UNIVERSIDAD DE BURGOS

FACULTAD DE CIENCIAS

Lorentzian Poisson homogeneous spaces, quantum groups and noncommutative spacetimes

TESIS DOCTORAL

IVÁN GUTIÉRREZ SAGREDO

2019

PROGRAMA DE DOCTORADO: QUÍMICA AVANZADA

AGRADECIMIENTOS

La lista de personas a las que agradecer su apoyo y confianza durante todo el tiempo que ha llevado la realización de esta Tesis es demasiado larga como para no olvidar a alguien importante, por lo que comienzo pidiendo disculpas de antemano a cualquiera que debiese aparecer aquí y no lo haga.

En primer lugar, agradezco enormemente a mis directores de Tesis, Ángel Ballesteros y Francisco J. Herranz (más conocido como Vulpi), por todo su apoyo y guía en la realización de esta Tesis. Es obvio que sin ellos nada de esto hubiera sido posible, por lo tanto cualquier agradecimiento se queda corto.

A Pedro Naranjo, que durante estos años no ha parado de transmitirme su pasión por la gravedad cuántica.

A Catherine Meusburger y Bernd Schroers, por su amabilidad y apoyo durante mis estancias en Erlangen y Edimburgo, respectivamente. Obviamente también mi agradecimiento por todo lo que me enseñaron durante dichas estancias, y en el resto de oportunidades que hemos tenido de colaborar.

Igualmente, mi más sincero agradecimiento a todas aquellas personas con las que he tenido el placer de discutir sobre los contenidos de esta Tesis, o sobre otras muchas ideas. Esta lista incluye a un gran número de gente, entre los que se encuentran Giulia Gubitosi, Flavio Mercati, todo el grupo de Física Matemática de Castilla y León, y todos los miembros de las redes 'Geometría, Mecánica y Control' (GMC) y 'Quantum structure of spacetime' (qSpace). Un agradecimiento especial a los coordinadores de estas dos redes, por todo su trabajo, sin las que esta Tesis solamente sería una pequeña parte de lo que es.

A todos los integrantes del Departamento de Física de la Universidad de Burgos, y en especial a Andrés, Chus y Nicolás, que me han apoyado durante estos cuatro años.

Quiero también agradecer de una forma especial a todos mis amigos, que han sido una fuente de apoyo continua, y sin los que todo esto no hubiera sido posible.

Finalmente, a mi familia, y especialmente a mis padres, a quienes nunca podré agradecer lo suficiente su apoyo incondicional a lo largo de toda mi vida.

Publications

The original research results contained in this Thesis have already been presented in the following publications:

1. Ballesteros A, Gubitosi G, Gutierrez-Sagredo I, Herranz FJ. Curved momentum spaces from quantum groups with cosmological constant. *Phys. Lett B.* **773** (2017) 47-53.

2. Ballesteros A, Gutierrez-Sagredo I, Herranz FJ, Meusburger C, Naranjo P. Quantum groups and noncommutative spacetimes with cosmological constant. *J. Phys. Conf. Ser.* **880** (2017) 012023.

3. Ballesteros A, Gubitosi G, Gutierrez-Sagredo I, Herranz FJ. Curved momentum spaces from quantum (anti-)de Sitter groups in (3+1) dimensions. *Phys. Rev D.* **97** (2018) 106024.

4. Ballesteros A, Gutierrez-Sagredo I, Herranz FJ. The Poincaré group as a Drinfel'd double. *Class. Quantum. Gravity.* **36** (2019) 025003.

5. Ballesteros A, Gutierrez-Sagredo I, Herranz FJ. Drinfel'd double structures for Poincaré and Euclidean groups. *J. Phys. Conf. Ser.* **1194** (2019) 012041.

6. Herranz FJ, Ballesteros A, Gutierrez-Sagredo I, Santander M. Cayley-Klein Poisson homogeneous spaces. *Geom. Integr. Quantization.* **20** (2019) 161-83.

7. Ballesteros A, Gutierrez-Sagredo I, Herranz FJ. Noncommutative spaces of worldlines. *Phys. Lett B.* **792** (2019) 175-81.

8. Ballesteros A, Gutierrez-Sagredo I, Herranz FJ. The κ-(A)dS noncommutative spacetime. *Phys. Lett B.* **796** (2019) 93-101.

9. Gutierrez-Sagredo I, Ballesteros A, Gubitosi G, Herranz FJ. Quantum groups, noncommutative Lorentzian spacetimes and curved momentum spaces, to appear in *Proceedings of the First Hermann Minkowski Meeting on the Foundations of Spacetime Physics*, C. Duston, M. Holman (Editors). arXiv: 1907.07979.

10. Ballesteros A, Gutierrez-Sagredo I, Mercati F. Coreductive Lie bialgebras and dual homogeneous spaces (2019). arXiv: 1909.01000.

Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'

—Michael Atiyah, *Mathematics in the 20th Century*

Contents

Chapter 1

Introduction

The most accurate description of the physical world provided by contemporary physics is based on two fundamentally different theories. On the one hand, general relativity describes the gravitational field and shows how it is intrinsically tied to the geometry of spacetime. On the other hand, quantum mechanics provides a description of the electromagnetic, weak and strong fields. These four fields, also known as fundamental interactions or fundamental forces, completely describe any physical phenomena ever observed. In particular, quantum mechanics and its modern developments in the form of quantum field theory, gauge theory and finally the Standard Model of particle physics [1, 2, 3, 4] have provided an extremely accurate and unified description of the electromagnetic, weak and strong interactions. The Standard Model of particle physics has predicted with a huge accuracy a number of particles before they had ever been observed. A recent and important example concerns the Higgs mechanism, predicted in [5, 6, 7], which led to the discovery by the LHC of the Higgs boson [8, 9]. Similarly, general relativity [10, 11] has also received accurate experimental confirmations. We just mention two of them, which have been recently performed: the first direct measurement of gravitational waves [12] and the first direct observation of a black hole [13]. The experimental data obtained confirm the predictions of general relativity with an extraordinarily high accuracy.

Although both general relativity and quantum mechanics are internally consistent physical theories with an almost incredible level of accuracy that are able to explain every experimental observation to date, fundamental problems arise when trying to unify these two theories. Of course, an important consideration is whether this unification is necessary. We have at least two important reasons to assert that this is indeed the case. On the one hand, from a purely theoretical point of view it would be highly desirable to have *one* self-consistent physical theory, valid to explain any physical observation. This would arguably provide a much more satisfying description of Nature than two different theories whose range of application is limited to certain regimes. On the other hand, there exist certain regimes whose physics is not well understood, and those regimes are precisely the ones that require of both general relativity and quantum theory to be described, because the physical phenomena taking place involve gravitational and quantum mechanical contributions of comparable strength. A paradigmatic example of such a regime is the early universe [14]. Even less extreme situations, like regions of spacetime very close to a very massive object, are good examples of physical situations in which gravitational and quantum effects are of the same order of magnitude.

The need for a quantum theory of gravity was already proposed in 1936 by M. Bronstein [15], who was the first to note that concentrating a sufficiently large amount of energy in a tiny region of spacetime would inevitably produce a gravitational collapse giving rise to a micro black hole, and thus imposing a fundamental limit to our capacity to measure the strength of the gravitational field. This is a remarkable difference of the gravitational field with respect to the other fundamental interactions, which is intrinsically linked to the relation between the gravitational field and the curvature of spacetime described by Einstein's field equations.

This unique feature of the gravitational field is at the heart of the problems that any attempt to quantize gravity has found until this date. In particular, a perturbative quantization of gravity has not been possible due to the fact that this theory is nonrenormalizable, because of the presence of unmanageable ultraviolet divergences.

Before proceeding further, it is useful to have an idea of the scale at which gravitational and quantum effects are of a comparable order of magnitude, and so under which conditions a theory of quantum gravity is needed. This scale receives the name of *Planck scale*, and it is the one that results from the combination of the three natural constants that appear in the relevant physical theories: the speed of light c (special relativity), Planck's constant $\hbar$ (quantum mechanics) and Newton's constant G (gravity). As pointed out by Max Planck [16], with these three fundamental constants of Nature, one can define natural units of time, length and mass, namely

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-43}\ s, \qquad l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\ m, \qquad m_P = \sqrt{\frac{\hbar c}{G}} \sim 1.2 \cdot 10^{19}\ \mathrm{GeV/c^2}. \tag{1.1}$$

These quantities are known as the *Planck time*, the *Planck length* and the *Planck mass*, respectively. A comparison of the Planck mass with the highest energy collision performed in LHC, which is 'only' $1.3 \cdot 10^4$ GeV/c^2, gives a clear idea of what is the order of magnitude of quantum gravity effects. However, as we have already commented, in the first moments of the universe, when time was of the same order of magnitude of the Planck time, these quantum gravity effects are thought to be important. In fact, any serious attempt to explain the first instants of our universe should include, at least, general relativity together with some quantum gravity corrections.

The key assumption that underlies this Thesis is that, at the Planck scale, at which a quantum theory of gravity would describe the universe, it would be natural to expect that spacetime could present some kind of 'discreteness' or 'fuzziness'. This is what is generally understood by *quantum spacetime*, term which is employed in a broad sense depending on the concrete approach being considered. This 'discreteness' could appear from two fundamentally different points of view:

- The first one is a fundamental theory of quantum gravity that has a discrete spacetime as one of its essential ingredients.

- The second point of view consists on thinking about this discreteness approach as a phenomenological one, amenable to describe some of the features of the quantum structure of spacetime without having at hand a fundamental theory of quantum gravity.

The first viewpoint is, up to date, not available, and it will be the second point of view the one we follow in this Thesis. In particular, we take as an initial assumption that the quantum nature of the gravitational field induces some kind of 'discreteness' or 'fuzziness', not necessarily fundamental, of the spacetime. This assumption rapidly raises, among many others, questions regarding the fate of the spacetime symmetries, specially Lorentz covariance. It seems very reasonable to think that any loss of smoothness for the spacetime should imply modified spacetime symmetries. Note that this is true irrespectively of the fact that the 'discreteness' is fundamental (our 'discrete' spacetime emerges from a full quantum gravity theory) or effective (we do not have a full quantum gravity theory, so we just take a 'discrete' model).

A natural, and indeed largely studied, framework that naturally encompass all the previous considerations is the one of noncommutative geometry [17, 18], and in particular, its incarnation in the theory of quantum groups and the noncommutative spacetimes covariant under them. In fact, the relation between quantum groups and quantum gravity has been suggested from different viewpoints since the introduction of the former by Drinfel'd [19] more than three decades ago. In particular, we will be aiming to describe the 'quantum' structure of the geometry of spacetime at the Planck scale through a noncommutative algebra of 'quantum spacetime coordinates' [20, 21, 22, 22, 23]. This framework has a large number of important advantages:

- It is independent of the underlying fundamental quantum gravity theory, or equivalently, it provides models that should be recovered (if they turn out to be correct) in an appropriate limit by any candidate of quantum gravity theory.

- Related to the previous point is the fact that these models could realistically provide some testable scenarios of quantum gravity effects, providing in this way experimental data/experimental limits, that any quantum gravity candidate should predict/respect.

- The theory of quantum groups provides a very natural setting for the inclusion in the theory of a new fundamental physical constant, most of the time thought to be (related to) the *Planck energy* E_P *or Planck length* l_P, which mathematically is just the deformation parameter of the quantum group. Recall that E_P in terms of (1.1) is given by $E_P = m_P c^2 = \hbar/t_P$. This implies that it is the parameter that governs the noncommutativity of the spacetime algebra, thus generating uncertainty relations between noncommuting coordinates that can be used in order to describe a 'fuzzy' or 'discrete' nature of the spacetime at very small distances or high energies [20, 24, 25, 26, 27, 22, 28, 29, 30].

- The construction of noncommutative spacetimes provides mathematically consistent models with quantum group covariance which provide much extra information than those models based on generic noncommutative algebras.

Another important feature of the quantum group approach to the construction of effective models of quantum gravity effects is more 'philosophical', and is related with the question of what are the most appropriate or natural mathematics to describe a quantum spacetime. The answer of the approach based on quantum groups is clear and refers to the noncommutative geometry appoach for the study of 'noncommutative manifolds'. The underlying idea is simple: when studying a topological manifold, Gelfand and Naimark proved that all the topological structure can be recovered from its (commutative) algebra of functions and that, in fact, there is a duality between topological spaces and commutative algebras, in the sense that every commutative algebra is the algebra of functions for some topological space. This theorem shows a clear path to generalize the notion of topological space (and in particular of smooth manifold) to some kind of 'noncommutative space', just by considering noncommutative algebras. Although, unfortunately the topological counterpart is lost in this approach, if we think on noncommutative algebras as 'deformations' of commutative ones, we could think that the underlying space would be a 'noncommutative deformation' of the initial topological space (although rigorously there is no such a thing).

