Casimir Energy through Transfer Operators for Sine-Gordon Backgrounds

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The quantum vacuum interaction energy between a pair of semitransparent two-dimensional plates represented by Dirac delta potentials and its first derivative, embedded in the topological background of a sine-Gordon kink, is studied through an extension of the *TGTG*-formula (developed by O. Kenneth and I. Klich in the scattering approach). Quantum vacuum oscillations around the sine-Gordon kink solutions are interpreted as a quantum scalar field theory in the spacetime of a domain wall. Moreover, the relation between the phase shift and the density of states (the well-known Dashen–Hasslacher–Neveu or DHN formula) is also exploited to characterize the quantum vacuum energy.

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Subject Index   B3, A10, A11

1. Introduction

The vacuum state of an arbitrary quantum field theory (QFT) is the state of minimum energy of the Hamiltonian. This state is not empty but contains electromagnetic waves and infinite pairs of virtual short-lived particles and antiparticles [1]. These pairs annihilate each other very quickly in accordance with the Heisenberg energy–time uncertainty principle. Nevertheless, when some objects are introduced into the space, sometimes nearby particles collide with them and get reflected in such a way that they do not combine again with their antipaticles. Quantum forces may thus appear in the system as a result of the presence of these frontiers. The physical properties of the vacuum state and the vacuum energy show a strong dependence on the type of boundaries. One of the most important boundary phenomena is the Casimir effect [2], in which the presence of two parallel, uncharged and conducting plates restricts the modes of the fluctuations of the electromagnetic field between them. Vacuum polarization occurs and a finite force between both plates appears. This effect, predicted in 1948 by H.B.G Casimir, was experimentally measured for the first time in 1958 by M.J. Sparnaay [3]. Since then, many studies have been performed for bodies with different geometries and materials [4,5]. This effect has numerous applications in nanoelectronic devices [6,7], in absorption phenomena in carbon nanotubes [8], and in inflation processes [9], among many others.

A crucial point of the two-plate theory is that the vacuum energy is ultraviolet-divergent. It needs to be regularized and renormalized in order to obtain the finite contribution related to

the interaction betwen objects. There are several approaches to achieve this goal. For instance, it is worth mentioning:

- Computing the 00-component of the energy-momentum tensor in terms of Green's function and the scattering data and then subtracting the first terms in its Born expansion [10],
- calculating the transfer operator to apply the *TGTG*-formula [11], and
- using zeta functions, complex integrals and heat traces [12–14].

The *TGTG*-formula allows one to compute the quantum vacuum interaction energy between two separated objects represented by a smooth classical background in a flat spacetime. It is a formalism based on transition operators (more precisely, the so-called Lippmann–Schwinger *T*-operator) and Green's functions. It has been used, for instance, to compute the vacuum interaction energy between two sine-Gordon kinks [15], and between two plates mimicked by $\delta\delta'$ potentials [16], like the ones that will be discussed in the following sections. The major advantage of using the *TGTG*-formalism instead of the other ones aforementioned is that only the scattering problem for the background potential together with one single object is necessary to compute the vacuum interaction energy between the pair of bodies. This could be a crucial factor whenever the scattering problem for the complete potential with two objects in a classical background is hard to solve. Furthermore, using the *TGTG*-formula significantly reduces the complexity of the analytical and numerical computation. For both these reasons, it is the method that is used in the present work.

The main objective here is to find a generalization of the *TGTG*-formula to study the quantum vacuum interaction energy between a pair of two-dimensional homogeneous plates located[1] at $z = a, b$, and immersed in a weak curved background potential. This background is constituted by a sine-Gordon kink centred at the origin of the direction orthogonal to the plates. Thus, the background potential will be the Pöschl–Teller (PT) one:

$$V_{PT}(z) = -2\operatorname{sech}^2 z = -\frac{2}{\cosh^2 z} = -\frac{8}{(e^z + e^{-z})^2}. \tag{1}$$

It models the propagation of mesons moving in a sine-Gordon kink background [17–20]. Notice that kinks in $1 + 1$ dimensions can be embedded into a $3 + 1$ dimensional theory as solutions which are independent of all but one spatial direction. These types of solutions with finite energy per unit area are known as domain walls [18]. They were produced in the early stages of the Universe by the spontaneous breaking of discrete symmetries [21]. The propagation of mesons around this topological defect will be the focus in this article.

Sine-Gordon kinks have attracted much attention to-date. Previous works concerning the 00-component of the energy momentum tensor $T_{\mu\nu}$ in a system with a kink in the real line, and the scattering problem of two delta potentials symetrically placed around a kink can be found in Refs. [20,22]. Moreover, the one-soliton sine-Gordon solution can be used to construct (locally) a metric describing the curved spacetime of a black hole [23], to generate 4D Einstein gravity solutions [24], and to provide braneworld scenarios [25]. In this last case, the scalar field acts as a source of gravity around the brane. So far, much is made of quantum forces in different curved backgrounds and in gravitational contexts by Saharian [26], Bordag [27], Milton and Fulling [28], and Elizalde and Odintsov [29], among others. There are relevant dif-

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[1]The position four-vector will be expressed from now on as $x^\mu = (t, \vec{x}_\parallel, z) \in \mathbb{R}^{1,3}$. Notice that $\vec{x}_\parallel \in \mathbb{R}^2$. Likewise, the four-momentum will be $K^\mu = (E, \vec{k}_\parallel, k)$. Here, z and k are the position and the momentum coordinates in the direction orthogonal to the surfaces of the plates.

ferences between studying QFT for flat metrics and doing it for those that characterize curved manifolds. One of the fundamental problems in curved spaces [30,31] is that only for globally hyperbolic curved spacetimes endowed with a global temporal Killing vector is the spacetime a fiber bundle with a set of spatial slices or Cauchy surfaces which evolve in time. For each fixed value of the temporal coordinate, one could solve the spectra of the Laplacian–Beltrami operator in the spatial slice as done for the Minkowski metric. However, in another, more general case, if the curved spacetime is such that the fiber bundle does not allow an interpretation in terms of particle spectra independent of the observer, talking about scattering is ambiguous. Consequently, computing the quantum vacuum energy either from the 00-component of the energy-momentum tensor or from the transfer operators defined in terms of the scattering data will not offer a universal outcome independent of the observer. In fact, there are not many results of the *TGTG*-formula in curved backgrounds, and sometimes it does not even exist [32]. However, there is a special case to be taken into account. When the frequencies of the particles created by the gravitational background are much smaller than the Planck frequency, one could use the perturbation theory for this curved spacetime as a semiclassical approach to quantum gravity [33,34]. In doing so, this weak gravitational backgrounds are treated classically and the matter fields are the ones which will be quantized. The key point is that this gravity would be strong enough to produce some effects on the quantum matter, but not so strong as to require its own quantization. This is exactly the case that is going to be considered in this work.

Here, I deal with the Casimir force between singular plates in a solitonic background. In particular, the plates are modeled by the Dirac point-like potential

$$V_{\delta\delta'}(z) = v_0\delta(z-a) + w_0\delta'(z-a) + v_1\delta(z-b) + w_1\delta'(z-b), \tag{2}$$

with $v_0, v_1, w_0, w_1, a, b \in \mathbb{R}$ and $a < b$. Above, δ' denotes the first derivative of the delta function. Dirac delta potentials are widely used as toy models for realistic materials like quantum wires [35], and to analyse physical phenomena such as Bose-Einstein condensation in periodic backgrounds [36], or light propagation in 1D relativistic dielectric superlattices [37]. Despite being a rather simple idealization of the real system, the δ function has been proved to correctly represent surface interactions in many models related to the Casimir effect [38]. For instance, Dirac δ functions have been set on the plates acting as the electrostatic potential [39], to represent two finite-width mirrors [6], or to describe the permittivity and magnetic permeability in an electromagnetic context, by associating them to the plasma frequencies in Barton's model on spherical shells [40,41]. On the other hand, the first derivative of the delta potential has been used to study mono-atomically thin polarizable plates formed by lattices of dipoles [42], and to analyse resonances in 1D oscillators [43]. There is some controversy in the definition of the δ' potential since different regularizations produce different scattering data (see Ref. [44] and references therein). Here, I will use the one presented in Ref. [45], in which the authors define it by introducing a Dirac delta potential at the same point to regularize the whole potential. As they explained, the major advantage of this choice is that it enables defining this singular potential in terms of matching conditions at the origin which do not depend on the choice of a regularization method.

In order to study the Casimir force between singular Dirac delta plates in a PT background, this article is organised as follows: Sections 2 and 3 involve the computation of the spectrum of scattering and bound states of the associated Schrödinger operator as well as the derivation of the Green's functions, respectively. In Sections 4 and 5, the *TGTG*-formula and the DHN one

[46] will be used to analyse the quantum vacuum interaction energy and to study the Casimir pressure between plates. Finally, Section 6 summarizes the main conclusions. The natural system of units $\hbar = c = 1$ will be used.

