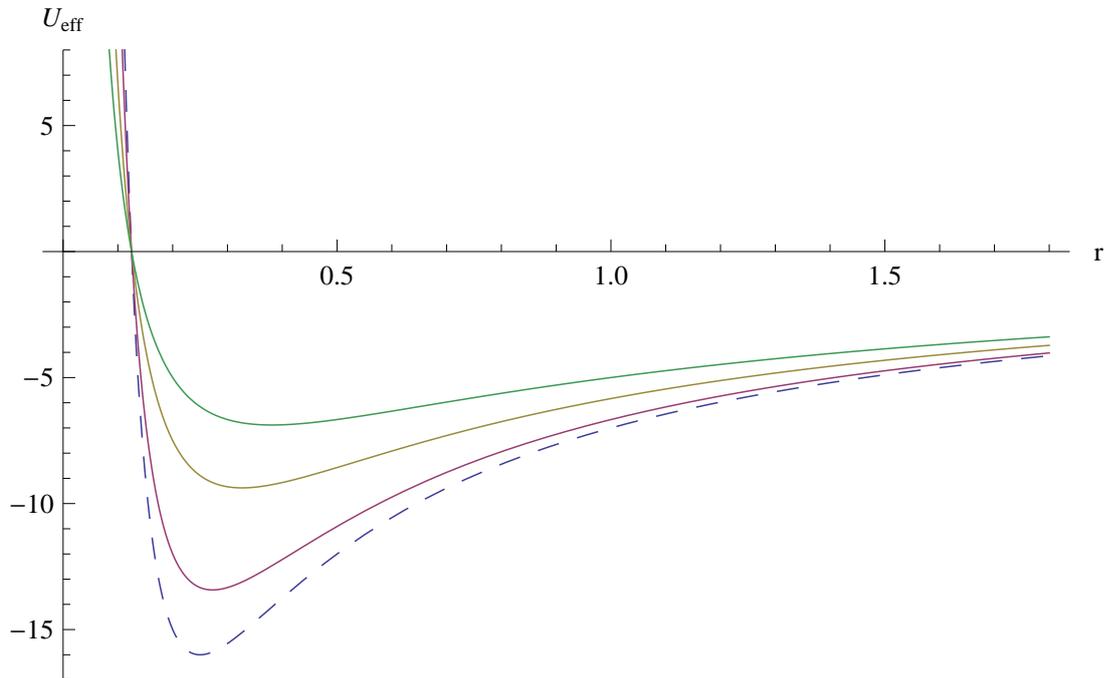


Figure 1. The effective potential U_{eff} (10) with $k = 8$ and $\mathbf{L}^2 = 2$, for $\eta = \{0, 0.05, 0.2, 0.4\}$. The dashed line corresponds to the KC potential ($\eta = 0$) and increasing values of η lead to higher values of the potential.

where the effective potential is thus given by

$$U_{\text{eff}}(Q(r)) = \frac{\mathbf{L}^2}{2r(\eta + r)} - \frac{k}{\eta + r}. \tag{10}$$

Consequently, the radial motion can be described as the 1D dynamics of a particle under the effective potential $U_{\text{eff}}(Q(r))$. As it can be appreciated from figure 1, the radial equation admits a hydrogen-like potential for $\eta > 0$ and $k > 0$, which can be interpreted as a genuine η -deformation of the effective potential for the KC system.

3. A maximally superintegrable quantization

In order to obtain the quantum analog of the kinetic energy term \mathcal{T}_η (1) we have to deal with the unavoidable ordering problems in the canonical quantization process that come from the nonzero curvature of the underlying space (see, e.g [7] and references therein). A detailed analysis of the different possible quantization prescriptions together with a proof of their equivalence through gauge transformations will be presented in a forthcoming paper [21]. One of this prescriptions consists in the so called ‘direct’ or Schrödinger quantization [15], under which the quantum Hamiltonian \mathcal{H}_η keeps the maximal superintegrability property and is therefore endowed with $(2N - 1)$ algebraically independent operators that commute with \mathcal{H}_η . This prescription has been already successfully used in the case of a curved (Darboux III) oscillator system [6, 7] and makes use of all the algebraic machinery coming from the symmetries of the classical Hamiltonian. This result can be stated as follows.