Before proceeding with our discussion about noncommutative spacetimes, let us recall a well-known example of a noncommutative algebra. It is well-known that in the Hamiltonian description of physical systems, noncommutative algebras play a prominent role. For instance, nonrelativistic quantum mechanics is based on a 'noncommutative phase space' in which position and momenta operators generate the Lie algebra

$$[\hat{x}^a, \hat{p}_b] = i\hbar\, \delta^a_b, \qquad [\hat{x}^a, \hat{x}^b] = 0, \qquad [\hat{p}_a, \hat{p}_b] = 0, \qquad a, b \in \{1, \ldots, N\}, \tag{1.2}$$

which is the direct sum of N copies of the Heisenberg-Weyl algebra. Here, noncommutativity is controlled by the fundamental constant $\hbar$, since the $\hbar \to 0$ limit of (1.2) leads to an abelian algebra, and the classical limit of (1.2) is defined by the Poisson bracket

$$\{x^a, p_b\} \equiv \lim_{\hbar \to 0} \frac{[\hat{x}^a, \hat{p}_b]}{i\hbar} = \delta^a_b. \tag{1.3}$$

In this way we recover the symplectic structure of the Hamiltonian formulation of Classical Mechanics, which can properly be said to be a Poisson-noncommutative theory. This apparently innocent example indeed underlies the idea of introducing noncommutative algebras in order to describe quantum spacetime, since it presents the following important insight: the switch from classical to quantum mechanics is encoded by a fundamental constant of Nature, in this case $\hbar$ (with units of action), in such a way that the quantum behavior of a system becomes relevant when its action is of the order of $\hbar$. Although the problem in which we are interested in this Thesis is essentially different, this well-known example shows how the shift from commutative to noncommutative objects can be mediated by the introduction of a new physical constant that divides two fundamentally different (but related) descriptions of reality.

When applied to the concrete problem of describing quantum spacetime, the arguments above should be sufficient to convince the reader that noncommutative geometry is a sensible approach to study the effective quantum spacetime arising from a fundamental theory of quantum gravity. Moreover, the remarks about the nonexistence of an underlying topological noncommutative spacetime are not important since there is no single reason why our universe must be a topological space (in particular a smooth manifold). In this sense, we will call *noncommutative spacetime* to some noncommutative algebra of functions depending analytically on some parameter (the *quantum or deformation parameter*) in such a way that in a certain limit we recover the commutative algebra of functions of a classical spacetime. By the latter we simply mean any Lorentzian smooth manifold with a metric which is a solution of Einstein's field equations of general relativity. To work in this way is useful because it allows to think about noncommutative spacetimes as analytic deformations of classical spacetimes, and in particular it allows to consider, for instance, how noncommutative Minkowski spacetimes could be constructed.

All the discussion above regarding noncommutative geometry is also relevant for describing the symmetries of these noncommutative spacetimes. As we have already mentioned, we will work with those noncommutative spacetimes which are covariant under quantum group symmetries, where quantum groups are just noncommutative versions of Lie groups. In fact the quantum groups appearing in this work will be deformations (depending analytically of some quantum parameter) of the Lie groups of isometries of the maximally symmetric Lorentzian spacetimes of constant curvature. There are three of these classical spacetimes, the so-called Minkowski, de Sitter and anti-de Sitter spacetimes (hereafter (A)dS spacetimes). Their isometry groups are called Poincaré, de Sitter and anti-de Sitter groups (hereafter (A)dS groups). In this way, objects such as noncommutative Minkowski or (A)dS spacetimes, and quantum Poincaré or (A)dS groups will appear repeatedly throughout the Thesis, usually accompanied by the relevant quantum parameter.

We recall that the so-called 'quantum' deformations of kinematical Lie groups and algebras (see [19, 31, 32, 33, 33, 34] and references therein) and their semiclassical counterparts (Poisson-Lie groups [19, 35]) will be the main tool employed in this Thesis. As argued above, they present many features that make them suitable to be considered in a quantum gravity scenario. In concrete, some of the most relevant for this work will be:

- They are Hopf algebra deformations of kinematical Lie groups in which the quantum deformation parameter can be related to a Planck scale parameter [36, 37, 38].
- They give rise to noncommutative spacetimes which are covariant under quantum group (co)actions. In this context, several notions related with 'quantum kinematical geometry' can rigorously be generalized, like the ones of Poisson and quantum homogeneous spaces [39, 40, 41, 42, 43, 44, 45, 46].
- Quantum groups can be thought of as Hopf algebra quantizations of Poisson-Lie groups, and the relevance of the latter in (2+1) gravity has strictly been established (see [47, 48, 49, 50, 51, 52, 53]).
- Deformed Casimir operators of quantum kinematical algebras can be interpreted as modified dispersion relations of the same type that the ones appearing in several phenomenological approaches to quantum gravity [54, 38, 55].

As it happens with ordinary Lie groups, quantum group techniques are specially useful to construct noncommutative analogues of spacetimes that can be obtained either as group manifolds or as homogeneous spaces (as discussed previously, Minkowski and (A)dS spacetimes fall into this class). Moreover, each Lie group admits a number of different quantum group deformations, and the quantum spacetime arising from each of them can essentially be different. The classification and explicit construction of such a plurality of quantum geometries constitutes one of the main issues in the theory of quantum kinematical groups, which is far from being completed. In general, we will show that noncommutative spacetimes with a non-vanishing cosmological constant Λ can be viewed as 'geometric' nonlinear deformations (with parameter Λ) of the noncommutative Minkowski spacetimes with quantum deformation parameter q related to either l_P or E_p.

In the following pages, we present a brief introduction to the main problems and results presented within this Thesis, which we intend to be useful in order to comprehend which are the main results included in this work, and which concrete problems they try to solve, together with a brief physical motivation to these problems.

Mathematical foundations

The general idea presented until now has been that 'fuzziness' or 'minimal length (energy) scale' scenarios for quantum gravity seem to point out in the direction of some kind of noncommutative spacetime, and that this noncommutativity is characterized by a shift from geometry to algebra if we attend to the natural mathematical tools that we posses to study these objects.

This Thesis follows this line of thought, and thus in Chapter 2 we introduce the essential tools from differential geometry that allows us to study the geometry of classical spacetimes. In particular, we first introduce some general notions about Lie groups and homogeneous spaces, and then we particularize them to the case of the three Lorentzian spacetimes of constant curvature. These three spacetimes, whose noncommutative versions constitute the main objects of this Thesis, are described in full detail, including the introduction of the appropriately chosen local coordinates that will be used in the rest of this Thesis. Spacetimes with a non-vanishing constant curvature are specially interesting from the point of view of noncommutative spaces, and they have not been considered in detail in the literature. In this regard, we will present important generalizations of well-known results for the flat (Minkowskian) case that nevertheless were still unknown.

As we have also mentioned, the philosophy of quantum groups is somehow to maintain some of this geometrical intuition. In this sense, we have already mentioned that quantum groups can be thought of as Hopf algebra quantizations of Poisson-Lie groups. Although all these concepts will be made precise in Chapter 3, for now it is sufficient to mention that a Poisson-Lie group is just a Lie group together with a Poisson structure (like the one in the phase space of classical mechanics), in such a way that the group multiplication and this Poisson structure are compatible [19]. In the same way, the noncommutative spacetimes mentioned above (which mathematically are quantum homogeneous spaces) have a Poisson version known as Poisson homogeneous spaces [39]. In a complete analogy with homogeneous spaces endowed with an action of a Lie group, they are just smooth manifolds endowed with a Poisson structure compatible with the action of the Poisson-

Lie group, and it is because of this compatibility that they are called covariant Poisson spacetimes.

The relevance of Poisson-Lie groups and Poisson homogeneous spaces is thus twofold: on the one hand, they are simpler mathematical objects than quantum groups/quantum homogeneous spaces, in the sense that one can apply all the well-known tools from differential geometry to study them, and so keeping the geometrical insights provided by the classical objects. On the other hand, Poisson-Lie groups/Poisson homogeneous spaces can be quantized to obtain their noncommutative counterparts. In other words, we have that quantum homogeneous spaces are (comodule algebra) quantizations of Poisson homogeneous spaces, which are covariant under Poisson-Lie group actions [39], and the quantization of the latter provides the corresponding quantum group symmetry. This second consideration is essential for our purposes, because sometimes the quantization process starting from a Poisson homogeneous space and arriving to a quantum homogeneous space is much simpler than the construction of the quantum homogeneous space by starting from the very beginning with noncommutative objects. This is the case for example in the construction of the quantum κ-(A)dS spacetime performed on Chapter 4.

Moreover, the Poisson approach has another huge advantage: even when the quantization is difficult, Poisson-Lie groups are just the first order, in the quantization parameter, of quantum groups (which is equivalent to say that quantum groups are quantizations of Poisson-Lie groups), and usually they provide the most relevant information regarding the deformation. This is specially true when having in mind applications to quantum gravity, in which the physical quantities related to these deformations are very small so, many times, no higher order contributions in these parameters seem to be necessary from a phenomenological point of view. As a consequence, the study and explicit construction of Poisson homogeneous spacetimes have been shown to be fruitful in order to construct quantum homogeneous spacetimes and, in general, noncommutative spaces with quantum group invariance (see [56, 57, 44, 43, 58, 59, 60, 61] and references therein).

A remark is in order here: we have mentioned two different parameters, the quantum (or deformation) parameter and the quantization parameter. These are two different parameters and they should not be confused: the quantum (or deformation) parameter is already present at the Poisson-Lie level and it is the parameter that we will interpret as related to the Planck length or energy (an example is the parameter κ of the κ-Minkowski spacetime). The quantization parameter is the parameter that appears when quantizing a Poisson-Lie group to obtain a quantum group (in the phase space example $\hbar$ would be the quantization parameter). This distinction is more important for conceptual clarity than from a physical point of view, since we really do not have any experimental data that allow us to interpret any of the parameters as a well-defined physical constant. In fact, it could happen that the physical Planck length (energy) will be related to the combination of these two parameters.

Minkowski and (A)dS noncommutative spacetimes

It is worth recalling that most of the quantum spacetimes that have been introduced so far in the literature are noncommutative versions of the Minkowski spacetime [62, 63, 64, 65, 66]. As a consequence, the construction of noncommutative spacetimes with non-vanishing cosmological constant arose as a challenging problem in order to describe the interplay between the non-vanishing curvature of spacetime and quantum gravity effects, having in mind the possible cosmological consequences of Planck scale physics, which have been considered in [67, 68, 69, 70, 71, 72].

Among these noncommutative spacetimes with quantum group symmetry, and in particular with a quantum Poincaré symmetry group, probably the most studied example is the well-known κ-Minkowski noncommutative spacetime

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\,\hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0, \qquad\qquad a, b = 1, 2, 3, \tag{1.4}$$

where κ is a parameter proportional to the Planck mass (see [63, 64, 65, 62]). The noncommutative algebra (1.4) defines a noncommutative spacetime which is covariant under the κ-Poincaré quantum group [65], a 'quantum deformation' of the group of isometries of Minkowski spacetime which is the dual (as a Hopf algebra) of the κ-Poincaré quantum algebra, that was obtained for the first time in [62] (see also [73, 74, 75, 76, 77]) by making use of quantum group contraction techniques [78, 79, 80] applied onto real forms of the Drinfel'd-Jimbo quantum deformation for appropriate complex simple Lie algebras [19, 31].

Since then, the κ-Minkowski spacetime has provided a privileged benchmark for the implementation of a number of models aiming to describe different features of quantum geometry at the Planck scale and their connections with ongoing phenomenological proposals. Without pretending to be exhaustive, κ-Minkowski spacetime has been studied in relation with wave propagation on noncommutative spacetimes [81], dispersion relations [82, 83, 84, 85], relative locality phenomena [86], curved momentum spaces and phase spaces [87, 88], noncommutative differential calculi [89, 90], star products [91], noncommutative field theory [92, 93, 94], representation theory [95, 96] and light cones [97]. Specially relevant for this Thesis, at least from a conceptual point of view, are deformed special relativity theories (formerly introduced in [98, 99, 100] and further developed in [101, 102, 103, 104, 36, 105, 106]) in which Planck mass m_P, or equivalently Planck length l_P (1.1), is introduced as a second relativistic invariant (besides the speed of light) which modifies the classical relativistic symmetries of the system, in order to make them compatible with the existence of a new fundamental scale.