2. Scattering data and spectrum

The Casimir force between plates is due to the coupling between the quantum vacuum fluctuations of the electromagnetic field with the charged current fluctuations of the plates [47]. For distances between plates rather larger than any other length scale concerning the electric response of the plates, only the long wavelength transverse modes of the electromagnetic field are relevant to the interaction. They can be mimicked by the normal modes of a scalar field whose dynamics is described by the action

$$S[\phi] = \frac{1}{2} \int d^4x \left[\partial_\mu \phi \partial^\mu \phi - V_{\delta\delta'}(z)\phi^2 - V_{PT}(z)\phi^2 \right]. \tag{3}$$

Consider for simplicity a real massless scalar field ϕ confined between two parallel $(D-1)$-dimensional plates separated by a distance $b-a$ in the axis orthogonal to the plates, i.e. the z-axis. For isotropic and homogeneous plates, there exists a translational symmetry along the surface of the plates, and the theory of free fields without boundaries is recovered for the parallel direction coordinates $\vec{x}_\parallel \in \mathbb{R}^{D-1}$. Splitting the spatial coordinate as $x = (\vec{x}_\parallel, z)$, with $\vec{x}_\parallel \in \mathbb{R}^{D-1}$ and taking into account the Fourier decomposition of the field

$$\phi(t, x) = \int d\omega \, e^{-i\omega t} \phi_\omega(x), \tag{4}$$

the equation for the modes of the fluctuations field is given by the non-relativistic Schrödinger separable eigenvalue problem

$$(-\Delta + V_{PT}(z) + V_{\delta\delta'}(z))\phi_\omega(x) = \omega^2 \phi_\omega(x). \tag{5}$$

In the equation above, $\Delta = \Delta_\parallel + \partial_z^2$, and the frequencies follow the dispersion relation given by $\omega^2 = \vec{k}_\parallel^2 + k^2$. Therefore, the problem is separable and only the direction orthogonal to the plates needs to be studied, as the spectrum is trivial in the other directions. From now on, I will consider $D = 3$, just for simplicity.

Notice that the non-relativistic Schrödinger operator related to the background in the dimension orthogonal to the plates, $\hat{K}_{PT} = -\partial_z^2 + V_{PT}(z)$, is not essentially self-adjoint in the Sobolev space of functions $W_2^2(\mathbb{R} - \{a, b\}, \mathbb{C})$. It is necessary to add some matching conditions concerning the continuity of the wave function and the discontinuity of its derivative at the boundary points $\{a, b\}$ in order to define the self-adjoint extensions of $\hat{K}_{PT}$ in the aforementioned domain. These boundary conditions will be determined by the specific potential that represents the plates. For instance, if they are mimicked by the Dirac delta potential $V_{\delta\delta'}$, the domain of the self-adjoint extension of $\hat{K}_{PT}$ is given by the suitable matching conditions:

$$\mathcal{D}_{\hat{K}_{PT}} = \left\{ \phi \in W_2^2(\mathbb{R} - \{a, b\}, \mathbb{C}) \left| \begin{pmatrix} \phi(a^+) \\ \phi'(a^+) \\ \phi(b^+) \\ \phi'(b^+) \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ \beta_0 & \alpha_0^{-1} & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \beta_1 & \alpha_1^{-1} \end{pmatrix} \begin{pmatrix} \phi(a^-) \\ \phi'(a^-) \\ \phi(b^-) \\ \phi'(b^-) \end{pmatrix} \right. \right\},$$

where $\qquad \alpha_i = \dfrac{1 + w_i/2}{1 - w_i/2}, \quad \beta_i = \dfrac{v_i}{1 - (w_i/2)^2}, \qquad i = 0, 1,$ $\qquad\qquad$ (6)

coming from the original work of Kurasov in one-dimensional systems [45,48]. The system of two plates in the chosen background has an open geometry so the positive energy spectrum will

be continuous. Scattering states correspond to solutions of the Schrödinger equation

$$(\hat{K}_{PT} + V_{\delta\delta'}(z))\phi_{\omega,\vec{k}_\parallel}(z) = k^2\phi_{\omega,\vec{k}_\parallel}(z), \tag{7}$$

with $k \in \mathbb{R}$ (such that $k^2 > 0$), and keeping in mind that the delta potential must be understood as the aforementioned boundary conditions at $z = a, b$. Given a linear momentum k, there are two independent scattering solutions to be found. In order for the Casimir energy between plates to be a non-negligible magnitude, the two objects have to be very close to each other. If the distance between plates is smaller than the support of the background potential in such a way that the plates are placed within this support, the scattering solutions for particles incoming to the system from the left (also called right-handed solutions) and from right (left-handed solutions), are respectively of the form

$$\psi_k^R(z) = \begin{cases} f_k(z) + r_R f_{-k}(z), & \text{if} \quad z < a, \\ B_R f_k(z) + C_R f_{-k}(z), & \text{if} \quad a < z < b, \\ t_R f_k(z), & \text{if} \quad z > b, \end{cases} \tag{8}$$

$$\psi_k^L(z) = \begin{cases} t_L f_{-k}(z), & \text{if} \quad z < a, \\ B_L f_k(z) + C_L f_{-k}(z), & \text{if} \quad a < z < b, \\ r_L f_k(z) + f_{-k}(z), & \text{if} \quad z > b. \end{cases} \tag{9}$$

Notice that $f_k(z) = e^{ikz}(\tanh(z) - ik)$ are the eigenfunctions of the operator $-\partial_z^2 + V_{PT}(z)$ (i.e. plane waves times first-order Jacobi polynomials). It is relevant to highlight that the transmission amplitudes $t_R(k)$, $t_L(k)$ are identical to each other due to the time-reversal invariance of the Schrödinger operator. Consequently, they will be substituted by $t(k)$ from now on. Replacing (8) and (9) in the matching conditions (6) and solving the resulting two systems of equations with unknowns $\{r_R, r_L, t, B_R, B_L, C_R, C_L\}(k)$, the scattering data are obtained [they are collected in Eq. (A2) in Appendix A].

The denominator of all the scattering parameters, $\Upsilon(k)$, is the spectral function (see its derivation in Appendix A). The set of zeroes of $\Upsilon(k)$ can be the poles of the scattering matrix $S(k)$. Notice that the $S(k)$-matrix admits an analytic continuation to the entire complex momentum plane. The zeroes of the spectral function on the positive imaginary axis in the complex momentum k-plane gives the bound states of the spectrum of the non-relativistic Schrödinger operator. Making $k \to i\kappa$ in $\Upsilon(k) = 0$, one can study the bound states as the intersections between an exponential and a rational function via the transcendent equation

$$-\frac{\Upsilon_1(\kappa)}{\Upsilon_2(\kappa)} = e^{-2\kappa(b-a)}, \tag{10}$$

where

$$\Upsilon_1(\kappa) = 4\,\Sigma(v_0, w_0, a, \kappa)\,\Sigma(v_1, w_1, b, \kappa),$$

$$\Upsilon_2(\kappa) = 16(\kappa - \tanh a)(\kappa + \tanh b)\,\Lambda(v_0, w_0, a, \kappa)\Lambda(-v_1, w_1, -b, \kappa),$$

$$\Lambda(v_i, w_i, x, \kappa) = -2w_i\,\mathrm{sech}^2 x - (v_i - 2w_i\kappa)(\kappa - \tanh x),$$

$$\Sigma(v_i, w_i, x, \kappa) = [2v_i + \kappa(4 + w_i^2)](\kappa^2 - 1) + 2\,\mathrm{sech}^2 x\,(v_i + 2w_i\tanh x). \tag{11}$$

Once the momenta of the bound states are determined, their energies can be computed by means of $E = (i\kappa)^2 < 0$. The lowest energy state will be characterized by E_{min}. For some type of potentials (such the pure delta plates, i.e. $w_0 = w_1 = 0$), it is possible to give an analytic formula to bound the energy of the states with the negative energy of the spectrum. In the pure delta

case, this energy takes the value

$$E_{min} = -\frac{1}{16}\left[(|v_1| + |v_0|) + \sqrt{(|v_1| + |v_0|)^2 + 16}\right]^2, \tag{12}$$

for all a, b, v_0, $v_1 \neq 0$. In other configurations, E_{min} has to be obtained numerically.

Finding the *TGTG*-formula implies focusing only in the scattering problem for one single plate. If one of the delta plates is removed (for instance $v_1 = w_1 = 0$), the spectral function of the reduced system is

$$(4 + w_0^2)\kappa^3 + 2v_0\kappa^2 - (4 + w_0^2)\kappa - 2v_0 \tanh^2 a + 4w_0 \tanh a \operatorname{sech}^2 a = 0. \tag{13}$$

By studying the asymptotic behavior of Eq. (13), as well as its maxima and minima for different values of the parameters, it can be seen that there may be two cases: only one bound state, or two bound states. There is no zero mode because the state with wave vector $k = 0$ does not constitute a pole of the $S(k)$-matrix. The scattering data for the reduced system can be obtained from equation (A2), and they are given by

$$t^\ell = \frac{-\alpha_0 W}{\alpha_0 f_{-k}(a)\left(\alpha_0 f_k'(a) - \beta_0 f_k(a)\right) - f_k(a)f_{-k}'(a)},$$

$$r_R^\ell = \frac{-f_k(a)\left[(\alpha_0^2 - 1)f_k'(a) - \alpha_0\beta_0 f_k(a)\right]}{\alpha_0 f_{-k}(a)\left(\alpha_0 f_k'(a) - \beta_0 f_k(a)\right) - f_k(a)f_{-k}'(a)},$$

$$r_L^\ell = \frac{-f_{-k}(a)\left[(\alpha_0^2 - 1)f_{-k}'(a) - \alpha_0\beta_0 f_{-k}(a)\right]}{\alpha_0 f_{-k}(a)\left(\alpha_0 f_k'(a) - \beta_0 f_k(a)\right) - f_k(a)f_{-k}'(a)}, \tag{14}$$

where α_0, β_0 are as defined in Eq. (6), and W is the Wronskian

$$W \equiv W[f_k(a), f_{-k}(a)] = -2ik(k^2 + 1). \tag{15}$$

From now on, the superscript ℓ, r in the scattering data indicates which plate is being considered: ℓ for the plate placed on the left of the system and r for the one placed on the right. The subscripts R, L refer to "*diestro*" (right-handed) and "*zurdo*" (left-handed) scattering. Notice that when $v_1 = w_1 = 0$, the scattering data is such that $B_R^\ell = t^\ell$, $B_L^\ell = r_L^\ell$, $C_R^\ell = 0$, $C_L^\ell = 1$. Similarly, by setting $v_0 = w_0 = 0$ one obtains the reduced scattering data when the plate on the right is the only one present in the system. In this case, $\{t^r, r_R^r, r_L^r\}$ are given by Eq. (14) but replacing $\alpha_0 \to \alpha_1$, $\beta_0 \to \beta_1$, $a \to b$. Furthermore, $C_L^r = t^r$, $B_L^r = 0$, $B_R^r = 1$, $C_R^r = r_R^r$.