Theorem 1. Let $\hat{\mathcal{H}}_\eta$ be the quantum Hamiltonian given by

$$\hat{\mathcal{H}}_\eta = \frac{|\hat{\mathbf{q}}|}{2(\eta + |\hat{\mathbf{q}}|)} \hat{\mathbf{p}}^2 - \frac{k}{\eta + |\hat{\mathbf{q}}|} = \frac{|\mathbf{q}|}{2(\eta + |\mathbf{q}|)} \left(-\hbar^2 \nabla^2 - \frac{2k}{|\mathbf{q}|} \right), \tag{11}$$

where $\hat{\mathbf{q}} = \mathbf{q}$, $\hat{\mathbf{p}} = -i\hbar\nabla$ and $\nabla = (\partial_1, \dots, \partial_N)$ such that $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. For any value of η and k it is verified that:

(i) $\hat{\mathcal{H}}_\eta$ commutes with the following operators ($m = 2, \dots, N; i = 1, \dots, N$)

$$\hat{\mathcal{C}}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{\mathcal{C}}_{(m)} = \sum_{N-m \leq i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad (12)$$

$$\hat{\mathcal{R}}_i = \frac{1}{2} \sum_{j=1}^N \hat{p}_j (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) + \frac{1}{2} \sum_{j=1}^N (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) \hat{p}_j + \frac{\hat{q}_i}{\mathbf{q}} \left(\eta \hat{\mathcal{H}}_\eta + k \right),$$

where $\hat{\mathcal{C}}^{(N)} = \hat{\mathcal{C}}_{(N)} \equiv \hat{\mathbf{L}}^2$ is the total quantum angular momentum and

$$\hat{\mathbf{R}}^2 = \sum_{i=1}^N \hat{\mathcal{R}}_i^2 = 2\hat{\mathcal{H}}_\eta \left(\hat{\mathbf{L}}^2 + \hbar^2 \frac{(N-1)^2}{4} \right) + \left(\eta \hat{\mathcal{H}}_\eta + k \right)^2.$$

(ii) Each of the three sets $\{\hat{\mathcal{H}}_\eta, \hat{\mathcal{C}}^{(m)}\}$, $\{\hat{\mathcal{H}}_\eta, \hat{\mathcal{C}}_{(m)}\}$ ($m = 2, \dots, N$) and $\{\hat{\mathcal{R}}_i\}$ ($i = 1, \dots, N$) is formed by N algebraically independent commuting operators.

(iii) The set $\{\hat{\mathcal{H}}_\eta, \hat{\mathcal{C}}^{(m)}, \hat{\mathcal{C}}_{(m)}, \hat{\mathcal{R}}_i\}$ for $m = 2, \dots, N$ with a fixed index i is formed by $2N - 1$ algebraically independent operators

(iv) $\hat{\mathcal{H}}_\eta$ is formally self-adjoint on the Hilbert space $L^2(\mathcal{M}^N)$, endowed with the scalar product

$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}^N} \overline{\Psi(\mathbf{q})} \Phi(\mathbf{q}) \left(1 + \frac{\eta}{|\mathbf{q}|} \right) d\mathbf{q}.$$

The proof of this result can be obtained through direct computation. Recall that in [7] we have considered a similar problem on the conformally flat Darboux III space.

Therefore, the Hamiltonian (11) leads to the following Schrödinger equation

$$\left(\frac{-\hbar^2 |\mathbf{q}|}{2(\eta + |\mathbf{q}|)} \nabla^2 - \frac{k}{\eta + |\mathbf{q}|} \right) \Psi(\mathbf{q}) = E \Psi(\mathbf{q}),$$

which in hyperspherical variables (8) turns into

$$\frac{r}{2(\eta + r)} \left(-\hbar^2 \partial_r^2 - \frac{\hbar^2(N-1)}{r} \partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} - \frac{2k}{r} \right) \Psi(r, \boldsymbol{\theta}) = E \Psi(r, \boldsymbol{\theta}), \quad (13)$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{N-1})$.

We remark that all the above results are well defined for any value of the parameters η and k . Nevertheless, the explicit solution of the quantum Hamiltonian depends on the sign of both of them. In particular, hereafter we shall restrict ourselves to consider $\eta > 0$, which implies that the variable $r \in (0, +\infty)$, and $k > 0$ which corresponds to the proper ‘curved’ hydrogen atom potential.

4. Spectrum and eigenfunctions

In view of the effective potential introduced in (10), one should expect that the quantum Hamiltonian (11) should have both a discrete and a continuous spectrum, and this is indeed the case. When the Schrödinger equation (13) is expressed in hyperspherical variables, it can be solved by factorizing the wave function into radial and angular components

$$\Psi(r, \boldsymbol{\theta}) = \psi(r) Y(\boldsymbol{\theta}),$$

and by considering the separability provided by the first integrals $\hat{C}_{(m)}$ (12) with eigenvalue equations given by

$$\hat{C}_{(m)}\Psi = c_m\Psi, \quad m = 2, \dots, N.$$

From it, we obtain that $Y(\boldsymbol{\theta})$ solves completely the angular part and this corresponds, as expected, to the hyperspherical harmonics satisfying that