In the light of the interest devoted to the κ-Poincaré quantum algebra, it is somehow surprising that analogous deformations for (A)dS groups had not been presented so far. In particular, from a physical perspective they are specially relevant when cosmological distances are involved, because then the interplay between gravity and quantum spacetime should take into consideration the spacetime curvature [69, 107, 108, 70]. Therefore a natural (maximally symmetric) noncommutative spacetime to be considered in this context should be the quantum analogue of the (A)dS spacetime.

This will be in fact the first important result of this Thesis, presented in the first half of Chapter 4: we have proven that there is essentially only one possible generalization (under reasonable physical assumptions) of the κ-Minkowski spacetime to the case of a non-vanishing cosmological constant, and we have explicitly constructed it. It should be noticed that while the problem of constructing noncommutative (A)dS spacetimes, which are covariant under the appropriate quantum kinematical group, has recently attracted some attention (see [109, 110, 111, 112, 113, 114, 60]), probably the most important case given by the κ-(A)dS spacetime was still lacking.

The name of this noncommutative example is due to the fact that in the limit of vanishing cosmological constant $\Lambda \to 0$ one recovers the κ-Poincaré algebra, while in the limit $\kappa \to \infty$ one recovers the algebra of symmetries of the (A)dS spacetime. In this way, the structure of the first part of Chapter 4 is the following: first, we recall both the well-known κ-Poincaré quantum algebra, and then we construct the κ-Minkowski noncommutative spacetime (1.4), starting by its construction as a Poisson homogeneous space and followed by its quantization (which in this case is straightforward). Afterwards, we prove that the generalization of this deformation to the (A)dS spacetime is unique if we impose that the time translation generator is primitive, and we recall the associated quantum algebra, which had been presented in [114]. Afterwards, we explicitly construct the noncommutative κ-(A)dS spacetime, by following the same approach of quantizing the semiclassical limit provided by the Poisson homogeneous space structure. However, due to the presence of a non-vanishing cosmological constant, the κ-(A)dS spacetime is highly nonlinear and therefore, in order to obtain a quantization in term of simple expressions, we firstly quantize its first order. Afterwards we introduce ambient space coordinates, in which the Poisson structure is quadratic, and in terms of which the quantization takes a simple form. All these results were recently presented in [115].

Noncommutative spaces of worldlines

So far we have introduced the idea of noncommutative spacetimes that are covariant under the appropriate quantum group symmetries. However, there exists a number of approaches to quantum gravity in which the fundamental object is not spacetime itself but momentum or phase spaces. An example is the relative locality approach [116, 117], whose basic assumption is that the only physical quantities that can truly be measured are momenta and energy of particles. In this context, it is indeed a natural question to ask whether quantum groups, understood as deformed symmetries of noncommutative spacetimes, have any role to play in the description of these momentum or phase spaces. In fact they do, and the second part of Chapter 4 is devoted to this subject.

For the simplest case of free massive particles moving on some spacetime, the phase space can be identified with the space of oriented time-like geodesics (worldlines of free massive particles). If this spacetime is taken as the Minkowski one, time-like geodesics are simply straight lines inside the lightcone. However for a general spacetime, i.e. a smooth manifold endowed with a Lorentzian metric, the situation is much more complicated, as will be detailed in Chapter 4. For the present work our interest is focused in the maximally symmetric spacetimes of constant curvature, i.e. Minkowski and (A)dS spacetimes, for which the space of time-like geodesics is indeed a homogeneous space.

As we will show in the second part of Chapter 4, the construction of a noncommutative space of worldlines that is covariant under the appropriate quantum group of isometries follow the same lines as in the case of spacetime. In this Chapter we firstly introduce a general procedure to construct noncommutative spaces of worldlines associated to any quantum deformation (with the only obvious requirement that the quantum homogeneous space of worldlines to be well-defined). We remark here that an important step in this construction, which greatly simplifies our future work, is to choose local coordinates on this space in such a way that the canonical Poisson-Lie structure on the group can be obtained by canonical projection, so we firstly introduce these local coordinates.

In order to illustrate this construction, we consider the particular κ-Poincaré deformation, which as we will see, has some remarkable properties. In particular, the Poisson homogeneous space obtained is of Poisson subgroup type, and thus it is arguably simpler than the corresponding spacetime, which is coisotropic. As it is the common procedure during this Thesis, we firstly look at the Poisson counterparts of any construction in order to get a semiclassical geometrical description, and only later we try to quantize it to obtain the full noncommutative structures. In the case of noncommutative phase spaces, this Poisson approach is even more natural since phase spaces are naturally symplectic manifolds. The consequences of this Poisson structure induced by a quantum deformation of the isometry groups for the dynamics of free massive particles on Lorentzian homogeneous spaces certainly deserves further attention, but it is remarkable that in the κ-Poincaré case this Poisson structure is (almost) a symplectic structure and so its quantization is trivial to perform. All this procedure is presented in Chapter 4 and is based on the results presented in [61].

Curved momentum spaces

While we have previously considered the role of quantum deformations in phase space, in Chapter 5 we study the other aforementioned space: momentum space, and in particular the nontrivial structure induced on it by the quantum κ-Poincaré and κ-(A)dS groups. In order to understand the ideas behind this construction, we should remark that recent developments in quantum gravity research have given new substance to the long-forgotten idea that momentum space should have a nontrivial geometry, an intuition originally due to Max Born [118].

However, such an idea was not seriously considered in the bibliography until the introduction, more than a decade ago, of deformed special relativity (DSR) in [98, 100]. Nevertheless, it is now understood that a nontrivial geometry of momentum space is a general feature of DSR theories [104, 103, 119, 117, 116, 120]. This is intimately related with the presence of the Planck energy as a second relativistic invariant (besides the speed of light), that can play the role of a curvature scale of the momentum manifold [121, 122]. Nontrivial properties of momentum space emerge also in (2+1)-dimensional quantum gravity, where explicit computations show that the effective description of quantum gravity coupled to point particles is given by a theory with curved momentum space and noncommutative spacetime coordinates [123, 37, 124, 125]. Of more direct interest for the results present in this Thesis are those models of noncommutative geometry, where the space of momenta (that are dual to the noncommutative spacetime coordinates) is curved [126, 81, 127, 87].

The nontrivial geometry of momentum space is also interesting from a phenomenological point of view, since for instance, it is related to an energy-dependent correction to the time of flight of free particles [128] (known as dual redshift) or to dual-gravity lensing [129]. These Planck-scale corrections can be tested by astrophysical observations [130]. Nevertheless, most of these studies are not able to describe situations in which both spacetime and momentum space curvature are present, although the most promising observations involve propagation of particles over cosmological distances, for which spacetime curvature cannot be neglected [54, 131, 132, 70, 133]. In the past few years several proposals, aimed at extending relativistic models with curved momentum space, were put forward in order to include non-vanishing spacetime curvature [107, 134, 135, 136, 137, 138]. The general understanding coming from these approaches is that when both momentum space and spacetime have non-vanishing curvature they become so intertwined that it is not possible to give a neat geometrical description of the properties of momentum space on its own.

In this Thesis we show that this is not necessarily the case. Indeed, in Chapter 5 we are able to explicitly construct the curved momentum space generated by quantum-deformed spacetime symmetries in presence of a non-vanishing cosmological constant. We achieve this result by enlarging the momentum space so that the latter is not only the manifold of momenta associated to translations on spacetime, but it also includes the 'hyperbolic' momenta associated to the boost transformations and the angular momenta associated to rotations. Within this construction we can also show that in the vanishing cosmological constant limit the Lorentz sector is not involved in the structure of the momentum space because it decouples from the energy-momentum sector, thus recovering previous results in the literature.

In particular, we again make use of Hopf algebras, which have proved to be a very useful mathematical framework to model certain DSR symmetries. As mentioned previously, the most studied example is the κ-Poincaré Hopf algebra [64, 76, 139], the investigation of which provided inspiration and more precise understanding of several features of DSR models. For example, it can be explicitly shown that the manifold of momenta associated to the κ-Poincaré translation generators is a ('portion' of a) dS manifold, whose curvature is determined by the quantum deformation scale κ [87] and whose metric determines the free particle dispersion relation that is indeed compatible with the κ-Poincaré symmetries, thus showing that the phenomenology associated to the κ-Poincaré algebra fits very naturally within the framework of relative locality [140, 87, 141].

In Chapter 5 we present a generalization of all these results to the case of a non-vanishing cosmological constant by working with the κ-deformation of the (A)dS algebra, previously constructed in Chapter 4 (see also [74, 62, 142, 143, 144, 68, 114]). The approach here presented starts by considering the lower (1+1) and (2+1)-dimensional cases [145], in which the situation is simpler, and then to work out the physically realistic case of (3+1) dimensions [146]. In the low dimensional cases, the situation is qualitatively similar to the flat case: the curved momentum space associated to this quantum deformation has the geometry of a ('portion' of a) dS manifold, although now enlarged with 'hyperbolic' and angular momenta associated to boosts and rotations, respectively. However, in the (3+1)-dimensional case, the qualitative picture changes, showing that the geometry of the momentum space is different dependending on the sign of the cosmological constant. The method employed [147, 59] to obtain these results makes use the Poisson version of the 'quantum duality principle' in order to construct the dual Poisson-Lie group associated to the κ-(A)dS deformation. This procedure, which is a generalization of the one presented in [103], has the advantage that we can obtain deformed dispersion relations as Casimir functions for the Poisson-Lie structure on the dual group. All these results have given rise to the publications [145, 146, 148].

(2+1) noncommutative spacetimes from Drinfel'd doubles

All the constructions commented above have something in common: they are applications of quantum groups (Hopf algebras) to quantum gravity problems from a phenomenological or effective point of view. However, quantum groups have also arised in different approaches to quantum gravity, such as the combinatorial quantization of gravity in (2+1) dimensions or path integral approaches such as state sum models or spin foams (see, for instance [149, 150, 151, 152, 153, 154, 155, 156], and references therein).

While, as we have commented, in any dimension quantum groups are natural candidates to describe the symmetries of a quantum theory of gravity, in the case of (2+1) dimensions their role is much better understood. Moreover, quantum gravity in (2+1) dimensions is often consider as a suitable toy model which incorporates some conceptual key points that a full quantum gravity theory is supposed to address (see [157] for a nice introduction to the subject). As it is well-known, gravity in (2+1) dimensions is quite different from the full (3+1)-dimensional theory [158, 157]. The source of this difference can be traced back to the fact that in three dimensions the Ricci tensor completely determines the Riemann tensor. Therefore, every solution of the vacuum Einstein field equations is locally isometric to one of the three maximally symmetric spacetimes of constant curvature, Minkowski or (A)dS, only depending on the value of the cosmological constant [157]. As a consequence, gravity in (2+1) dimensions is a topological theory in which gravitational waves do not exist. In fact, (2+1)-dimensional gravity admits a description as a Chern-Simons theory with gauge group given by the group of isometries of the corresponding spacetime model [49, 50]. In this context the phase space structure of (2+1)-gravity is related with the moduli space of flat connections on a Riemann surface whose symmetries are given by certain Poisson-Lie (PL) groups [47, 48], which as already stated are the semiclassical counterpart of quantum groups. The relevant Poisson structure on this latter space admits a natural description in terms of coboundary Lie bialgebras associated with the gauge group. It is the presence of these PL groups playing the role of classical symmetries what makes clearer how quantum groups should enter in the game.

Given the above considerations, while the generic role of PL and quantum groups in (2+1)-gravity is clear, the question of which quantum deformations are the relevant ones from the physical viewpoint is a matter of intense investigation [68, 52, 159, 160, 161, 162]. In this context, both Lorentzian and Euclidean groups have been considered [154, 52, 159, 21, 163, 164, 165, 51]. Moreover, there is evidence that relevant quantum deformations are the ones coming from a classical r-matrix arising from a Drinfel'd double (DD) structure, since this ensures that the Fock-Rosly condition [48] is fulfilled, thus allowing a consistent definition of the Poisson structure on the moduli space of flat connections (see [110] and references therein). These works made evident that a systematic study of all the possible

DD structures for the isometry groups of spacetime models for (2+1)-gravity was needed. However, while the DD structures for the (A)dS groups have been fully described [110, 109, 112, 111], that is not the case for DD structures for the Poincaré group, thus preventing the complete understanding of Lorentzian DD structures and their relationships under Lie bialgebra contraction procedures [80], for which the inclusion of the cosmological constant Λ as an explicit parameter has been proven to be very useful. Given that models of Euclidean gravity in three dimensions have also been considered [159], it certainly would be interesting to have an explicit description not only of the DD structures for the Poincaré group, but also for the Euclidean group, and in this way be able to compare both theories and their possible consequences.