A rather important fact is that, due to the PT background potential, the translational invariance of the system is broken. $V_{PT}(z)$ breaks the isotropy of the space, and consequently if $f_k(z)$ is a eigenfunction of the non-relativistic Schrödinger operator $\hat{K}_{PT}$, then $f_k(z + a)$ with $a \in \mathbb{R} - \{0\}$ will no longer be another. This means that the scattering data explicitly depend on the position of the plates in a non-trivial way.

For computing the vacuum interaction energy in the corresponding QFT for the general case in which $v_0, v_1, w_0, w_1 \in \mathbb{R} - \{0\}$, the value of the energy for the lowest energy bound state of the quantum mechanical problem explained in this section is essential. Since the bound state with the lowest energy is characterized by E_{min}, the mass of the fluctuations in the theory has to be equal to this value of the energy for making fluctuation absorption impossible. The unitarity of the QFT sets this lower bound for the mass of the quantum vacuum fluctuations, such that the total energy of the lowest energy state of the spectrum will be zero. Thus, the spectrum of the associated QFT will consist of a set of discrete states with energies in the gap $[0, |E_{min}|]$, and a continuum of scattering states with energies above the threshold $E = |E_{min}| = m^2$. The number of discrete states will be determined by the value of the coefficients $\{v_0, v_1, w_0, w_1\}$, with a

maximum of three being possible.[2] It is important to highlight that the value of E_{min} must be computed for the whole system of two objects plus the background potential. This is not contradictory to the fact that if the Casimir energy is calculated with the *TGTG*-formula, only the transmission and reflection coefficients of the reduced system of an object in the background potential are needed. This will be discussed in detail in Section 4.

3. Green's function

Once the spectral problem has been solved, the usual second quantization procedure could be applied to promote the non-relativistic quantum mechanical theory to a QFT in which to study the quantum vacuum interaction energy between objects. The *TGTG*-formula is based on two main elements: the Green's function, and the transfer operators. The characteristic Green's function can be obtained by solving the differential equation

$$\left[\partial_\mu \partial^\mu - 2\,\mathrm{sech}^2 z + V_{\delta\delta'}(z) + m^2 \right] G(x^\mu, y^\mu) = \delta(x^\mu - y^\mu) \tag{16}$$

for the complete Green's function

$$G(x, x') = \int \frac{d^2 k_\parallel}{(2\pi)^2} e^{i\vec{k}_\parallel(\vec{x}_\parallel - \vec{x}'_\parallel)} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_k(z, z'), \tag{17}$$

or equivalently by solving

$$\left[-\partial_{z_1}^2 - k^2 - 2\,\mathrm{sech}^2 z_1 + V_{\delta\delta'}(z_1) \right] G_k(z_1, z_2) = \delta(z_1 - z_2) \tag{18}$$

for the reduced Green's function. Solving this differential equation requires assuming the continuity of $G_k(z_1, z_2)$ and the discontinuity of its first derivative at the points $\{a, b\}$, as well as imposing an exponentially decaying behaviour of the solutions at infinity. Another way to compute the reduced Green's function in the spatial dimension orthogonal to the surfaces of the plates is by using [16]

$$G_k(z, z') = \frac{\mathrm{u}(z - z')\psi_k^R(z)\psi_k^L(z') + \mathrm{u}(z' - z)\psi_k^R(z')\psi_k^L(z)}{W[\psi_k^R, \psi_k^L]} \tag{19}$$

with the two linear independent scattering solutions given in Eqs. (8)–(9) for the complete system of two plates in the PT background. Note that $\mathrm{u}(z - z')$ is the unit or Heaviside step function. Both aforementioned methods yield the same solution for the correlator.

Moreover, the Wronskian $W[\psi_k^R, \psi_k^L]$ has to be the same for the three zones in which the two delta plates divide the space. This imposes the following relation between the scattering coefficients: $t = B_R C_L - C_R B_L$. This relation is useful to simplify the solutions of the Green's function in the different zones, and to rewrite them as

$$G_k(z_1, z_2) = G_k^{PT}(z_1, z_2) + \Delta G_k(z_1, z_2), \tag{20}$$

where $\Delta G_k(z_1, z_2)$ is given by Eq. (A4) in Appendix A.

Notice that the Green's function for the kink potential centered at the origin without any delta interactions (i.e. $v_0 = w_0 = v_1 = w_1 = 0$) takes the form

$$G_k^{PT}(z_1, z_2) = \frac{1}{W} f_{-k}(z_<) f_k(z_>) = \frac{e^{ik|z_1 - z_2|}}{W} \left(k^2 + ik|\tanh z_1 - \tanh z_2| + \tanh z_1 \tanh z_2 \right),$$

$$\tag{21}$$

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[2] Although each $\delta\delta'$ potential can hold at most two bound states, since the two plates are very close together, the whole system of two plates in the PT background acts as a well that is not deep or wide enough to accommodate four bound states, but which can accommodate three.

where $z_<$ and $z_>$ are the lesser or the greater of z_1 and z_2. It plays the same role as the Green's function $G_k^0(z_1, z_2) = e^{ik|z_1-z_2|}/(-2ik)$ in free plain backgrounds. This is due to the fact that the PT potential is transparent (there is no additional reflection with respect to the free case). Furthermore, since the PT potential breaks the isotropy of the space, the Green's function is such that $G_k^{PT}(z_1, z_2) \neq G(z_1 - z_2)$. In fact, $G^{PT}(z, z)$ is not a constant as happens in the free flat case, but depends on the spatial orthogonal coordinate in a non-trivial way, and thus spatial translations are no longer symmetries of the system.

4. Casimir energy and *TGTG* formula

The quantum vacuum energy per unit area of the plates is computed as the summation over all the frequencies $\omega = \sqrt{\vec{k}_\parallel^2 + k^2 + m^2}$ of the fields modes. Notice that

$$\omega^2 \in \sigma(-\partial_{\vec{x}_\parallel}^2 + \hat{K}_{PT} + V_{\delta\delta'}(z)), \tag{22}$$

where σ refers to the spectrum of the operator. However,

$$\frac{\tilde{E}_0}{A} = \frac{1}{2} \oint_k \int_{\mathbb{R}^2} \frac{d\vec{k}_\parallel}{(2\pi)^2} \sqrt{m^2 + k^2 + \vec{k}_\parallel^2}, \tag{23}$$

has a dominant ultraviolet divergence because the energy density of the free theory in the bulk gives an infinite result. This is so because the bulk in this case involves the space between the plates, and although the plates are separated by a finite small distance, their surface area is intended to be infinitely long. Thus the volume of the bulk is infinitely large. The fact that the plates have infinite area also means that the self-energy of each plate is a divergence of a lesser degree than that of the bulk. This divergence is often called subdominant divergence [49]. The aim of this work is to compute the quantum vacuum interaction energy, which is the only part of the quantum vacuum energy $\tilde{E}_0$ that depends on the distance between plates, and gives a finite result. In order to obtain this quantity, an exponentially decaying function acting as a regulator is introduced to remove the aforementioned divergences. In such a way, one rewrites the integration over the parallel modes as

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \frac{d\vec{k}_\parallel}{(2\pi)^2} \sqrt{\vec{k}_\parallel^2 + k^2 + m^2}\, e^{-\epsilon(\vec{k}_\parallel^2 + k^2 + m^2)} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \chi(k, \epsilon)\, e^{-\epsilon(k^2 + m^2)}, \tag{24}$$

with

$$\chi(k, \epsilon) = \frac{\sqrt{\pi}}{4\,\epsilon^{3/2}} + \frac{\sqrt{\pi}(m^2 + k^2)}{4\sqrt{\epsilon}} - \frac{1}{3}\left(k^2 + m^2\right)^{3/2} + o(\epsilon). \tag{25}$$

Eliminating the contribution of the parallel modes to the dominant and subdominant divergences means removing the first two terms in $\chi(k, \epsilon)$ before performing the limit. In this way

$$\frac{\tilde{E}_0}{A} = -\frac{1}{2} \oint_k \frac{(m^2 + k^2)^{3/2}}{6\pi}. \tag{26}$$

In the equation above, the sum over modes of the spectrum in the orthogonal direction splits into the summation over a finite number of states with positive energy in the gap $[0, |E_{min}|]$ (coming from the bound states of the associated quantum mechanical problem) and the integral over the continuous states with energies greater than $|E_{min}|$.