$$\hat{C}_{(N)}Y(\boldsymbol{\theta}) \equiv \hat{\mathbf{L}}^2Y(\boldsymbol{\theta}) = \hbar^2l(l + N - 2)Y(\boldsymbol{\theta}), \quad l = 0, 1, 2 \dots$$

where l is the angular momentum quantum number. It can be proven [7] that the eigenvalues c_m are related with the $N - 1$ quantum numbers of the angular observables through

$$c_k \leftrightarrow l_{k-1}, \quad k = 2, \dots, N - 1, \quad c_N \leftrightarrow l,$$

which means that

$$Y(\boldsymbol{\theta}) \equiv Y_{c_{N-1}, \dots, c_2}^{c_N}(\theta_1, \theta_2, \dots, \theta_{N-1}) \equiv Y_{l_{N-2}, \dots, l_1}^l(\theta_1, \theta_2, \dots, \theta_{N-1}).$$

As a consequence, the radial Schrödinger equation (13) is given by

$$\frac{r}{2(\eta + r)} \left(-\hbar^2 \frac{d^2}{dr^2} - \frac{\hbar^2(N - 1)}{r} \frac{d}{dr} + \frac{\hbar^2l(l + N - 2)}{r^2} - \frac{2k}{r} \right) \psi(r) = E\psi(r),$$

which can be written in the form

$$\left(-\hbar^2 \frac{d^2}{dr^2} - \frac{\hbar^2(N - 1)}{r} \frac{d}{dr} + \frac{\hbar^2l(l + N - 2)}{r^2} - \frac{2K}{r} \right) \psi(r) = 2E\psi(r), \quad (14)$$

where the new ‘coupling constant’ K turns out to be energy-dependent

$$K = k + \eta E. \quad (15)$$

In this way we find that the equation (14) is formally equivalent to the Schrödinger equation of the radial hydrogen atom, whose bounded eigenfunctions are given, in terms of the generalised Laguerre polynomials L_n^α , by:

$$\psi_{n,l}(r) = r^l \exp\left(-\frac{Kr}{\hbar^2(n + l + \frac{N-1}{2})}\right) L_n^{2l+N-2}\left(\frac{2Kr}{\hbar^2(n + l + \frac{N-1}{2})}\right). \quad (16)$$

Notice that the eigenfunctions do not only depend on the usual quantum numbers n, l but also on the eigenvalue E through K (15).

If we substitute these functions within the equation (14), the following algebraic equation is obtained:

$$E = -\frac{K^2}{2\hbar^2(n + l + \frac{N-1}{2})^2} = -\frac{(k + \eta E)^2}{2\hbar^2(n + l + \frac{N-1}{2})^2}.$$

By solving such a quadratic equation in terms of E we obtain the bounded discrete spectrum of the system, whose eigenvalues depend both on the ‘deformation’ parameter η and on the quantum numbers n, l . Namely,

$$E_{n,l}^\eta = \frac{-\hbar^2(n + l + \frac{N-1}{2})^2 - \eta k + \sqrt{\hbar^4(n + l + \frac{N-1}{2})^4 + 2\eta k \hbar^2(n + l + \frac{N-1}{2})^2}}{\eta^2}. \quad (17)$$

In this way, the eigenfunctions, $\psi_{n,l}^\eta(r)$, can be explicitly obtained by introducing (17) into K (15) and, next, by substituting the latter in (16). Note that in the limit $\eta \rightarrow 0$ the following well-known expression for the energies is recovered:

$$E_{n,l}^0 = -\frac{k^2}{2\hbar^2 \left(n + l + \frac{N-1}{2}\right)^2}.$$

And the first-order effect of the deformation on the spectrum can be appreciated through a power series expansion in η :

$$E_{n,l}^\eta = E_{n,l}^0 + \eta \frac{k^3}{2\hbar^4 \left(n + l + \frac{N-1}{2}\right)^4} + o(\eta^2). \quad (18)$$

Let us remark that since the spectrum of $\hat{\mathcal{H}}_\eta$ is bounded from below (as the classical effective potential indicates), for a sufficiently large k we can safely assume that $K = k + \eta E > 0$. Moreover the condition $\Psi \in L^2(\mathcal{M}^N)$ translates to

$$\int_{\mathbb{R}^N} |\Psi(\mathbf{q})|^2 \left(1 + \frac{\eta}{|\mathbf{q}|}\right) d\mathbf{q} < \infty.$$

Note also that the degeneracy of this spectrum is exactly the same as in the ND hydrogen atom, which is a strong signature of the maximal superintegrability of this quantum system. The explicit expressions for the wave functions corresponding to the continuous spectrum will be given in the forthcoming paper [21].