In Chapter 6 some of these problems will be faced, and in particular the quantum deformations arising from DD structures will be put into correspondence with the full classification given in [166] for both Lorentzian and Euclidean signatures (note that analogous classifications for the (A)dS cases can be found in [167, 168]), and the analysis of their contraction from the DD structures for the (A)dS Lie algebras, which were provided in [110], will also be given. Moreover, each DD structure provides a canonical quantum deformation of the Poincaré or Euclidean group, completely characterized by its canonical r-matrix. We therefore explicitly construct and analyze each of the associated Poisson homogeneous spaces associated to these DD structures, and find that they indeed have quite different properties. A similar analysis is performed for the (1+1)-non-trivially centrally extended Poincaré group, obtaining that it has two possible DD structures. We also argue that this plurality of DD structures of the Poincaré group is exceptional among the kinematical groups, for which we present a discussion of the possibility of existence of DD structures for kinematical groups in (2+1) and other dimensions. In the publications [169, 170], all these new results can be found.

Dual Poisson homogeneous spaces

A fundamental difference between a commutative and a noncommutative spacetime that we have not sufficiently emphasized until this moment is the existence of uncertainty relations in the latter. This is a well-known fact, and indeed a good example is the quantum mechanical phase space (1.2). However, in (1.2) the noncommutativity only arises between momenta and position operators while, in the case of a noncommutative spacetime there will be an extra noncommutativity among the position operators themselves. For a noncommutative spacetime that is covariant under some quantum group, these uncertainty relations are tied to this symmetry. Due to the physical relevance of these uncertainty relations, the problem of how quantum group symmetries are related to the uncertainty relations of the corresponding covariant noncommutative spacetime certainly deserves some attention.

With this in mind, in Chapter 7 we introduce the concepts of coreductive and cosymmetric deformations, which have the property that the geometry of their dual spaces, which determines the nature of these uncertainty relations, have a specially simple form. More in detail, the Lie bialgebra associated to a coreductive deformation defines a dual space which is a reductive homogeneous space, while a cosymmetric deformation generates a dual symmetric space. In order to illustrate these concepts with some explicit examples, we consider the κ-Poincaré and κ-(A)dS deformations analyzed in Chapters 4 and 5 and we study the geometry of their associated dual spaces. In general, these dual spaces are completely different to the associated spacetimes. In the particular examples considered, they do not even admit invariant metrics, so with the aim of studying their geometry we make use of the framework of K-structures, which was introduced in Chapter 2, and their associated connections. It is found that while in (2+1) dimensions the κ-Lie bialgebra is coreductive for the Poincaré and (A)dS algebras, in (3+1) dimensions coreductivity is only admissible for the κ-Poincaré Lie bialgebra, since the introduction of this condition turns out to be incompatible with the existence of a non-vanishing cosmological constant parameter Λ. Regarding the dual homogeneous spaces, we obtain that while in (2+1) dimensions their associated Poisson structures turn out to be Λ-deformations of the Lorentz Lie algebra $\mathfrak{so}(2,1)$, in (3+1) dimensions the dual of the κ-Minkowski space has a Poisson structure isomorphic to the non-deformed Lorentz Lie algebra $\mathfrak{so}(3,1)$.

This Chapter 7, of a more mathematical nature, shows that the study of dual PL groups and other associated geometric structures provides a completion of the landscape of new symmetries associated to quantum groups. These results are included in the paper [171].

Chapter 2

Lie groups and homogeneous spaces

The aim of this Chapter is to provide the most relevant differential geometric notions that will be needed during the rest of this Thesis. In particular, it will deal with the basics of Lie groups, Lie algebras and their actions on smooth manifolds. This latter concept will be specially important for our work, so we introduce the notion of a G-space, i.e. a manifold endowed with an action of a Lie group G.

Certain type of actions of a Lie group on a manifold, namely transitive actions, define the important notion of homogeneous space, which will be a key concept for this Thesis. We also state some well-known results that allow us to construct models of homogeneous spaces from coset spaces of Lie groups by closed subgroups, where these subgroups are the ones fixing a chosen point from the homogeneous space. Among homogeneous spaces, the so-called symmetric spaces will be treated in more detail, since most of the geometric objects appearing in this Thesis belong to this subset.

In §2.1 we introduce the concepts of Lie group and Lie algebra, show the relation among them and define several important related notions. Afterwards we define the notion of homogeneous space for a Lie group and study its geometry. We finish this Section by considering the special cases of reductive and symmetric homogeneous spaces, and by presenting their geometrical properties in more detail.

In §2.2 we study the Lie algebras of the groups of isometries for the three maximally symmetric Lorentzian spaces of constant curvature, namely Minkowski and (anti-)de Sitter, which we refer as Lorentzian spacetimes of constant curvature (or simply Lorentzian spacetimes for brevity). Here we introduce the kinematical basis we employ in the rest of this Thesis. We give explicit expressions for the (3+1), (2+1) and (1+1)-dimensional situations, since they will be heavily used during this work.

In §2.3 we study the Minkowski and (anti-)de Sitter spacetimes as coset spaces of their Lie groups of isometries (motion groups), namely the Poincaré $ISO(3,1)$, anti-de Sitter $SO(3,2)$ and de Sitter $SO(4,1)$ groups. Our description will be based on the construction of Lorentzian spacetimes as cosets of the motion group by the so-called Lorentz group. We will provide appropriate local coordinates on them, as well as ambient space coordinates. In particular, these local coordinates are defined in such a way that our future constructions follow in the simplest possible way. Similarly to the previous Section we give explicit descriptions for the different dimensional cases (1+1), (2+1) and (3+1). We finish this Section by describing the metric properties of these spaces, and by providing the expressions for left- and right-invariant vector fields in the chosen local coordinates.

2.1 Lie groups, Lie algebras and homogeneous spaces

During this Section we follow [172, 173, 174, 175, 176, 177, 178, 179]. Let us start by defining some standard notation. During the rest of this work, we define an n-dimensional smooth real manifold M (which we shall simply call a manifold or smooth manifold) as a paracompact Hausdorff topological space M equipped with an atlas $\{(U_\mu, \varphi_\mu) | M = \cup_\mu U_\mu,\ \varphi_\mu : U_\mu \to \mathbb{R}^n\}$ such that every transition function

$$\varphi_{\mu\nu} = \varphi_\nu \circ \varphi_\mu^{-1}\Big|_{\varphi_\mu(U_\mu \cap U_\nu)} : \varphi_\mu(U_\mu \cap U_\nu) \to \varphi_\nu(U_\mu \cap U_\nu) \tag{2.1}$$

is smooth, i.e. $\varphi_{\mu\nu} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$. In fact, most of the time, manifolds appearing in this work will be real analytic. Because M is paracompact and Hausdorff, then it admits partitions of unity, and hence it admits both Riemannian metrics and connections. For any manifold M we denote by TM and T^*M to its tangent and cotangent bundles. Local sections of TM are called vector fields and denoted by $\Gamma(TM) = \mathfrak{X}(M)$, while local sections of T^*M are called one-forms and denoted by $\Gamma(T^*M) = \Omega^1(M)$. In general, sections of any bundle B over M will be denoted by $\Gamma(B)$. For a vector field evaluated in a point $m \in M$ we write $X_m \in T_m M$.

Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map between them. Then we denote by $f_* : TM \to TN$ to the differential (pushforward) of f, defined as a bundle map such that for each $m \in M$ defines the linear map $f_{*,m} : T_m M \to T_{f(m)} N$ given by

$$f_{*,m}(X_m)(F) = X_m(F \circ f) \tag{2.2}$$

for all $X_m \in T_m M$ and $F \in \mathcal{C}^\infty(\mathbb{R})$. If $(x^1, \ldots, x^n)$ is a set of local coordinates on an open subset $U \subset M$, then we write $\{\partial_a\big|_m = \frac{\partial}{\partial x^a}\Big|_m | a \in \{1, \ldots, n\}\}$ to the associated basis of $T_m M$. Then, the set of vector fields $\{\frac{\partial}{\partial x^a} |\ a \in \{1, \ldots, n\}\}$ give a local parallelization of $U \subset M$.

2.1.1 Lie groups

Consider a set G and let us endow it with a group structure. We define the *multiplication map* by

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\to gh \end{aligned} \tag{2.3}$$

and the *inverse map* by

$$\begin{aligned} \iota : G &\to G \\ g &\to g^{-1} \end{aligned} \tag{2.4}$$

for all $g, h \in G$. The set G endowed with this group structure will also be called G, since no confusion should arise. The identity element of a group G will be hereafter denoted by e, unless otherwise stated.

Definition 2.1. A *Lie group* over the field k is a group G equipped with the structure of a smooth manifold in such a way that the group multiplication map is a smooth map.

Note that this definition implies that the inverse map is also smooth, which is deduced by applying the inverse map theorem to the map $G \times G \to G \times G, (g, h) \to (g, gh)$ whose differential clearly induces an isomorphism between the tangent spaces. In this work the base field k will be either $\mathbb{R}$ or $\mathbb{C}$, and the groups are understood to be finite-dimensional unless otherwise stated. Let us fix $g \in G$ and define three operations on G that will play a significant role, namely *left multiplication, right multiplication and conjugation*, namely

$$\begin{aligned} L_g : G &\to G \\ h &\to g\, h, \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_g : G &\to G \\ h &\to h\, g \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} C_g : G &\to G \\ h &\to g\, h\, g^{-1}, \end{aligned} \tag{2.7}$$

respectively. In fact, it is easy to check that these maps are diffeomorphisms of G. Note that $L_g = \mu(g, \cdot)$, $R_g = \mu(\cdot, g)$ and $C_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$, since L_g and R_g commute.

Definition 2.2. A subgroup H of the Lie group G is called a *Lie subgroup* if it is a submanifold of G.

Hereafter we write $H < G$ if we want to emphasize that H is a Lie subgroup of G, while by $H \subset G$ we only mean that H is contained in G as a subset.

Theorem 2.1. *[179] Every closed subgroup H of G is a Lie subgroup.*

Definition 2.3. Let G_1 and G_2 be Lie groups. A map $\phi : G_1 \to G_2$ is called a *Lie group homomorphism* if it is both a group homomorphism and a smooth map. If in addition ϕ is a group isomorphism and a diffeomorphism, then it is called an *isomorphism of Lie groups.*

Definition 2.4. A *one parameter subgroup* of a Lie group G is a smooth curve

$$\begin{aligned} g : \mathbb{R} &\to G \\ t &\to g(t) = g_t \end{aligned} \tag{2.8}$$

satisfying

$$\begin{aligned} g(t_1)g(t_2) &= g(t_1+t_2) \\ g(0) &= e. \end{aligned} \tag{2.9}$$

Clearly, for each one parameter subgroup, the differential of g at $0 \in \mathbb{R}$ defines a unique $X = g_{*,0} \in T_eG$.

Let M be a smooth manifold and G a Lie group. We can define different actions of G as a group on M as a set. However the extra structure of both M and G motivates the following more restrictive definition.

Definition 2.5. An *action of a Lie group* G on a smooth manifold M is a homomorphism $\alpha : G \to \mathrm{Diff}(M)$ such that the map $\alpha(g) : M \to M, m \to \alpha(g)\, m$ is smooth for all $g \in G$. We say that an action α is *transitive* if for all $m, n \in M$, there exists at least one element $g \in G$ such that $\alpha(g)m = n$. The action α is called *faithful* if for every $g \in G$, $\alpha(g)m = m$ for all $m \in M$ implies $g = e$. We call a manifold endowed with a Lie group action a *G-space*.

With the notation of the previous definition, it is clear that an action induces a smooth map $G \times M \to M$ which, by a slight abuse of notation, we also denote α and we call the action of G on M, since no confusion should arise. This map is given by

$$\begin{aligned} \alpha : G \times M &\to M \\ (g, m) &\to \alpha(g)(m). \end{aligned} \tag{2.10}$$

This smooth map $\alpha : G \times M \to M$ induces the following two smooth maps

$$\begin{aligned} \alpha_g : M &\to M \\ m &\to \alpha_g(m) = \alpha(g)(m) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \alpha_m : G &\to M \\ g &\to \alpha_m(g) = \alpha(g)(m). \end{aligned} \tag{2.12}$$

for all $g \in G$ and $m \in M$. The first one is just the smooth map $\alpha(g)$ of Definition 2.5 that defines the action and assigns, for each fixed $g \in G$ the point $m' \in M$ that results of applying such a transformation. The second one assigns, for any fixed $m \in M$, to every $g \in G$ the point $m' \in M$ resulting from applying the transformation corresponding to $g \in G$. In general, it should be clear to which of the previous maps we are referring in each moment. From the condition of $\alpha : G \to \mathrm{Diff}\,(M)$ being a group homomorphism (of course the group operation on $\mathrm{Diff}\,(M)$ is the composition of diffeomorphisms), we have that $\alpha(g\,h) = \alpha(g)\,\alpha(h)$ for all $g, h \in G$, and therefore it is clear that $\alpha_{(g\,h)}\,(m) = \alpha_g\,(\alpha_h\,(m))$.