Now, it is necessary to remove from Eq. (26) the contribution to the divergences coming from the modes in the orthogonal direction, where the one-dimensional kink lives. These contributions are different from the ones of the modes in the parallel directions, because in the

orthogonal direction the space is no longer a free one. The usual method is to put the system into a very large box of length L with periodic boundary conditions (p.b.c.) at its edges:

$$\psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right), \qquad \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right). \tag{27}$$

By so doing, the entire spectrum of the Schrödinger operator $\hat{K} = -\partial_z^2 - 2\,\mathrm{sech}^2(z) + V_{\delta\delta'}(z)$ becomes discrete. This fact will be taken into account below.

On the one hand, the contribution of the discrete set of N states in the gap to the vacuum interaction energy is

$$-\frac{1}{2}\sum_{j=1}^{N} \frac{\left(\sqrt{(i\kappa_j)^2 + m^2}\right)^3}{6\pi}. \tag{28}$$

The frequencies of these bound states for each configuration (v_0, v_1, w_0, w_1, a, b) will be determined numerically by solving $\Upsilon(i\kappa) = 0$, $\kappa > 0$ from the scattering problem. If there were half-bound states in the spectrum (i.e. states with energies that lie in the threshold $E = |E_{min}|$), they would have to be accounted for with a weight of 1/2. However, this will not be the case covered in the example of the two Dirac plates in a PT background.

On the other hand, concerning the states with energy $E > m^2$, it is necessary to compute

$$-\frac{1}{2}\sum_{k_n \in \sigma^+(\hat{K})} \frac{\left(\sqrt{k_n^2 + m^2}\right)^3}{6\pi}, \tag{29}$$

where $\sigma^+(\hat{K}) = \{k_n \in \sigma(\hat{K}) \,|\, k_n^2 + m^2 > |E_{min}|\}$. The differential equation $\hat{K}\psi(z) = k_n^2\psi(z)$ with periodic boundary conditions at $\pm L/2$ must be solved. Notice that now the wave function $\psi(z) = A\psi_k^R(z) + B\psi_k^L(z)$ is a linear combination of the scattering solutions (8) and (9). The resulting system of equations admits a solution whenever the following spectral equation holds:

$$h_{p.b.c.}(k, L) \equiv 2tW - 2(t^2 - r_R r_L)f_k\left(\frac{L}{2}\right)f_k'\left(\frac{L}{2}\right) + 2f_{-k}\left(\frac{L}{2}\right)f_{-k}'\left(\frac{L}{2}\right)$$

$$+ (r_R + r_L)\left[f_{-k}\left(\frac{L}{2}\right)f_k'\left(\frac{L}{2}\right) + f_k\left(\frac{L}{2}\right)f_{-k}'\left(\frac{L}{2}\right)\right] = 0. \tag{30}$$

The scattering data are again given in Eq. (A2) of Appendix A. The discrete set of zeroes $\{k_n\}$ of the secular function $h_{p.b.c.}(k, L)$ on the real axis will be the frequencies of the modes over which one has to perform the summation in Eq. (29). This sum can be computed through a complex integral over a contour enclosing all the zeroes of $h_{p.b.c.}(k, L)$. By using the residue theorem in complex analysis and also by taking into account the states in the gap, the total quantum vacuum interaction energy reads

$$\frac{E_0}{A} = -\frac{1}{2}\left[\oint_\Gamma \frac{dk}{2\pi i}\frac{(m^2 + k^2)^{3/2}}{6\pi}\partial_k \log h_{p.b.c.}(k, L)\right] - \frac{1}{2}\sum_{j=1}^{N} \frac{\left(\sqrt{(i\kappa_j)^2 + m^2}\right)^3}{6\pi}, \tag{31}$$

where Γ is the contour represented in Fig. 1. It can be proved that the integration over the circumference arc of the contour is zero in the limit $R \to \infty$. Hence, the integration over the whole contour Γ reduces to the integration over the straight lines $\xi_\pm = \pm i\xi + m$ where the parameter fulfils $\xi \in [0, R]$.

Moreover, the dominant and subdominant divergent terms associated with the orthogonal modes and caused by the confinement of the system in a very large box must be subtracted as
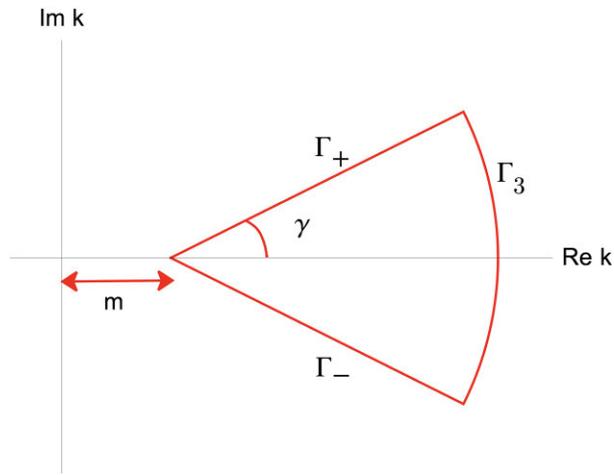
Fig. 1. Complex contour that encloses all the zeroes of the spectral function when $R \to \infty$. In this contour, $\Gamma_{\pm} = \{m + \xi e^{\pm i\gamma} \mid \xi \in [0, R]\}$ and $\Gamma_3 = \{m + R e^{i\nu} \mid \nu \in [-\gamma, \gamma]\}$. The angle $\gamma = \pi/2$ is chosen. When integrating, the contour will be run counterclockwise.

explained in Refs. [50–52], i.e. by computing

$$\int_0^R \frac{d\xi}{12\pi^2 i} \left\{ (m^2 + \xi_+^2)^{3/2} \left[L - L_0 - \partial_\xi \log \frac{h_{p.b.c.}(\xi_+, L)}{h_{p.b.c.}(\xi_+, L_0)} \right] \right. \tag{32}$$
$$\left. - (m^2 + \xi_-^2)^{3/2} \left[L - L_0 - \partial_\xi \log \frac{h_{p.b.c.}(\xi_-, L)}{h_{p.b.c.}(\xi_-, L_0)} \right] \right\}$$

in the limits $L_0, R \to \infty$. The result of the integration does not depend on the box size, and consequently one could study the limit $L \to \infty$. At this point, reversing the change of variable $\xi \to -ik$, due to the Wick rotation on the momentum, yields the DHN formula [46]

$$E_0 = -\frac{A}{12\pi^2} \int_m^\infty dk \left(\sqrt{k^2 + m^2} \right)^3 \frac{d\delta(k)}{dk} - \frac{A}{2} \sum_{j=1}^N \frac{\left(\sqrt{-(\kappa_j)^2 + m^2} \right)^3}{6\pi}, \tag{33}$$

with $m^2 = |E_{min}|$, and where $\delta(k)$ is the phase shift related to the scattering problem in the direction orthogonal to the plate:

$$\delta(k) = \frac{1}{2i} \log_{-\pi} \left[t^2(k) - r_R(k) r_L(k) \right]. \tag{34}$$

So far, I have considered the scattering problem for waves interacting with a system inside a large box, without specifying the type of system I was working with. In addition, the divergences related to putting the system in a box were eliminated. However, notice that the system is composed by two infinitely large plates and a background potential. Consequently, the subdominant divergences associated with the plates are still present and a renormalization mode-by-mode is necessary. This step is achieved by subtracting from the phase shift of the whole system with two plates the corresponding phase shifts associated with a reduced problem with only one delta plate:

$$\tilde{\delta}(k) = \delta_{v_0 w_0, v_1 w_1}(k) - \delta_{v_0 w_0}(k) - \delta_{v_1 w_1}(k). \tag{35}$$

It is worth highlighting that this equation constitutes a subtraction mode-by-mode of the spectrum to complete the renormalization. This method is different from the frequently used one of setting a cutoff in the integral over modes to remove the high energetic part of the spectrum

that does not feel the background. The phase shift (35) has to be used in the DHN formula (33) to obtain a finite result.

Nevertheless, instead of using the aforementioned approach with the derivative of the phase shifts acting as the density of states, the *TGTG*-formula will be used. The results will be the same using either of these two procedures. However, with the *TGTG* representation we do not have to work with the scattering of the complete problem, but with that of the reduced problem of a single object in the background. Hence, the numerical computational effort is much lower.

In the seminal paper [11], O. Kenneth and I. Klich give the following formula for the quantum vacuum interaction energy between two compact bodies (1 and 2) in one spatial dimensional flat spacetime:

$$E_0 = -i \int_0^\infty \frac{d\omega}{2\pi} \operatorname{tr} \log\left(1 - T_1 G_{12} T_2 G_{21}\right). \tag{36}$$

Notice that the case the authors considered does not present bound states with negative energy in the spectrum of the corresponding Schrödinger operator in quantum mechanics. On the contrary, these type of states must be included in the case I am considering. Furthermore, in appendices B and C of Ref. [11] the authors prove that for any pair of disjoint finite bodies separated by a finite distance, and any Green's function that is finite away from the diagonal, the *TGTG*-operator is trace-class. The modulus of its eigenvalues is less than one and hence $\log(1 - TGTG)$ is well defined. A similar reasoning can be followed here for the system of a pair of two-dimensional plates, which are assumed not to touch, in the background of a kink. Thus, the only step left to be taken is to calculate the *T*-operators for each one of the plates.