Finally let us stress that if we consider the following reparametrization

$$\eta \rightarrow \frac{1}{\sqrt{\lambda}}, \quad k \rightarrow \frac{\omega^2}{\sqrt{\lambda}}.$$

the spectrum (17) turns out to have the same dependence on the principal quantum number (which in this case corresponds to $\mathcal{N} = n + l$) as the bounded spectrum for the intrinsic oscillator on a Darboux III space that was given in [6, 7]. Indeed, this fact can be understood in terms of the classification of intrinsic KC and oscillator systems on non-Euclidean spaces provided in [30]. According to such classification, the Taub-NUT and the Darboux III oscillator are superintegrable systems of type II with the same parameters λ, δ (Taub-NUT: $\gamma = \frac{1}{2}, \lambda = 0, \delta$; Darboux III: $\gamma = 1, \lambda = 0, \delta$). However as showed in [31] for the intrinsic KC systems, the parameter γ only affects the form of the principal quantum number \mathcal{N} (Taub-NUT $\mathcal{N} = n + l$; Darboux III: $\mathcal{N} = 2n + l$) but not the overall dependence on \mathcal{N} which indeed turns out to be the same.

Acknowledgments

This work was partially supported by the Spanish MINECO through the Ramón y Cajal program (A.E.) and under grants MTM2010-18556 (A.B and F.J.H.), AIC-D-2011-0711 (MINECO-INFN) (A.B, F.J.H. and O.R.) and FIS201-22566 (A.E.), by the ICMAT Severo Ochoa under grant SEV-2011-0087 (A.E.), by Banco Santander-UCM under grant GR35/10-A-910556 (A.E.), and by a postdoctoral fellowship by the Laboratory of Mathematical Physics of the CRM, Université de Montréal (D.R.).

References

- [1] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 *SIGMA* **7** 048
- [2] Kalnins E G, Kress J M and Miller W Jr 2005 *J. Math. Phys.* **46** 053510
- [3] Kalnins E G, Kress J M and Miller W Jr 2006 *J. Math. Phys.* **47** 043514
- [4] Ballesteros A, and Herranz F J 2009 *J. Phys. A: Math. Theor.* **42** 245203
- [5] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2008 *Physica D* **237** 505
- [6] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 *Phys. Lett. A* **375** 1431
- [7] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 *Ann. Phys.* **326** 2053
- [8] Manton N S 1982 *Phys. Lett. B* **110** 54
- [9] Atiyah M F and N.J. Hitchin N J 1985 *Phys. Lett. A* **107** 21
- [10] Gibbons G W and Manton N S 1986 *Nucl. Phys. B* **274** 183
- [11] Fehér L G and Horváthy P A 1987 *Phys. Lett. B* **183** 182
- [12] Gibbons G W and Ruback P J 1988 *Comm. Math. Phys.* **115** 267
- [13] Iwai T and Katayama N 1994 *J. Phys. A: Math. Gen.* **27** 3179
- [14] Iwai T and Katayama N 1995 *J. Math. Phys.* **36** 1790
- [15] Iwai T, Uwano Y and Katayama N 1996 *J. Math. Phys.* **37** 608
- [16] Bini D, Cherubini C and Jantzen R T 2002 *Class. Quantum Grav.* **19** 5481
- [17] Bini D, Cherubini C, Jantzen R T and Mashhoon B 2003 *Class. Quantum Grav.* **20** 457
- [18] Gibbons G W and Warnick C M 2007 *J. Geom. Phys.* **57** 2286
- [19] Jezierski J and Lukasik M 2007 *Class. Quantum Grav.* **24** 1331
- [20] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 *Ann. Phys.* **324** 1219
- [21] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D In preparation
- [22] Bertrand J 1873 *C.R. Acad. Sci. Paris* **77** 849
- [23] Perlick V 1992 *Class. Quantum Grav.* **9** 1009
- [24] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2008 *Class. Quantum Grav.* **25** 165005
- [25] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 *Commun. Math. Phys.* **290** 1033
- [26] Enciso A and Peralta-Salas D 2007 *J. Geom. Phys.* **57** 1679
- [27] Enciso A and Peralta-Salas D 2009 *Indiana Univ. Math. J.* **58** 1947
- [28] Li P and Tam L F 1987 *Amer. J. Math.* **109** 1129
- [29] Li P and Tam L F 1995 *J. Differential Geom.* **41** 277
- [30] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2013 *Nankai Series in Pure, Appl. Math. Theor. Phys.* **11** ed C Bai, J P Gazeau and Mo-Lin Ge (Singapore: World Scientific) p 211, arXiv:1304.4544
- [31] Riglioni D, *J. Phys. A: Math. Theor.* **46** 265207