Associated to any Lie group action there are a number of important notions, which we now summarize. For each $m \in M$ we call the set

$$G_m = \{g \in G \,|\, \alpha_g\,(m) = m\} \tag{2.13}$$

the stabilizer of m, which is the set of elements of G that keeps the point m invariant. The following Theorem states that this is indeed always a subgroup of G. The set of points in M that can be reached from a fixed point $m \in M$ by the action of G is called *the orbit of* m, so

$$O_m = \{\alpha_g\,(m)\,|\,g \in G\}. \tag{2.14}$$

When studying G actions on M the subset of functions belonging to $\mathcal{C}^\infty(M)$ which is invariant by G plays an important role and we denote it by

$$\mathcal{C}^\infty(M)^G = \{f \in \mathcal{C}^\infty(M) \mid f(\alpha_g\,(m)) = f(m) \ \ \forall m \in M, g \in G\}. \tag{2.15}$$

Theorem 2.2. *[177] Let α be an action of a Lie group G on a smooth manifold M. For any point $m \in M$ the map*

$$\begin{aligned} \alpha_m : G &\to M \\ g &\to \alpha(g)(m) \end{aligned} \tag{2.16}$$

has constant rank k and

i) the stabilizer G_m is a Lie subgroup of codimension k in G and $T_e(G_m) = \mathrm{Ker}\,(\alpha_m)_*$;

ii) for some neighborhood U of $e \in G$ the set $\alpha(U)m = \{\alpha_g(m)\,|\,g \in U\}$ is a submanifold of dimension k in M, and $T_m(\alpha(U)m) = (\alpha_m)_*(T_e(G))$;

iii) if the orbit $\alpha(U)$ is a submanifold in M, then $\dim \alpha(G)m = k$.

For a transitive action α the stabilizers G_m, G_n of any two points $m, n \in M$ are isomorphic. Moreover, they are conjugate subgroups, so there exists some $g \in G$ such that $G_n = g\,G_m\,g^{-1}$ for any $m, n \in M$. This fact is very useful, because for transitive actions, it is sufficient to compute the stabilizer G_m for a given $m \in M$. We will use this property during the rest of this work in order to construct coset spaces. The fact that the stabilizer G_m of $m \in M$ is a Lie subgroup explains the alternative notation *isotropy subgroup*.

Three actions of G on itself are associated to (2.5), (2.6) and (2.7) in a canonical way, namely

$$\begin{aligned} L : G \times G &\to G \\ (g, h) &\to L_g(h) = g\,h, \end{aligned} \tag{2.17}$$

$$\begin{aligned} R : G \times G &\to G \\ (g, h) &\to R_g(h) = h\,g, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} C : G \times G &\to G \\ (g, h) &\to C_g(h) = g\,h\,g^{-1}, \end{aligned} \tag{2.19}$$

and we denote them as *left, right and adjoint actions of* G, respectively. These three actions define maps $G \to G$, which we call L_g, R_g, C_g respectively, by fixing their first argument.

2.1.2 Left- and right-invariant vector fields

Let M be a smooth manifold and $\mathfrak{X}(M)$ the set of vector fields on M. This set forms an infinite dimensional Lie algebra when endowed with the Lie bracket for vector fields, defined by its action on functions

$$[X,Y]f = X(Yf) - Y(Xf), \tag{2.20}$$

for all $X, Y \in \mathfrak{X}(M)$ and all $f \in \mathcal{C}^\infty(M)$. In the particular case where $M = G$ is a Lie group two subsets of $\mathfrak{X}(M)$ are particularly important: the left- and right invariant vector fields, which are defined by

$$\begin{aligned}\mathfrak{X}^L(G) &= \left\{ X^L \in \mathfrak{X}(M) | (L_h)_* X^L \Big|_g = X^L \Big|_{hg} \right\} = \\ &= \left\{ X^L \in \mathfrak{X}(M) | (X^L f) \circ L_h = X^L (f \circ L_h) \right\},\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathfrak{X}^R(G) &= \left\{ X^R \in \mathfrak{X}(M) | (R_h)_* X^R \Big|_g = X^R \Big|_{gh} \right\} = \\ &= \left\{ X^R \in \mathfrak{X}(M) | (X^R f) \circ R_h = X^R (f \circ R_h) \right\},\end{aligned} \tag{2.22}$$

respectively. By definition, each left- (right-) invariant vector field is completely defined by its value at the identity of G, since for example $(X^L f)(g) = (X^L(f \circ L_g))(e)$, and so they admit the following explicit description. For any given vector $T_a \in T_e G$ we have

$$(X^L_a f)(g) = \frac{d}{dt}\Big|_{t=0} f\left(g\, e^{tT_a}\right), \qquad (X^R_a f)(g) = \frac{d}{dt}\Big|_{t=0} f\left(e^{tT_a} g\right), \tag{2.23}$$

for all $f \in \mathcal{C}^\infty(G)$ and all $g \in G$. The vector space of left-invariant vector fields equipped with a Lie bracket is called the Lie algebra $\mathfrak{g}$ of G. The previous argument shows that there exists an isomorphism from the vector space $T_e G$ to the vector space of left-invariant vector fields on G, so due to the invariance of the Lie bracket under diffeomorphisms, we can identify $\mathfrak{g} \simeq T_e G$. From now on we will use this identification. This construction inspires the abstract definition of a Lie algebra.

2.1.3 Lie algebras

Let V be a vector space over the field k. A *Lie bracket on* V is a bilinear map $[\,,\,] : V \otimes V \to V$ satisfying the following two conditions

(L1) $[X,Y] = -[Y,X]$;

(L2) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$;

for all $X, Y, Z \in V$. Condition (L1) is called the antisymmetry condition, while (L2) is the Jacobi identity. Note that the antisymmetry condition (L1) is equivalent, in presence of the Jacobi identity (L2), to the condition $[X,X] = 0$ for all $X \in V$.

Definition 2.6. A Lie algebra $\mathfrak{g}$ is a pair $(V, [\,,\,]_{\mathfrak{g}})$ where V is a vector space and $[\,,\,]_{\mathfrak{g}}$ is a Lie bracket.

If there is no uncertainty about the Lie algebra under consideration, we can omit the subscript in the bracket and write just $[\,,\,]$ instead of $[\,,\,]_{\mathfrak{g}}$. We call the *dual of a Lie algebra* $\mathfrak{g}$, and denote it by $\mathfrak{g}^*$, the dual vector space V^*. It should be noticed that in principle the Lie algebra structure does not induce any extra structure on $\mathfrak{g}^*$. Let $\mathfrak{g}$ be an n-dimensional Lie algebra and choose an algebraic basis of $\mathfrak{g}$, say $\{X_1, \ldots, X_n\}$. Then we define an algebraic basis on $\mathfrak{g}^*$, say $\{x_1, \ldots, x_n\}$, by means of the canonical pairing $\langle X_i, x_j\rangle = \delta_{ij}$ between vector spaces. In this work we are mainly interested with finite dimensional Lie algebras, so unless otherwise stated this will be the case for all the Lie algebras under consideration.

Let us consider an n-dimensional Lie algebra $\mathfrak{g}$ and take a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ in which the *structure constants* are c_{ij}^k, i.e. $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$. Then the antisymmetry condition (L1) is equivalent to $c_{ij}^k = -c_{ji}^k$ for all $i, j, k \in \{1, \ldots, n\}$. The Jacobi identity (L2) is equivalent to

$$\sum_{l=1}^{n} \left(c_{jk}^l c_{il}^m + c_{ij}^l c_{kl}^m + c_{ki}^l c_{jl}^m\right) = 0, \tag{2.24}$$

for all $i, j, k \in \{1, \ldots, n\}$. In fact, these equations can be just expressed as

$$\sum_{l} \sum_{(i,j,k)\in\Sigma_3} \left(c_{jk}^l c_{il}^m\right) = 0, \tag{2.25}$$

for all $i, j, k \in \{1, \ldots, n\}$. $\sum_{(i,j,k)\in\Sigma_3}$ denotes a cyclic permutation of the indices i, j, k.

Definition 2.7. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A *Lie algebra homomorphism* ϕ from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi \circ [\,,\,]_{\mathfrak{g}_1} = [\,,\,]_{\mathfrak{g}_2} \circ (\phi \otimes \phi)$.

The identification $T_eG \simeq \mathfrak{g}$ allows the definition of the *exponential map*

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\to g_1, \end{aligned} \tag{2.26}$$

where g_t are the one-parameter subgroups from Definition 2.4. This allows us to rigorously exponentiate Lie algebra elements X to one-parameter subgroups of $g_t \subset G$, and by the properties given in Definition 2.4 it is fully justified (and hereafter we will do so) to write $g_t = \exp(tX)$ for all $X \in \mathfrak{g}$.

2.1.4 Representations of Lie groups and Lie algebras

The adjoint and coadjoint representations of a Lie group and a Lie algebra will play an important role in the rest of the Thesis, so we now describe it. Let V be a finite dimensional vector space over the field k.

Definition 2.8. A *representation of a Lie group* G on V is a smooth group homomorphism $\rho : G \to GL(V)$.

Since k will be either $\mathbb{R}$ or $\mathbb{C}$, we will also write $\rho : G \to GL(n, \mathbb{R})$ or $\rho : G \to GL(n, \mathbb{C})$ to specify whether the representation is real or complex. In complete analogy with the notion of a Lie group representation, we define

Definition 2.9. A *representation of a Lie algebra* $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to GL(V)$.

In the previous definitions it should be noticed that the Lie algebra structure $GL(V)$ is given by the usual matrix commutator $[A, B] = A \cdot B - B \cdot A$.

Given that a Lie algebra is by itself a particular a vector space, then we can indeed consider representations of both G and $\mathfrak{g}$ on $GL(\mathfrak{g})$, and so consider Lie group and Lie algebra representations of the form

$$\rho : \mathfrak{g} \to GL(\mathfrak{g}), \tag{2.27}$$

and

$$\rho : G \to GL(\mathfrak{g}). \tag{2.28}$$

If the homomorphism ρ defining the representation is injective, then we shall say that the representation is *faithful*.

There is a canonical representation of G on its Lie algebra $\mathfrak{g}$ namely, the *adjoint representation*, associated to the adjoint action of G on itself, and defined by

$$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) \\ g &\to \mathrm{Ad}_g \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ T &\to \left.\frac{d}{dt}\right|_{t=0} \left(g e^{tT} g^{-1}\right) \end{aligned} \tag{2.30}$$

for all $T \in \mathfrak{g}$ and $g \in G$. The differential of the adjoint representation of a Lie group G is the so-called adjoint representation of its Lie algebra $\mathfrak{g}$. Differentiating the representation Ad we get

$$\mathrm{ad}_S(T) = \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{\exp(s\,S)}(T) = \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \left(e^{sS} e^{tT} e^{-sS}\right) = [S, T], \tag{2.31}$$

and this motivates the following

Definition 2.10. The *adjoint representation of* $\mathfrak{g}$ is the Lie algebra homomorphism

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to GL(\mathfrak{g}) \\ X &\to \mathrm{ad}_X \end{aligned} \tag{2.32}$$

where the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, called the *adjoint action of* $\mathfrak{g}$ *on itself*, is given by $\mathrm{ad}_X(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

The adjoint representation satisfies

$$\mathrm{ad}([X,Y]_{\mathfrak{g}}) = [\mathrm{ad}_X, \mathrm{ad}_Y]_{GL(\mathfrak{g})}\,, \tag{2.33}$$

or equivalently

$$\mathrm{ad}([X,Y])Z = \mathrm{ad}_X(\mathrm{ad}_Y Z) - \mathrm{ad}_Y(\mathrm{ad}_X Z), \tag{2.34}$$

for all $X, Y, Z \in \mathfrak{g}$, which is just the the Jacobi identity (condition (L2) from the definition of the Lie bracket).