From the well-known Lippmann–Schwinger equation

$$\Delta G_k(z_1, z_2) = -\int dz_3 \, dz_4 \, G_k^{PT}(z_1, z_3) T_k(z_3, z_4) G_k^{PT}(z_4, z_2), \tag{37}$$

and $\tilde{K}_{z_1} G_k^{PT}(z_1 - z_2) = \delta(z_1 - z_2)$, it is easy to see that

$$-\tilde{K}_{z_2} \tilde{K}_{z_1} \Delta G_k(z_1, z_2) = \int dz_3 \, dz_4 \, \delta(z_1 - z_3) T_k(z_3, z_4) \delta(z_4 - z_2), \tag{38}$$

where $\tilde{K}_z = \hat{K}_{PT}(z) - k^2$. Notice that in the above formula, $\Delta G_k(z_1, z_2)$ corresponds to the Green's function of only one plate in the PT potential, i.e. the one given in Eq. (A4) with the coefficients of one of the plates equal to zero. Due to the absolute values contained in G_k^{PT}, to obtain the transfer matrix $T_k(z_1, z_2)$ corresponding to one plate, the only non-trivial contribution comes from the case in which one point is on the left and the other one on the right of the plate. Hence, since in that case the Green's function is given by $\Delta G_k(z_1, z_2) = (t - 1) G_k^{PT}(z_1, z_2)$, one needs to compute

$$T_k(z_1, z_2) = -(t - 1)\tilde{K}_{z_2} \tilde{K}_{z_1} G_k^{PT}(z_1, z_2). \tag{39}$$

In order to obtain the transfer matrix associated with the plate on the right[3], one assumes the plate sitting at the origin ($b = 0$) for simplicity, and hence one of the coordinates z_1, z_2 will be

---

[3]For computing the transfer matrix of the left-hand plate, one sets $v_1 = w_1 = 0$ in the transmission coefficient (A2) involved in the Green's function (A4) and considers the case in which the left-hand plate is centered at $a = 0$.

greater than zero and the other will be less than zero. Taking into account that

$$e^{ik|z_1-z_2|} = e^{ik(|z_1|+|z_2|)},$$

$$|\tanh z_1 - \tanh z_2| = \tanh|z_1| + \tanh|z_2|,$$

$$\tanh z_1 \tanh z_2 = -\tanh|z_1|\tanh|z_2|, \tag{40}$$

(in both the cases $z_1 < 0$, $z_2 > 0$ and $z_1 > 0$, $z_2 < 0$), it is possible to rewrite the free Green's function in the background of the kink as

$$G_k^{PT}(z_1, z_2) = -\frac{1}{W} f_k(|z_1|) f_k(|z_2|). \tag{41}$$

Because the Green's differential equation

$$\left(-\partial_{|z|}^2 - k^2 - 2\,\mathrm{sech}^2 z\right) f_k(|z|) = 0 \tag{42}$$

holds, and using the formulas for the derivatives of functions depending on absolute values

$$\frac{df_k(|z|)}{dz} = f_k'(|z|)\,\mathrm{sign}\,z,$$

$$\frac{d^2 f_k(|z|)}{dz^2} = f_k''(|z|) + f_k'(|z|)\,2\,\delta(z), \tag{43}$$

the transfer matrix for the right-hand plate can be written as

$$T_k(z_1, z_2) = \frac{t-1}{W} 4\,\delta(z_1)\,\delta(z_2)\,f_k'(|z_1|)f_k'(|z_2|) = -4\,\delta(z_1)\,\delta(z_2)\,\Delta G(z_1, z_2)\frac{f_k'(|z_1|)}{f_k(|z_1|)}\frac{f_k'(|z_2|)}{f_k(|z_2|)}$$

$$= \frac{|W|^2}{k^4}\delta(z_1)\delta(z_2)\Delta G_k(z_1, z_2). \tag{44}$$

The Green's function for one delta plate is defined as the following piece-wise function:

$$\Delta G_k(z_1, z_2) = \begin{cases} \frac{r_L}{W} f_k(z_1) f_k(z_2), & \text{if } z_1, z_2 > b, \\ \frac{r_R}{W} f_{-k}(z_1) f_{-k}(z_2), & \text{if } z_1, z_2 < b, \\ \frac{t-1}{W} f_k(z_>) f_{-k}(z_<), & \text{otherwise,} \end{cases} \tag{45}$$

with the scattering data as given in Eq. (A2) but for the case $v_0 = w_0 = 0$. Consequently,

$$T_k(z_1, z_2) = -\frac{W^*}{k^2}\delta(z_1)\delta(z_2) \begin{cases} r_L(b=0), \\ r_R(b=0), \\ 1 - t(b=0). \end{cases} \tag{46}$$

Here the asterisk means the complex conjugate. The Lippmann–Schwinger operator is related to the scattering matrix by $S = 1 - i\delta(\omega - \omega')T$. This implies normalizing T so that the factor k^{-2} cancels out.

When the delta potential which mimics the plate is evaluated at another point different from the origin, the just-computed result for $T_k(z_1, z_2)$ is valid once after performing the translations $z_1 \mapsto z_1 - b$ and $z_2 \mapsto z_2 - b$. Notice that in the definition of T at a point different to $z = 0$, the translation must be understood as replacing the scattering coefficients at $z = 0$ contained in its definition by the ones at $z = b$. Due to the PT potential, the isotropy of the spacetime is broken. Hence, $r_{R,L}(b) \neq r_{R,L}(0)e^{ikb}$, so the general translations $z_1 \mapsto z_1 - b$ and $z_2 \mapsto z_2 - b$ aforementioned must not be interpreted in this usual sense. The T operator for the right-hand side plate is thus given by

$$T(z_1, z_2) = -W^*\delta(z_1 - b)\delta(z_2 - b) \begin{cases} r_L(b), & z_1, z_2 \to b^+, \\ r_R(b), & z_1, z_2 \to b^-, \\ 1 - t(b), & \text{otherwise,} \end{cases} \tag{47}$$

and analogously for the plate located at $z = a$.

Notice that the T-matrix is the probability amplitude for a particle to interact with the potential but without propagation. Hence, in the system of two plates mimicked by point-like delta potentials in the background of a kink, the definition of the T-operator must depend on ΔG evaluated at the point at which the delta potential is centered, as it is the case in Eq. (47). It could not depend on an arbitrary point of the spacetime for causality not to be violated. Notice that in more general cases, when the potential representing each object is not supported at a point but a compact interval, T is local. Although in this case T would depend on the points that constitute the support of the potential, it does not violate causality because it does not depend on arbitrary points.

By definition $G^{PT}(z_1, z_2) = \langle \mathcal{T} \phi(z_1)\phi(z_2) \rangle$, an expression in which the time-ordering operator product has been considered. All eigenstates of $\hat{K}_{PT}$ with fixed energy k^2 can be described in terms of the orthonormal basis of left- and right-hand PT free waves. By labeling $R = f_k(z)$ and $L = f_{-k}(z)$, the Green's function or propagator can be written in this basis as

$$G^{PT}(a, b) = \frac{1}{W}|L(a)\rangle\langle L(b)|, \qquad G^{PT}(b, a) = \frac{1}{W}|R(b)\rangle\langle R(a)|, \tag{48}$$

where $a < b$, and the trace of the $TGTG$-operator behaves as

$$|L(a)\rangle\langle L(b)|T^r|R(b)\rangle = r_L^\ell(v_0, w_0, a, k)\, r_R^r(v_1, w_1, b, k). \tag{49}$$

It has been taken into account that $T|R\rangle = |L\rangle$ and vice versa. Thus, it is clear that the $TGTG$-formula will involve the reflection coefficients, which depend explicitly on the position of each plate. The above formula is the only product of the T-matrix components that allows coincidences of the z_1, z_2 points in $[a, b]$ and contributes to the quantum vacuum interaction energy between plates. In the special case discussed here, the $TGTG$ operator has a rank equal to one. This implies that

$$\text{tr}\log(1 - TGTG) = \log\det(1 - TGTG) = \log(1 - \text{tr}\,TGTG). \tag{50}$$

For other cases, if the modulus of the eigenvalues of the $TGTG$ operator is less than one, it is still possible to use

$$\text{tr}\log(1 - TGTG) = \log\det(1 - TGTG) \approx \log(1 - \text{tr}\,TGTG) \tag{51}$$

as a good approximation up to first order to simplify Eq. (36). The demonstration is collected in Appendix B. In summary, replacing Eqs. (49) and (50) in Eq. (36), and generalizing it to three dimensions, leads to the final expression (52) with $m^2 = |E_{min}|$:

$$
\begin{aligned}
\frac{E_0}{A} &= -\frac{1}{2}\sum_{j=1}^{N}\frac{\left(\sqrt{-\kappa_j^2 + m^2}\right)^3}{6\pi} + \frac{1}{8\pi^2}\int_m^\infty d\xi\, \xi\, \sqrt{\xi^2 - m^2}\, \log(1 - \text{Tr}\,TGTG_\xi), \\
&= -\frac{1}{2}\sum_{j=1}^{N}\frac{\left(\sqrt{-\kappa_j^2 + m^2}\right)^3}{6\pi} + \frac{1}{8\pi^2}\int_m^\infty d\xi\, \xi\, \sqrt{\xi^2 - m^2}\, \log\left[1 - r_L^\ell(v_0, w_0, a, i\xi)\, r_R^r(v_1, w_1, b, i\xi)\right].
\end{aligned}
\tag{52}
$$