For a vector space V we denote its n-fold tensor product by $V^{\otimes n} = \overbrace{V \otimes \cdots \otimes V}^{n}$. The extension of the adjoint action of $\mathfrak{g}$ on itself to $\mathfrak{g}^{\otimes n}$ will be specially useful. So we define

$$\begin{aligned} \mathrm{ad}_X : \quad & \mathfrak{g}^{\otimes n} \quad \to \mathfrak{g}^{\otimes n} \\ & Y_1 \otimes \cdots \otimes Y_n \to \sum_{a=1}^{n} Y_1 \otimes \cdots \otimes \mathrm{ad}_X(Y_a) \otimes \cdots \otimes Y_n. \end{aligned} \tag{2.35}$$

The dual of (2.29) is the *coadjoint representation* $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$, defined by $\mathrm{Ad}^*_g = (\mathrm{Ad}_{g^{-1}})^*$ where the duality is defined with respect to the canonical pairing between dual vector spaces, i.e.

$$\langle \mathrm{Ad}^*_g x, X\rangle = \langle x, \mathrm{Ad}_{g^{-1}} X\rangle \tag{2.36}$$

for all $g \in G$, $X \in \mathfrak{g}$ and $x \in \mathfrak{g}^*$. The differential of Ad^* in $e \in G$ will be denoted by $\mathrm{ad}^* : \mathfrak{g} \to GL(\mathfrak{g}^*)$ and is given by $\mathrm{ad}^*_X = -(\mathrm{ad}_X)^*$, thus satisfying

$$\langle \mathrm{ad}^*_Y x, X\rangle = \langle x, -\mathrm{ad}_Y X\rangle \tag{2.37}$$

for all $X, Y \in \mathfrak{g}$ and $x \in \mathfrak{g}^*$.

A Lie algebra $\mathfrak{g}$ can act on a smooth manifold by means of an *infinitesimal action*, by which we mean a Lie algebra morphism from $\mathfrak{g}$ to the Lie algebra of vector fields $\mathfrak{X}(M)$.

Example 2.1. (**Matrix Lie groups**) Let $\mathrm{Mat}(n \times n, k)$ be the set of n-dimensional square matrices with coefficients in the field $k = \mathbb{R}, \mathbb{C}$. The general linear group is defined as $GL(n,k) = \{M \in \mathrm{Mat}(n \times n, K) \,|\, \det M \neq 0\}$, i.e. the ones with an inverse. Its Lie algebra is $\mathfrak{gl}(n,k) = \mathrm{Mat}(n \times n, K)$. By Theorem 2.1 every closed subgroup of $GL(n,k)$ is a Lie group, and they are called *matrix Lie groups* (or *linear Lie groups*). Note that $GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$. Some important real matrix groups are listed below, together with their Lie algebras.

- Orientation group:

$$GL^+(n,\mathbb{R}) = \{M \in GL(n,\mathbb{R}) \,|\, \det M > 0\}\,, \qquad \mathfrak{gl}^+(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R}). \tag{2.38}$$

- Special linear group:

$$SL(n,\mathbb{R}) = \{M \in GL(n,\mathbb{R}) \,|\, \det M = 1\}\,, \qquad \mathfrak{sl}(n,\mathbb{R}) = \{M \in \mathfrak{gl}(n,\mathbb{R}) \,|\, \mathrm{tr} M = 0\}\,. \tag{2.39}$$

- Orthogonal group:

$$O(n) = \left\{M \in GL(n,\mathbb{R}) \,|\, M^t = M^{-1}\right\}, \qquad \mathfrak{o}(n) = \left\{M \in \mathfrak{gl}(n,\mathbb{R}) \,|\, M^t = -M\right\}. \tag{2.40}$$

- Special orthogonal group:

$$SO(n) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R}), \qquad \mathfrak{so}(n) = \mathfrak{o}(n) \cap \mathfrak{sl}(n,\mathbb{R}). \tag{2.41}$$

- Indefinite orthogonal groups:

$$\begin{aligned} O(p,q) &= \left\{M \in GL(n,\mathbb{R}) \,|\, M^t I_{p,q} M = I_{p,q}\right\}, \\ \mathfrak{o}(p,q) &= \left\{M \in \mathfrak{gl}(n,\mathbb{R}) \,|\, M^t I_{p,q} + I_{p,q} M = 0\right\}, \end{aligned} \tag{2.42}$$

  where $p+q=n$, $I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$ and I_m the $m \times m$ identity matrix.

- Special indefinite orthogonal groups:

$$SO(p,q) = O(p,q) \cap SL(p+q,\mathbb{R}), \qquad \mathfrak{so}(p,q) = \mathfrak{o}(p,q) \cap \mathfrak{sl}(n,\mathbb{R}). \tag{2.43}$$

  Note that we can see (special) orthogonal groups as particular cases of (special) indefinite orthogonal groups, with $p = 0$ and thus $O(0,n) = O(n)$ and $SO(0,n) = SO(n)$.

- Symplectic groups:

$$\begin{aligned} Sp(n,\mathbb{R}) &= \left\{M \in GL(2n,\mathbb{R}) \,|\, M^t S_n M = S_n\right\}, \\ \mathfrak{sp}(n) &= \left\{M \in \mathfrak{gl}(2n,\mathbb{R}) \,|\, M^t S_n + S_n M = 0\right\}, \end{aligned} \tag{2.44}$$

  where $S_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Some important complex matrix groups are:

- Special linear group:

$$SL(n,\mathbb{C}) = \{M \in GL(n,\mathbb{C}) \,|\, \det M = 1\}, \qquad \mathfrak{sl}(n,\mathbb{C}) = \{M \in \mathfrak{gl}(n,\mathbb{C}) \,|\, \mathrm{tr} M = 0\}. \tag{2.45}$$

- Unitary groups:

$$U(n) = \left\{M \in GL(n,\mathbb{C}) \,|\, M^\dagger M = I\right\}, \qquad \mathfrak{u}(n) = \left\{M \in \mathfrak{gl}(n,\mathbb{C}) \,|\, M^\dagger = -M\right\}. \tag{2.46}$$

  where † denotes Hermitian conjugation (complex conjugation followed by transposition).

- Special unitary groups:

$$SU(n) = U(n) \cap SL(n,\mathbb{C}), \qquad \mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C}). \tag{2.47}$$

♢

All the groups explicitly used in this Thesis are matrix groups.

2.1.5 Geometry of homogeneous spaces

In the following we will briefly recall some useful results concerning the geometry of homogeneous spaces, considering with special care the particular cases of reductive and symmetric spaces. We stress that on the one hand, all the spacetime manifolds that we study during this work are symmetric spaces, and on the other hand, in Chapter 5 we will introduce certain deformations of spacetime symmetries for which its dual space is a reductive or symmetric space. For the sake of brevity, in the sequel we use an algebraic approach to the geometry of these spaces.

Definition 2.11. Let M be a smooth manifold. A *connection* ∇ *on* M is a bilinear map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y) &\to \nabla_X Y \end{aligned} \tag{2.48}$$

satisfying

i) $\nabla_{fX} Y = f \nabla_X Y$,

ii) $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$,

for all $X, Y \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. The vector field $\nabla_X Y$ is called the *covariant derivative of* Y *with respect to* X.

By using partitions of unity it is easy to see that connections can be defined on every smooth manifold. Associated to every connection on M there are two objects, the Riemann curvature and torsion tensors, that measure the extent to which the connection is not commutative (curvature), and how much it differs from the Lie bracket $[\cdot,\cdot]$ (torsion). These two tensors are defined as follows.

Definition 2.12. Let M be a smooth manifold and ∇ a connection on M. The *curvature tensor* R *for the connection* ∇ is the map

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y,Z) &\to R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \end{aligned} \tag{2.49}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.13. Let M be a smooth manifold and ∇ a connection on M. The *torsion tensor* T *for the connection* ∇ is the map

$$\begin{aligned} T : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y) &\to \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned} \tag{2.50}$$

for all $X, Y \in \mathfrak{X}(M)$.

We say that a connection is *torsion-free* if its torsion tensor T vanishes identically. Let (M,g) be a indefinite Riemannian manifold, i.e. a manifold M equipped with a indefinite

Riemannian metric g, defined by $g_m(X,Y) = \langle X_m, Y_m\rangle$ for all $X, Y \in \mathfrak{X}(M)$ and $m \in M$. In terms of the linear map

$$\begin{aligned} \varphi : T_mM &\to T_mM \\ X &\to R(X_m, Y_m)Z_m \end{aligned} \tag{2.51}$$

for all $X_m, Y_m, Z_m \in T_mM$, the *Ricci tensor* R is defined by

$$R(Y_m, Z_m) = \operatorname{tr}\varphi, \tag{2.52}$$

for all $X_m, Y_m \in T_mM$.

Definition 2.14. We say that a connection is *(indefinite) Riemannian, (indefinite) metric compatible* or simply that it is a *metric connection*, if

$$\nabla_X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{2.53}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $\langle\cdot,\cdot\rangle : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}$ is the (indefinite) metric on M.

It is a well-known fact that in any Riemannian manifold, there is a unique torsion-free connection compatible with the metric (see for example [172]), called the *Levi-civita connection*. The same is true for indefinite Riemannian manifolds.

Proposition 2.1. *[176] Let (M, g) be a (indefinite) Riemannian manifold. Then there is a unique connection ∇ on M satisfying the following two properties*

i) ∇ *is metric compatible;*

ii) ∇ *is torsion-free.*

After these general definitions, let us consider the geometry of homogeneous spaces. The starting point is given by the construction in the following Proposition.

Proposition 2.2. *[180] Let G be a Lie group and H a closed subgroup of G. Then the quotient space G/H admits a structure of a real analytic (in particular smooth) manifold in such a way that the left action of G on G/H is real analytic, that is, the mapping $G \times G/H \to G/H$ which maps $(g, g'H)$ into $(gg')H$ is real analytic. In particular, the projection*

$$\begin{aligned} p : G &\to G/H \\ g &\to gH \end{aligned} \tag{2.54}$$

is a real analytic quotient map. In addition p is a locally trivial fibre bundle.

The previous result, which states the properties of coset spaces, together with the following one, which identifies homogeneous spaces with certain cosets, will be heavily used in the rest of this Thesis.

Theorem 2.3. *[177] Let α be a transitive action of a Lie group G on a smooth manifold M. Then for any $m \in M$ the map*

$$\begin{aligned} \beta_m : G/H_m &\to M \\ gH_m &\to \alpha_g m \end{aligned} \tag{2.55}$$

is a diffeomorphism which commutes with the action of G. (Here it is assumed that the group G acts on G/H_m by left translations.)

Definition 2.15. Let $\alpha : G \to \mathrm{Diff}\,(M)$ be a transitive action of the Lie group G on the smooth manifold M. Then we say that (M, G, α) is a *homogeneous space*. Under the identification provided by the diffeomorphism (2.55) of Theorem 2.3, we write $M = G/H_m$, where H_m is the stabilizer of some $m \in M$, and we also call this coset a homogeneous space.

In the above Definition, for every homogeneous space there is a choice of a point $m \in M$. This is not a problem, since if $m' \in M$ is a different point, then by transitivity of α, there is an element $g \in G$, such that $\alpha_g(m) = m'$ and $H_{m'} = C_g(H_m) = gH_m g^{-1}$, and G/H_m and $G/H_{m'}$ are diffeomorphic. In this way we usually write $M = G/H$ in order to denote a homogeneous space. Theorem 2.3 together with Proposition 2.2 identify coset spaces of Lie groups by closed subgroups as models of homogeneous spaces for transitive Lie group actions.

In the following, it will be useful to characterize certain vector fields associated to Lie group actions on homogeneous spaces.

Definition 2.16. For any homogeneous space $M = G/H$ with an action $\alpha : G \to \mathrm{Diff}\,(M)$ we have the associated *action vector fields* $X^\alpha \in \mathfrak{X}(M)$, defined by

$$(X^\alpha_a f)(m) = \left.\frac{d}{dt}\right|_{t=0} f(\alpha_{\exp(-tT_a)} m) \tag{2.56}$$

for all $T_a \in \mathfrak{g}$, $m \in M$ and $f \in \mathcal{C}^\infty(M)$.

Action vector fields X^α on $M = G/H$ are related to right invariant vector fields (2.23), for if $f \in \mathcal{C}^\infty(M)$, then $f \in \mathcal{C}^\infty(G)^H$ and we have that

$$\begin{aligned} (X^\alpha_a f)(gH) &= \left.\frac{d}{dt}\right|_{t=0} f(\alpha_{\exp(-tT_a)}(gH)) = \left.\frac{d}{dt}\right|_{t=0} f(e^{-tT_a} gH)) = \\ &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tT_a} g)) = -\left.\frac{d}{dt}\right|_{t=0} f(e^{tX} g)) = -(X^R_a f)(g). \end{aligned} \tag{2.57}$$

In particular, action vector fields for the action $\alpha : G \to \mathrm{Diff}\,(M)$ form a finite dimensional Lie subalgebra $\mathfrak{X}^\alpha(M)$, which is isomorphic to $\mathfrak{g}$, of the infinite dimensional Lie algebra $\mathfrak{X}(M)$.

For any homogeneous space $M = G/H$ we have the identification $T_o(M) = T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$, induced by the projection p of Proposition 2.2. We write $o = eH$ for the origin of a homogeneous space. A specially beautiful and conceptually clear treatment of the geometry of homogeneous spaces is given by the theory of K-structures [181, 182, 183, 184], which will be used in the last Chapter of the Thesis.

Definition 2.17. Let M be an n-dimensional smooth manifold and $K < \mathrm{GL}(n,\mathbb{R})$ a matrix Lie group. A *K-structure* is a reduced subbundle P of the frame bundle $F(M)$ to the Lie group K.

Example 2.2. Some important types of geometric structures on smooth manifolds admit a unified description in the language of K-structures. In particular, K-structures associated to the matrix Lie groups introduced in Example 2.1 are specially interesting, and we just mention here briefly some of them:

- For $K = GL^+(n,\mathbb{R})$ we have orientations.
- For $K = SL(n,\mathbb{R})$ we have volume elements.
- For $K = O(n)$ we have Riemannian metrics.
- For $K = O(p, n-p)$ we have indefinite Riemannian metrics (in the particular case $p = 1$ we say that such a indefinite Riemannian metric is a *Lorentzian metric*).
- For $K = \mathrm{Sp}(2n)$ we have (almost) symplectic structures.
- For $K = GL(n,\mathbb{C})$ (seen as a subset of $GL(2n,\mathbb{R})$) we have (almost) complex structures.
- For $U(n)$ we have (almost) Kähler structures.

In the language of K-structures, some geometric properties can be explained in a very elegant and simple manner, for example:

i) From $SL(n,\mathbb{R}) \subset GL^+(n,\mathbb{R})$ we get that a volume element induces an orientation.

ii) From $Sp(n,\mathbb{R}) \subset SL(2n,\mathbb{R})$ we get that an (almost) symplectic form induces a volume element.

iii) From $GL(n,\mathbb{C}) \subset GL^+(2n,\mathbb{R})$ we get that an (almost) complex structure induces an orientation.

iv) From $U(n) = O(2n) \cap Sp(2n,\mathbb{R}) \cap GL(n,\mathbb{C})$ we get that an (almost) Kähler structure is nothing more that a Riemannian metric, an (almost) symplectic structure and an (almost) complex structure (together with some compatibility conditions between them).

In the previous examples, 'almost' means that these structures are not necessarily integrable.

$\diamondsuit$

The following relevant result relates quadratic forms with the geometry of manifolds endowed with K-structures.

Theorem 2.4. *[185, 186] For a closed subgroup K of* $\mathrm{GL}(n,\mathbb{R})$*, with $n \geq 3$, the following two statements are equivalent:*

i) K is the group of all matrices which preserve a certain non-degenerate quadratic form of any signature,

ii) For every n-dimensional manifold M and for every reduced subbundle P of $F(M)$ with group K, there is a unique torsion-free connection in P.

This Theorem essentially says that there is a one-to-one correspondence between non-degenerate quadratic forms on tangent spaces to manifolds and linear torsion-free connections on them. In the following we consider the particular case $K = O(p, n-p)$ of the previous Theorem, which corresponds to indefinite Riemannian metrics. Specially important will be the case $K = O(1, n-1)$, which gives rise to Lorentzian metrics, which are the relevant ones in general relativity, and in particular in the description of the maximally symmetric spacetimes of constant curvature, introduced in the next Section, and whose quantum deformations are the main topic of this Thesis.

G-invariant indefinite Riemannian metrics on homogeneous spaces

We have mentioned that an indefinite Riemannian metric is a special case of K-structure, in particular it is an $O(p, n-p)$-structure, since the Lie group $O(p, n-p)$ preserves the quadric

$$\sum_{i=1}^{p}(x^i)^2 - \sum_{j=p+1}^{n}(x^j)^2, \tag{2.58}$$

defining the indefinite Riemannian metric. For homogeneous spaces G/H, transitivity of the action allows us to describe G-invariant indefinite Riemannian metrics in terms of its restriction to the origin eH of G/H. We have the following

Proposition 2.3. *[172] There is a natural one-to-one correspondence between the G-invariant indefinite Riemannian metrics g on $M = G/H$ and the Ad_H-invariant non-degenerate symmetric bilinear forms $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{h}$. The correspondence is given by*

$$\langle \bar{X}, \bar{Y}\rangle = g(X,Y)_{eH}, \tag{2.59}$$

for all $X, Y \in \mathfrak{g}$, where $\bar{X}$ and $\bar{Y}$ are the elements of $\mathfrak{g}/\mathfrak{h}$ corresponding to X and Y, respectively.

A metric g is called Riemannian if and only if the the bilinear form $\langle\cdot,\cdot\rangle$ of the previous Proposition is positive definite, i.e has signature $(+\cdots+)$. A metric g is called *Lorentzian* if and only if the form $\langle\cdot,\cdot\rangle$ has signature $(+-\cdots--)$. In the general case with signature $(\underbrace{++\cdots+}_{p}\underbrace{-\cdots--}_{n-p})$ we have an indefinite Riemannian (also pseudo-Riemannian) metric.

2.1.6 Geometry of reductive homogeneous spaces

Among the set of homogeneous spaces, those that are reductive are simpler to describe. They are essentially those homogeneous spaces G/H for which their tangent space can be identified with some subset of the Lie algebra $\mathfrak{g}$ (recall that for a general homogeneous space we only have that $T_{gH}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$). In the case of reductive homogeneous spaces we do have an Ad_H-invariant isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{t}$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$, and so the description of these spaces becomes specially simple. Reductive spaces will be relevant in two points of this Thesis: firstly, the maximally symmetric Lorentzian spaces we will consider in following Section are symmetric spaces, and thus they are particular examples of reductive spaces. Secondly, and this is the main reason we describe their geometry in this Section, in Chapter §7 reductive spaces will appear as dual spaces associated to certain Lie bialgebra structures underlying quantum deformations.

Definition 2.18. Let $M = G/H$ be a homogeneous space. We say that M is *reductive* if $\mathfrak{g} = \mathrm{Lie}\,(G)$ may be decomposed into a vector space direct sum of $\mathfrak{h} = \mathrm{Lie}\,(H)$ and an Ad_H-invariant subspace $\mathfrak{t}$, that is, if

i) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}, \qquad \mathfrak{h} \cap \mathfrak{t} = 0,$

ii) $\mathrm{Ad}_H \mathfrak{t} \subset \mathfrak{t}.$

Condition *ii)* implies that $\mathfrak{t}$ is $\mathrm{ad}_{\mathfrak{h}}$-invariant, that is

ii)' $[\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}.$

If H is connected, then $ii)'$ implies $ii)$. Thus, for reductive homogeneous spaces, the structure of the Lie algebra $\mathfrak{g}$ can be written as

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}, \qquad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{h} \oplus \mathfrak{t}. \tag{2.60}$$

As mentioned before, Ad_H-invariance for $\mathfrak{t}$ implies that for a reductive homogeneous space $M = G/H$ we can further identify $T_o(M) = T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{t}$. In what follows, the following notations will prove useful: $[\cdot,\cdot]_{\mathfrak{h}}$ and $[\cdot,\cdot]_{\mathfrak{t}}$ define the projection of the Lie bracket $[\cdot,\cdot]$ to the respective subspaces. Then we write

$$[X, Y] = [X, Y]_{\mathfrak{h}} + [X, Y]_{\mathfrak{t}}, \tag{2.61}$$

where $[X, Y]_{\mathfrak{h}} \in \mathfrak{h}$ and $[X, Y]_{\mathfrak{t}} \in \mathfrak{t}$ are the projections of the Lie bracket on $\mathfrak{g}$ to $\mathfrak{h}$ and $\mathfrak{t}$, respectively.

For reductive spaces, given a G-invariant K-structure there is a one-to-one correspondence between the set of G-invariant connections and the set of linear mappings $\mathfrak{t} \to \mathfrak{g}$. The zero map defines the simplest of these connections, the so-called *canonical connection*, which has the following three important properties:

- If we call $g_{\mathfrak{t}} = \{e^{tX} | X \in \mathfrak{t}\}$ the set of one-parameter subgroups (see Definition 2.4), then $g_{\mathfrak{t}}$ is the set of geodesics starting at o for the canonical connection. Moreover, any other geodesic is just a translate by a Lie group element of one of the curves in $g_{\mathfrak{t}}$, so the complete set of geodesics is just $\{\alpha_g(g_{\mathfrak{t}}) \,|\, g \in G\}$.
- The canonical connection is complete.

- G-invariant tensor fields are parallel transported with respect to the canonical connection.

For the canonical connection, curvature and torsion tensors take a simpler form. Taking into account G-invariance, we get the following

Theorem 2.5. *[173] Let P be a G-invariant K-structure on a reductive homogeneous space $M = G/H$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$. Curvature and torsion tensors of the canonical connection in P satisfy*

i) $T(X,Y)_{eH} = -[X,Y]_{\mathfrak{t}}, \qquad \forall X,Y \in \mathfrak{t},$

ii) $(R(X,Y)Z)_{eH} = -[[X,Y]_{\mathfrak{h}}, Z], \qquad \forall X,Y,Z \in \mathfrak{t},$

iii) $\nabla T = 0,$

iv) $\nabla R = 0.$

As we can see from Condition $i)$ of the above Theorem, the canonical connection is not torsion free. However, in the following Section we will be interested in torsion-free Lorentzian connections, so let us introduce the so-called *natural torsion-free connection*, which is the unique connection on the reductive homogeneous space $M = G/H$ which has the same set of geodesics as the canonical connection and is torsion-free. Similarly to the canonical connection, the natural torsion-free connection is complete.

G-invariant indefinite Riemannian metrics on reductive homogeneous spaces

When $M = G/H$ is a reductive homogeneous space, the tangent description of indefinite Riemannian metrics given in Proposition 2.3 can be refined. Then we have the following

Proposition 2.4. *[173] If $M = G/H$ is reductive with an Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$, then there is a natural one-to-one correspondence between the G-invariant indefinite Riemannian metrics g on M and the Ad_H-invariant non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}$. The correspondence is given by*

$$\langle X, Y \rangle = g(X,Y)_{eH}, \tag{2.62}$$

for all $X, Y \in \mathfrak{t}$.

The Ad_H-invariance of B implies that

$$\langle [Z,X], Y \rangle + \langle X, [Z,Y] \rangle = 0, \tag{2.63}$$

for all $X, Y \in \mathfrak{t}$ and all $Z \in \mathfrak{h}$. If H is connected, the converse is also true.

Definition 2.19. A homogeneous space $M = G/H$ with a G-invariant indefinite Riemannian metric g is said to be *naturally reductive* if it admits an Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ satisfying the condition

$$\langle X, [Z,Y]_{\mathfrak{t}} \rangle + \langle [Z,X]_{\mathfrak{t}}, Y \rangle = 0 \tag{2.64}$$

for all $X, Y, Z \in \mathfrak{t}$.

Let g be the G-invariant metric corresponding to $\langle\cdot,\cdot\rangle$. Then the Riemannian connection for g coincides with the natural torsion-free connection if and only if $M = G/H$ is naturally reductive. Moreover, if $M = G/H$ is naturally reductive then the curvature tensor R of the Riemannian connection satisfies

$$g(R(X,Y)Y,X)_{eH} = \frac{1}{4}\langle [X,Y]_{\mathfrak{t}}, [X,Y]_{\mathfrak{t}}\rangle - \langle [[X,Y]_{\mathfrak{h}}, Y], X\rangle \tag{2.65}$$

for all $X, Y \in \mathfrak{t}$.

The following Theorem is very useful in order to construct naturally reductive homogeneous spaces starting from a Lie algebra $\mathfrak{g}$.