There are some details that are worth highlighting. First, if $f_\pm(z)$ were replaced by the usual plane waves, Kenneth and Klich's original $TGTG$-formula would be restored. The reason is that, in flat isotropic spacetimes, the scattering coefficients for plates placed at another point different from the origin are equal to the ones at $z = 0$ times an exponential factor that accounts

for the translation that has taken place:

$$r_L^\ell(a) = r_L^\ell(0)\, e^{-2iak} = r_L^\ell(0)\, W\, G_{-k}^{(0)}(-a, a),$$

$$r_R^r(b) = r_R^r(0)\, e^{2ibk} = r_R^r(0)\, W\, G_k^{(0)}(-b, b). \tag{53}$$

On the contrary, the main difference when working with weak and transparent curved space-times that break the isotropy of the space is that this rule no longer applies. Thus, the scattering coefficients for plates placed at another point different from the origin are equal to the ones evaluated at $z = 0$ times the quotient between the transmitted probability amplitude at the generic point and the one at $z = 0$, and multiplied by another function related to the configuration of the space:

$$r_L^\ell(a) = r_L^\ell(0)\, \frac{G_{-k}^{PT}(-a, a)\, t(a)\, h^+(\alpha_0, \beta_0, -a)}{G_{-k}^{PT}(0, 0)\, t(0)\, h^+(\alpha_0, \beta_0, 0)},$$

$$r_R^r(b) = r_R^r(0)\, \frac{G_k^{PT}(-b, b)\, t(b)\, h^-(\alpha_1, \beta_1, b)}{G_k^{PT}(0, 0)\, t(0)\, h^-(\alpha_1, \beta_1, 0)}, \tag{54}$$

where $h^\pm(\alpha_i, \beta_i, x) = \alpha_i\beta_i \pm (\alpha_i^2 - 1)f'(k, x)/f(k, x)$. This fact is crucial to generalize the *TGTG*-formula in the case studied.

Secondly, by defining the potential $V_i(z)$ to describe each of the two plates as

$$V_i(z) = v_i\, \delta(z - z_i) + w_i \delta'(z - z_i) \tag{55}$$

(with $i = 0, 1$ and $z_0 = a$, $z_1 = b$), the Schrödinger operators $\hat{K}_i = -\partial_z^2 - V_i(z)$ are defined over a Hilbert space that, in general, is not isomorphic to that of $\hat{K}_{PT}$. Hence, G^{PT} and $T^{r,\ell}$ do not act in the same spaces, and $T^\ell G^{PT} T^r G^{PT}$ is ill-defined. To avoid this problem, a Wick rotation of the momentum k must be performed in order for all the operators to act in the same Hilbert space. The integral (52) is thus convergent and can be evaluated numerically with *Mathematica*. In the next section, the results of the Casimir energy for some configurations of the plates in the PT background potential will be discussed.

5. Casimir pressure

Once the quantum vacuum interaction energy is determined, one can study the Casimir force between plates as $F = -\partial E_0/\partial d$, where d is the distance between plates. Nevertheless, as already explained, the translational invariance is broken due to the PT background, which means that the scattering data for the plates explicitly depend on the position in a non-trivial way. Hence, when computing the Casimir force, a non-trivial contribution coming from the derivatives of the scattering amplitudes of one of the plates with respect to the position will appear. There is an ambiguity not yet clarified in this calculation. One can introduce the dependence on the distance between plates in three different ways:

(1) Putting the left-hand side plate at $z_1 = a$ and the right-hand side one at $z_2 = a + d$. In this case, only the scattering data of the plate on the right will depend on the distance d, and only the derivative of $r_R^r(v_1, w_1, a + d, k)$ with respect to d will appear.

(2) Considering the right-hand side plate placed at $z_2 = b$ and the left one at $z_1 = b - d$. Analogously to the previous case, the derivative of $r_L^\ell(v_0, w_0, b - d, k)$ with respect to the distance is the only possible contribution.

(3) When one of the plates is to the left of the origin and the other one to the right, one could describe the location of the plates as the left one being at $z_1 = -d + b$ and the other one
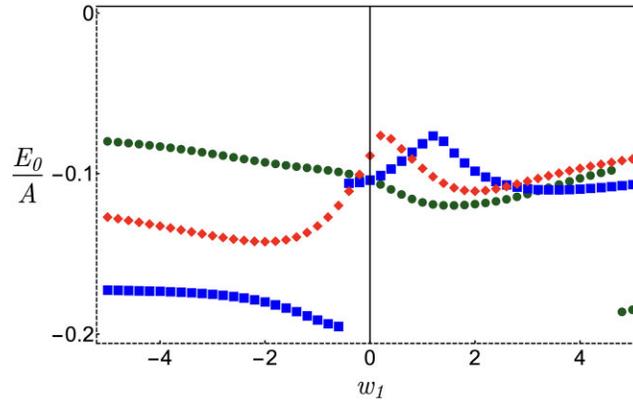
Fig. 2. Casimir energy per unit area between plates situated at $a = -0.2$, $b = 0.8$, as a function of the coefficient w_1. Different configurations are shown: (a) Pure δ' plates (i.e. $v_0 = v_1 = 0$) with $w_0 = 3$ (rhombi), (b) identical plates characterized by $v_0 = v_1 = 1$ and $w_0 = w_1$ (circles), and (c) opposite plates described by $v_0 = v_1 = 1$ and $w_0 = -w_1$ (squares).
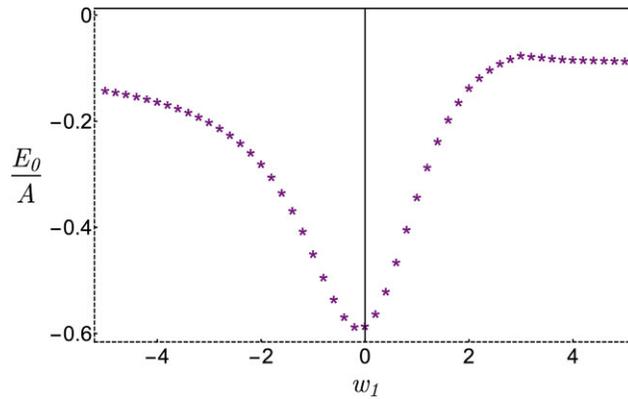
Fig. 3. Casimir energy per unit area between plates situated at $a = -0.2$, $b = 0.8$, as a function of the coefficient w_1. A generic $\delta\delta'$ potential with $v_0 = 1$, $v_1 = -4$, $w_0 = 2.5$ has been considered.

at $z_2 = d + a$, with $a < 0$ and $b, d > 0$. This case is different because the derivatives of the reflection coefficients of both plates $r_R^r(v_1, d + a, k)$ and $r_L^\ell(v_0, -d + b, k)$ will be taken into account.

It is work in progress to check that these three situations give rise to the same force. However, it seems reasonable to think that if the Casimir energy between plates has a change in the sign for some values of the parameters $\{v_0, v_1, w_0, w_1, a, b\}$, the Casimir force will present that too. Consequently, studying numerical results for the quantum vacuum interaction energy is enough to discuss whether this flip of sign appears as a consequence of the introduction of the δ' potential, as was the case in other configurations in flat spacetimes [16,53].

Figures 2, 3, and 4 show the quantum vacuum interaction energy per unit area of the plates for different configurations of the system of two plates in the PT background. As can be seen in the figures above, the energy is always negative, independently of the value of the δ and δ' co-efficients. Furthermore, it could be checked numerically that the quantum vacuum interaction energy is definitely negative, regardless of the relative position of the plates with respect to the kink center as well. This implies that the Casimir force between plates will always be attractive in this system. However, due to the changes of the spectrum of bound states as a function of
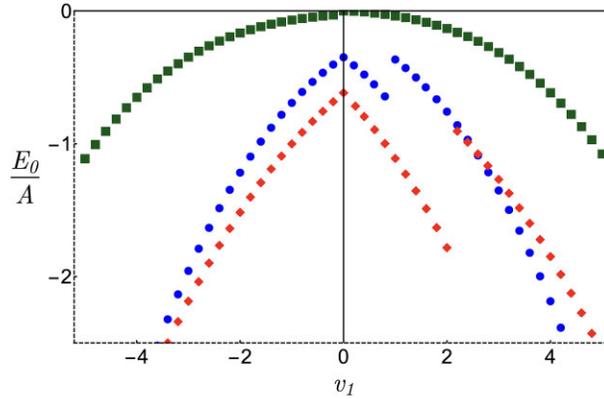
Fig. 4. Casimir energy per unit area between pure delta plates ($w_0 = w_1 = 0$) situated at $a = -b = -0.5$ as a function of v_1. In this plot $v_0 = -3$ (circles), $v_0 = 0.1$ (squares), and $v_0 = 4$ (rhombi).

$\{v_0, v_1, w_0, w_1\}$, and the relative position between the plates and the kink, there is a peculiar fact characteristic of the spectra shown in Figs. 2 and 4. The sudden discontinuities present in the plots are related to the 'loss' of one bound state with very low $k = i\kappa$ (nearly zero) in the spectrum of the system. At this point, it is necessary to realize that the whole system of the two plates together with the PT potential acts as a well with a fixed depth and width. Consequently, at the configuration in the space of parameters at which the jump appears, the resulting well is not deep enough to hold more bound states with large negative energy. This loss of a bound state, which becomes a state in the continuum, translates into a jump in the energy, because the terms related to the states within the gap give the major contribution to the total energy. The jump discontinuities in the energy appear for the $\delta\delta'$ plates case whenever $v_i > 0$. It can be checked that for the cases of identical and opposite plates, if $v_i < 0$, the results would be qualitatively similar to those shown in Fig. 3. Consequently, the introduction of the δ' potential modifies the spectrum in such a way that the well represented by the system can better accommodate bound states if $v_i < 0$. In these cases, for different configurations of the coefficients of the $\delta\delta'$ functions very close to each other, no bound states with momenta close to zero are lost.

Another important conclusion can be drawn. For flat spacetimes, it has been proved that the introduction of the δ' potential causes the sign of the force to change in different areas of the parameter space. This behaviour has been observed for instance for a scalar field and two concentric spheres defined by such a singular $\delta\delta'$ potential on their surfaces [53], or in Ref. [54] for $\delta\delta'$ plates. However, curiously, when considering this last configuration in a sine-Gordon kink spacetime, the change of sign in the energy disappears. Namely, one can consider a curved spacetime that is confining (in the sense that the background acts as a well) and the plates are within the support of that well. Then, even in the case where the plates act as very repulsive barriers, there are still negative energy states in the spectrum of the associated Schrödinger operator, and the Casimir energy between plates will be attractive. Consequently, although the background potential under consideration constitutes an example of a weak curved background, the results are quite different from the flat case.

When the PT potential is not confined at all between the plates and they are far from the kink centre, the system of two $\delta\delta'$-plates in flat spacetime is recovered. This is the reason that the numerical representation for this situation, which has already been studied in the literature [54], is not included here.

Finally, it is worth pointing out that even in the case where there is only one plate in the system, the other plate feels the interaction because there is still a non-zero quantum vacuum interaction energy in the system. This can be checked by looking at the non-zero values of the energy appearing in the vertical axis for the pure δ' plates (in this axis, $w_1 = 0$ and there is no right-hand plate in the system) in Fig. 2.

6.   Conclusions

In this work, a quantum scalar field between two parallel two-dimensional plates in a sine-Gordon background at zero temperature is presented. The main result is the generalization of the *TGTG*-formula for weak and transparent gravitational backgrounds in which the frequencies of the particles created by the gravitational background are much smaller than the Planck frequency, and the fields could be asymptotically interpreted as particles. The quantum vacuum interaction energy has been calculated using this formula, which only depends on the reflection coefficients associated with the scattering problem. They involve a dependence on the solutions of the Schrödinger equation for the specific background chosen, which play the same role as the plane waves in flat spacetimes. The Casimir energy thus depends on the parameters describing the potentials and on the distance from the plates to the center of the kink. To obtain the energy, it has been necessary to compute the Green's functions from the scattering data. The transfer matrix has been determined with complete generality too, in terms of the Green's function. Although the *TGTG*-formula has the advantage that it depends only on the scattering data of one of the plates, and it is not necessary to solve the scattering problem of the whole system, the well-known DHN formula has also been derived for completeness. The virtue of the *TGTG*-formula obtained here is that it can be easily generalized to other types of configurations, either for another background or for other potentials that could properly mimic the plates.

As an example, two plates mimicked by a Dirac δ-potential and its first derivative in a background of a topological PT kink is studied. The quantum vacuum fluctuations around the kink solution could be interpreted as mesons propagating in the spacetime of a domain wall. One of the relevant characteristics of the PT background is that the translational symmetry is broken and the space is anisotropic. This translates into the fact that the scattering coefficients, as well as the Green's function, will depend on the position of the plates in a non-trivial way. The wave functions of the continuous spectrum of states with positive energy have been characterized by means of these scattering data. The bound states have also been studied, setting a threshold for the minimum negative energy in the system. The unitarity of the QFT requires this lower bound be fixed as the mass of the quantum vacuum fluctuations, so that the total energy of the lowest energy state of the spectrum will be zero, making fluctuations in absorption impossible. It is worth highlighting that the quantum vacuum energy for this (3+1)-dimensional problem is only negative, independently of the value of the coefficients of the $\delta\delta'$ potentials and their location in relation to the kink center. This implies that the Casimir force between plates will be attractive in this system. Furthermore, even in the case where there is only one plate in the system, the other plate feels the Casimir interaction because there is still a non-zero quantum vacuum interaction energy in the system.

Once the Casimir energy between plates in the background of a sine-Gordon kink has been computed by using the *TGTG*-formula, it would be enlightening to obtain the same result but from the integration of the 00-component of the energy-momentum tensor. In this way one

could also study the spatial distribution of the energy density. This is left for further investigations.

I would like to finish by mentioning a possible future phenomenological application of the study collected in this work. Notice that in the Casimir effect, the electromagnetic force results from the computation of the one-loop quantum correction to the vacuum polarization in QED. Nevertheless, taking a quadratic approximation for the action implies that it does not depend on the coupling constant of the theory, and for that reason the force is relevant at the nanometre scale. By the same argument, if one-loop quantum corrections to the graviton propagator were calculated, the coupling constant of the gravitational theory would not appear either. And therefore, since the graviton has zero mass, one could think of studying the quantum interacting force between gravitational objects separated by a small distance, using the same quadratic approximation reasoning applied here. The force caused by non-massive quantum vacuum fluctuations around a classical solution for gravitons coupled to no matter which other massive field would constitute an example of quantum corrections to the gravitational field theory. The difficulty will lie in finding a material that is opaque to the gravitational waves [55]. Although interesting, this open problem is left for future research.

Appendix A. Scattering data and Green's function for two $\delta\delta'$ plates in a PT background

The scattering data as well as the spectral function are obtained directly when the system of Eqs. (8), (9) and (6) is solved with the software *Mathematica*. More specifically, for the left-to-right case, if one replaces Eq. (8) in the matching conditions of Eq. (6) one obtains the following system:

$$[B_R f_k(z) + C_R f_{-k}(z)] \mid_{z \to a} = \alpha_0 [f_k(z) + r_R f_{-k}(z)] \mid_{z \to a}$$

$$\frac{\partial}{\partial z}[B_R f_k(z) + C_R f_{-k}(z)] \mid_{z \to a} = \beta_0 [f_k(z) + r_R f_{-k}(z)] \mid_{z \to a} + \frac{1}{\alpha_0}\frac{\partial}{\partial z}[f_k(z) + r_R f_{-k}(z)] \mid_{z \to a}$$

$$t_R f_k(z) \mid_{z \to b} = \alpha_1 [B_R f_k(z) + C_R f_{-k}(z)] \mid_{z \to b} \tag{A1}$$

$$\frac{\partial}{\partial z}[t_R f_k(z)] \mid_{z \to b} = \beta_1 [B_R f_k(z) + C_R f_{-k}(z)] \mid_{z \to b} + \frac{1}{\alpha_1}\frac{\partial}{\partial z}[B_R f_k(z) + C_R f_{-k}(z)] \mid_{z \to b}.$$

Solving this system of equations, the scattering data for the non-relativistic mechanical problem of scalar fields propagating in the curved background of a topological PT kink while interacting with two Dirac $\delta\delta'$ plates are obtained. They are given by Eq. (A2). The right-to-left scattering can be obtain analogously by replacing Eq. (9) in Eq. (6), and it is easy to check that the denominator of the scattering data is the same as in the left-to-right case.

$$t(k) = \frac{1}{\Upsilon(k)}\alpha_0\alpha_1 W^2,$$

$$r_R(k) = \frac{1}{\Upsilon(k)}[-f_k(b)f_k(a)\left(f'_{-k}(b)\left(f'_k(a) - \alpha_0\mathcal{A}_0(a)\right) + \alpha_0^2 f'_{-k}(a)\left(-\alpha_1\mathcal{A}_1(b) + f'_k(b)\right)\right)$$

$$+ \alpha_1\mathcal{A}_1(b)f_k(a)f_{-k}(b)\left(f'_k(a) - \alpha_0\mathcal{A}_0(a)\right) - f_k(b)f_{-k}(a)\left(f'_k(a) + \alpha_0\beta_0 f_k(a)\right)\left(\alpha_1\mathcal{A}_1(b) - f'_k(b)\right)],$$

$$r_L(k) = -\frac{1}{\Upsilon(k)}[f_{-k}(b)f_k(a)f'_k(a)(-\alpha_1\mathcal{A}_1^*(b) + f'_{-k}(b)) + \alpha_0\beta_0 f_{-k}^2(a)(-f_{-k}(b)f'_k(b) + \alpha_1 f_k(b)\mathcal{A}_1^*(b))$$

$$+ f_{-k}(a)[f_{-k}(b)\alpha_0\mathcal{A}_0(a)(\alpha_1\mathcal{A}_1^*(b) - f'_{-k}(b)) + (-1 + \alpha_0^2)f'_{-k}(a)(-\alpha_1 f_k(b)\mathcal{A}_1^*(b) + f'_k(b)f_{-k}(b))]],$$

$$B_R(k) = \frac{1}{\Upsilon(k)}[\alpha_0 W\left(f_k(b)f'_{-k}(b) - \alpha_1\mathcal{A}_1(b)f_{-k}(b)\right)],$$

$$B_L(k) = -\frac{1}{\Upsilon(k)}[\alpha_1 W f_{-k}(a)\left(f'_{-k}(a) - \alpha_0\mathcal{A}_0^*(a)\right)],$$

$$C_R(k) = -\frac{1}{\Upsilon(k)}[\alpha_0 W f_k(b)\left(f'_k(b) - \alpha_1\mathcal{A}_1(b)\right)],$$

$$C_L(k) = -\frac{1}{\Upsilon(k)}[\alpha_1 W\left(\alpha_0\mathcal{A}_0(a)f_{-k}(a) - f_k(a)f'_{-k}(a)\right)],$$

$$\Upsilon(k) = -\left(\alpha_0\mathcal{A}_0^*(a) - f'_{-k}(a)\right)\left[f_k(b)\left(f_{-k}(a)\left(\alpha_1\mathcal{A}_1(b) - f'_k(b)\right) + f_k(a)f'_{-k}(b)\right) - \alpha_1\mathcal{A}_1(b)f_k(a)f_{-k}(b)\right]$$

$$+ \alpha_0^2 W\left(f_k(b)f'_{-k}(b) - \alpha_1\mathcal{A}_1(b)f_{-k}(b)\right). \tag{A2}$$

The notations $\mathcal{A}_i(z) = -\beta_i f_k(z) + \alpha_i f'_k(z)$, together with $W = -2ik(k^2 + 1)$, have been used to simplify the expressions.

Notice that the denominator of all the scattering parameters, $\Upsilon(k)$, is the Jost function. The spectral or Jost function is widely used in the literature [56–60] because of its connection to the denominator of the scattering coefficients. Notice that the Jost function $j(k)$ is related to the S-matrix by means of $\det S = t^2(k) - r_R(k)r_L(k) = e^{i2\delta(k)} = j^*(k)/j(k)$. It should be highlighted that the phase shift in the scattering problem is just minus the phase of the Jost function. Furthermore, the zeroes of the Jost function on the positive imaginary axis of the complex momentum plane characterize the wave vector of the bound states in the quantum mechanical problem. Consequently, the Jost function determines both the spectrum and the phase shift of the scattering problem. From the last expression of Υ in Eq. (A2), taking $\Upsilon(i\kappa) = 0$, $\kappa > 0$, and with a little bit of algebra, one could separate those terms involving an exponential from those with polynomials, obtaining the result given on page 5 [Eqs. (10) and (11)].

The Green's function of the associated QFT can be expressed in a compact way as

$$G_k(z_1, z_2) = G_k^{PT}(z_1, z_2) + \Delta G_k(z_1, z_2), \tag{A3}$$

with $\Delta G_k(z_1, z_2)$ given by Eq. (A4):

$$
\begin{cases}
\dfrac{r_L}{W} f_k(z_1) f_k(z_2), & \text{if } z_1, z_2 > b, \\[2ex]
\dfrac{r_R}{W} f_{-k}(z_1) f_{-k}(z_2), & \text{if } z_1, z_2 < a, \\[2ex]
\dfrac{B_R B_L}{t\,W} f_k(z_1) f_k(z_2) + \dfrac{C_R C_L}{t\,W} f_{-k}(z_1) f_{-k}(z_2) & \text{if } a < z_1 < b \text{ and } a < z_2 < b, \\[1ex]
+ \dfrac{C_R B_L}{t\,W} (f_{-k}(z_>) f_k(z_<) + f_k(z_>) f_{-k}(z_<)), & \\[2ex]
(t-1)\, G_k^{PT}(z_1, z_2), & \text{if } z_2 > b \text{ and } z_1 < a \, (\text{or } z_1 \leftrightarrow z_2), \\[2ex]
(C_L - 1)\, G_k^{PT}(z_1, z_2) + \dfrac{B_L}{W} f_k(z_1) f_k(z_2), & \text{if } z_2 > b \text{ and } a < z_1 < b \,(\text{or } z_1 \leftrightarrow z_2), \\[2ex]
(B_R - 1)\, G_k^{PT}(z_1, z_2) + \dfrac{C_R}{W} f_{-k}(z_1) f_{-k}(z_2), & \text{if } z_2 < a \text{ and } a < z_1 < b \,(\text{or } z_1 \leftrightarrow z_2).
\end{cases}
\tag{A4}
$$

The scattering data involved are collected in Eq. (A2). The points $\{a, b\}$ at which the plates are located are completely general and can be replaced by any other pair of points, independently of their position with respect to the origin, around which the PT kink is centered.

Appendix B. Proof of relation (51)

In appendices B and C of Ref. [11] the authors prove that if the Green's function $G(x, y)$ is smooth for $x \neq y$, then for any two disjoint objects A, B separated by a finite distance, G^{AB} is a trace-class operator. Moreover, T^A and T^B are bounded and $T^A G^{AB} T^B G^{BA}$ is a trace-class operator. They also prove that the modulus of the eigenvalues of $T^A G^{AB} T^B G^{BA}$ is less than one. The aim of this appendix is to demonstrate that

$$
\operatorname{tr} \log(1 - TGTG) = \log \det(1 - TGTG) \approx \log(1 - \operatorname{tr} TGTG)
\tag{B1}
$$

is satisfied whenever there are two separate objects in a weak curved background for which the aforementioned hypothesis for the $TGTG$ operator is satisfied, and rank $(TGTG) \neq 1$ is fulfilled.

The first equality in Eq. (B1) is general, and it can be proven by taking into account that any Hermitian matrix P representing an Hermitian operator can be transformed into a diagonal matrix P_D, so that $P_D = QPQ^{-1}$. In this way,

$$
e^{\operatorname{tr} \log P} = e^{\operatorname{tr} \log(Q^{-1} P_D Q)} = e^{\operatorname{tr}[Q^{-1}(\log P_D)Q]} = e^{\operatorname{tr} \log P_D} = e^{\sum_i \log \lambda_i}
$$
$$
= \prod_i \lambda_i = \det P_D = \det(QPQ^{-1}) = \det P.
\tag{B2}
$$

The cyclic property of the trace has been used above. The eigenvalues of the diagonal matrix P_D have been denoted by λ_i. On the other hand, if α_i are the eigenvalues of $M = TGTG$, it is easy to prove the second claim in Eq. (B1), because

$$
\log \det(1 - M) = \log \det[Q_1^{-1}(1 - M_D)Q_1] = \log \prod_i (1 - \alpha_i) = \log[1 - \sum_i \alpha_i + o(\alpha_i \alpha_j)]
$$
$$
\approx \log[1 - \operatorname{tr} M_D] = \log[1 - \operatorname{tr}(Q_1 M Q_1^{-1})] = \log[1 - \operatorname{tr} M].
\tag{B3}
$$

This constitutes a good approximation up to the first order since the norm of M is less than one.

In some references, another approach to understand the approximate computation can be found. If we consider dielectrics instead of the vacuum, if the dielectric function can be written as $1 + x$ with x being small (i.e., the dilute limit), $||TGTG|| < 1$, and one could make the formal expansion of $\log(1 - x)$ to finally write

$$\log \det(1 - TGTG) = -\sum_j \frac{1}{j} \text{Tr}\,(TGTG)^j \tag{B4}$$

(this is done in section V of the seminal paper by Kenneth and Klich [11] and in section 4.1 of Ref. [15]). In this last article, figure 4 shows the numerical difference between the contribution of the integral (36) for $j = 1$ and $j = 2$. As can be seen in the left-hand plot of the figure, considering $\text{Tr}\,(TGTG)^2$ gives a result numerically much smaller than the one involving $\text{Tr}\,(TGTG)$, so we can neglect it and still obtain a good approximation for the quantum vacuum correction to the Casimir energy up to the first order in $\hbar$. These types of approximations are widely used in the literature, for instance in Refs. [16,54].

There is another important detail to be considered. Notice that

$$\text{tr}\log(1 - TGTG) = \log \det(1 - TGTG) = c_{TGTG}(1) \tag{B5}$$

is always fulfilled. $c_{TGTG}(1)$ is the characteristic polynomial of the matrix representing the operator $TGTG$. This polynomial can be expressed in terms of the trace of the operator itself and the trace of matrices given by products of a finite number of times the operator, as

$$c_{TGTG}(1) = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \begin{vmatrix} \text{tr}\,TGTG & 1 & 0 & 0 & \dots & 0 \\ \text{tr}(TGTG)^2 & \text{tr}(TGTG) & 2 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \text{tr}(TGTG)^{k-1} & \text{tr}(TGTG)^{k-2} & \text{tr}(TGTG)^{k-3} & \text{tr}(TGTG)^{k-4} & \dots & k-1 \\ \text{tr}(TGTG)^k & \text{tr}(TGTG)^{k-1} & \text{tr}(TGTG)^{k-2} & \text{tr}(TGTG)^{k-3} & \dots & \text{tr}\,TGTG \end{vmatrix}. \tag{B6}$$

Thus the correct way to compute the Casimir energy between objects and obtain the exact result when rank $TGTG \neq 1$ is to evaluate numerically the following whole integral:

$$\int_m^\infty \xi \sqrt{\xi^2 - m^2}\,\log \det(1 - TGTG) = \int_m^\infty \xi \sqrt{\xi^2 - m^2}\,c_{TGTG}(1). \tag{B7}$$

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