Theorem 2.6. *[173] Let $M = G/H$ be a homogeneous space. If the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,(G)$ admits an Ad_G-invariant non-degeneate symmetric bilinear form $\langle\cdot,\cdot\rangle$ such that its restriction $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ to $\mathfrak{h}$ is non-degenerate. Then*

i) The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ defined by

$$\mathfrak{t} = \{X \in \mathfrak{g} \,|\, \langle X, Y\rangle = 0, \forall Y \in \mathfrak{h}\} \tag{2.66}$$

is Ad_H-invariant and the restriction $\langle\cdot,\cdot\rangle_{\mathfrak{t}}$ of $\langle\cdot,\cdot\rangle$ to $\mathfrak{t}$ is also non-degenerate and ad_H-invariant;

ii) The homogeneous space G/H is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ defined above and the G-invariant metric g defined by $\langle\cdot,\cdot\rangle_{\mathfrak{t}}$;

iii) The curvature tensor R defined by the metric g satisfies

$$g(R(X,Y)Y,X)_{\mathfrak{h}} = \frac{1}{4}\langle [X,Y]_{\mathfrak{t}}, [X,Y]_{\mathfrak{t}}\rangle_{\mathfrak{t}} + \langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}\rangle_{\mathfrak{h}} \tag{2.67}$$

for all $X, Y \in \mathfrak{t}$.

2.1.7 Geometry of symmetric homogeneous spaces

Let G be a Lie group and σ an automorphism of G. Lets call $G_\sigma \subset G$ to set the of elements which are fixed by sigma. Then $G_\sigma < G$ is a closed subgroup of G, and following [173], we define

Definition 2.20. A *symmetric space* is a triple (G, H, σ) consisting of a connected Lie group, a closed subgroup H of G and an involutive automorphism σ of G such that H lies between G_σ and the identity component of G_σ.

Obviously, if G_σ is connected then $H = G_\sigma$. The tangent space of a symmetric space naturally inherits the structure of a symmetric Lie algebra.

Definition 2.21. A *symmetric Lie algebra* is a triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ consisting of a Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h}$ and an involutive automorphism σ of $\mathfrak{g}$, where $\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \sigma(X) = X\}$.

For any symmetric Lie algebra, involutivity of σ directly implies that, as a linear map, its only possible eigenvalues are ± 1. If we call $\mathfrak{h}$ to the eigenspace corresponding to $+1$ and $\mathfrak{t}$ to the eigenspace corresponding to -1, i.e.

$$\begin{aligned} \sigma(X) &= X, \qquad && \forall X \in \mathfrak{h} \\ \sigma(X) &= -X, \qquad && \forall X \in \mathfrak{t} \end{aligned} \tag{2.68}$$

then it is straightforward to prove that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}, \qquad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{h}. \tag{2.69}$$

Proposition 2.5. *[173] Let (G, H, σ) be a symmetric space with tangent symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Then the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ is Ad_H-invariant, i.e. $\mathrm{Ad}_H \mathfrak{t} \subset \mathfrak{t}$.*

The Proposition above, together with (2.69), shows that every symmetric Lie algebra is reductive (2.60). Note that for every symmetric algebra $\mathfrak{h} \cap \mathfrak{t} = \emptyset$ trivially. In the same way, every symmetric homogeneous space is reductive.

Regarding the geometry of symmetric homogeneous spaces $M = G/H$, we have that the condition $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{h}$ implies a great simplification, since in that case the canonical connection coincides to the torsion-free connection. Moreover, this connection is the only connection on $M = G/H$ that is invariant by the symmetries of M. In the following Theorem some important properties of the canonical connection directly obtained by particularizing the corresponding results for reductive homogeneous spaces are stated.

Theorem 2.7. *[173] Let (G, H, σ) be a symmetric space, then the canonical connection on the homogeneous space $M = G/H$ satisfies*

i) If we call $g_{\mathfrak{t}} = \{e^{tX} | X \in \mathfrak{t}\}$ to the set of one-parameter subgroups (see Definition 2.4), then $g_{\mathfrak{t}}$ is the set of geodesics starting at eH for the canonical connection. Moreover, any other geodesic is just a translate by a Lie group element of one of the curves in $g_{\mathfrak{t}}$, so the complete set of geodesics is just $\{\alpha_g(g_{\mathfrak{t}}) \,|\, g \in G\}$;

ii) It is complete;

iii) G-invariant tensor fields are parallel transported;

iv) $T = 0$ and $\nabla T = 0$;

v) $R(X,Y)Z = \mathrm{ad}_Z[X,Y] = -[[X,Y],Z]$ for all $X, Y, Z \in \mathfrak{t} \simeq T_{eH}(M)$;

iv) $\nabla R = 0$.

G-invariant indefinite Riemannian metrics on symmetric homogeneous spaces

The following two Theorems completely define the geometry of the Lorentzian spaces we will introduce in the following Section.

Theorem 2.8. *[173] Let (G, H, σ) be a symmetric space. A G-invariant indefinite Riemannian metric on $M = G/H$, if there exists any, induces the canonical connection on M.*

Theorem 2.9. *[173] Let (G, H, σ) be a symmetric space with G semi-simple and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ be the canonical decomposition. Then the restriction of the Killing-Cartan form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ to $\mathfrak{t}$ defines a G-invariant indefinite Riemannnian metric on $M = G/H$ by*

$$g(X, Y)_{eH} = \langle X, Y \rangle \tag{2.70}$$

for all $X, Y \in \mathfrak{t}$.

In this Section we have described the geometry of homogeneous spaces, emphasizing the special cases of reductive and symmetric spaces, and we have stated all the mathematical background necessary to their construction as coset spaces from Lie groups. Now we shall use these results in order to describe in detail the three Lorentzian spaces, which will be the central objects during this Thesis.

2.2 Lie algebras of the Lorentzian groups

In the following two sections we study the three maximally symmetric Lorentzian spacetimes of constant curvature, i.e. anti-de Sitter (**AdS**), de Sitter (**dS**) and Minkowski (**M**) spacetimes, and their groups of isometries, namely the (A)dS and Poincaré groups. These three spaces and their motion groups admit a unified description in terms of the cosmological constant parameter Λ and in what follows we exploit this fact by presenting in a unified framework the Lie algebras, Lie groups and homogeneous spaces corresponding to them. In fact, we will also make use of the parameter

$$\eta^2 \equiv -\Lambda, \qquad\qquad \eta \equiv \sqrt{-\Lambda}. \tag{2.71}$$

In this way, when the cosmological constant $\Lambda > 0$ is positive (**dS**) we have that η will be a real parameter, while if the cosmological constant $\Lambda < 0$ is negative (**AdS**) then η will be a purely imaginary parameter. Clearly, in the vanishing cosmological constant case $\Lambda = 0$ (**M**) we have that $\eta = 0$. It should be noticed that, unless otherwise stated, all the expressions appearing in this Thesis will be analytic in the parameter η, so in particular the limit $\eta \to 0$ will always be well-defined.

These three maximally symmetric Lorentzian homogeneous spaces of constant curvature (together with their Lie groups of isometries and the respective Lie algebras) will be described in $(1 + 1)$, $(2 + 1)$ and $(3 + 1)$ dimensions, giving the explicit expressions in each case. For the sake of simplicity, hereafter we will refer to these spaces and groups simply as Lorentzian spaces (groups, algebras). Therefore, when we write 'Lorentzian space' we really mean a 'maximally symmetric Lorentzian homogeneous space of constant curvature'. More general manifolds endowed with a Lorentizian metric will be explicitly pointed out when appropriate.

We will see that while, for these spaces, some features are common to any dimension, some of their properties are strongly dimension-dependent. From now on we denote by n the spatial dimension of the spacetime, and so we denote by G_Λ^{n+1} to the (A)dS or Poincaré groups of dimensions $n+1$, and by $\mathfrak{g}_\Lambda^{n+1}$ to their Lie algebras, i.e. $\mathfrak{g}_\Lambda^{n+1} = \mathrm{Lie}(G_\Lambda^{n+1})$. We remark here that we usually omit the superscript denoting the dimension whenever no confusion is possible.

Hereafter we use latin indexes $a, b, c, \ldots$ to label spatial components and so they take values in the set $\{1, \ldots, n\}$. It should be sufficiently clear throughout the text which is the concrete value of n, but otherwise we will make it explicit. Greek indices will denote spacetime indices and so they take values in $\{0, \ldots, n\}$.

We denote spatial vectors by bold letters, for example $\mathbf{v} = (v^1, \ldots, v^n)$ and $\mathbf{w} = (w^1, \ldots, w^n)$. Then we write $\mathbf{v} \cdot \mathbf{w} = v^1 w^1 + \cdots + v^n w^n$ and $\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}$ for the scalar product of two vectors and the square of a vector, respectively. Whenever appropriate (dimensions two and three) we also write $\mathbf{v} \times \mathbf{w}$ for their vector product. We also denote by ϵ_{ab} and ϵ_{abc} the Levi-Civita symbols such that $\epsilon_{12} = 1$ and $\epsilon_{123} = 1$, in dimensions two and three, respectively. For spacetime vectors, i.e. vectors whose indices run from 0 to n, we use the notation $\bar{v} = (v^0, v^1, \ldots, v^n)$.

We now give a unified description of the Lie algebras of the three Lorentzian groups in terms of the cosmological constant. The basis we use to describe these Lie algebras will be called *the kinematical basis*, because it corresponds to a particular kinematical assignment of the generators of the one parameter transformation subgroups, as described below.

2.2.1 (3+1) dimensional Lorentzian algebras

Let us denote by $\mathfrak{g}_\Lambda^{3+1}$ the family of Lie algebras with the following commutation relations

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & [K_a, K_b] = -\epsilon_{abc} J_c, \\ [P_0, P_a] = -\Lambda K_a, & [P_a, P_b] = \Lambda \epsilon_{abc} J_c, & [P_0, J_a] = 0. \end{array} \tag{2.72}$$

It corresponds to the Lie algebra of the anti-de Sitter, Poincaré and de Sitter groups in $(3+1)$ dimensions when $\Lambda < 0$, $\Lambda = 0$ or $\Lambda > 0$, respectively. We call this basis the kinematical basis due to the clear physical interpretation of the generators: J_a are the generators of the three spatial rotations, K_a are the generators of the three Lorentz boosts while P_0 is the generator of the time translation and P_a are the generators of the three spatial translations. For $\Lambda > 0$ we have that $\mathfrak{g}_\Lambda^{3+1} \simeq \mathfrak{so}(4,1)$, for $\Lambda = 0$ it is $\mathfrak{g}_\Lambda^{3+1} \simeq \mathfrak{iso}(3,1)$[1] and for $\Lambda < 0$ it is $\mathfrak{g}_\Lambda^{3+1} \simeq \mathfrak{so}(3,2)$. Along this Thesis we will use the notation $\mathfrak{g}_\Lambda^{3+1}$ when we refer to this one-parametric family of algebras and we will only write $\mathfrak{so}(4,1)$, $\mathfrak{iso}(3,1)$ or $\mathfrak{so}(3,2)$ when we need to singularize one of them.

The family of Lie algebras (2.72) has two Casimir elements [187]. The first one is quadratic and comes from the Killing-Cartan form. It is given by

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda(\mathbf{J}^2 - \mathbf{K}^2). \tag{2.73}$$

The second one is a forth order invariant,

$$\mathcal{W}^2 = W_0^2 - \mathbf{W}^2 - \Lambda(\mathbf{J} \cdot \mathbf{K})^2, \tag{2.74}$$

[1]This notation is a shortcut for the semidirect sum of Lie algebras. For example in this case $\mathfrak{iso}(3,1) \equiv \mathfrak{so}(3,1) \ltimes \mathbb{R}^4$. The same notation will be used for the semidirect product of Lie groups, so for example $ISO(3,1) \equiv SO(3,1) \ltimes \mathbb{R}^4$.

where $W_0 = \mathbf{J}\cdot\mathbf{P}$ and $W_a = -J_a P_0 + \epsilon_{abc} K_b P_c$ are the components of the (anti-)de Sitter analogue of the Pauli-Lubanski four-vector. In the Poincaré case $\Lambda = 0$ the invariant $W_0^2 - \mathbf{W}^2$ provides the square of the spin/helicity operator, which in the rest frame is proportional to the square of the angular momentum operator. It is worth emphasizing that the presence of a non-vanishing cosmological constant Λ implies that the quadratic invariant has a new contribution coming from the Lorentz sector of the Lie algebra.

Two subalgebras of the Lorentzian Lie algebras (2.72) play an important role in the rest of the paper. The first one is the well-known Lorentz algebra $\mathfrak{l}^{3+1}$ containing boosts K_a and rotations J_a, which is Λ-independent and isomorphic to $\mathfrak{so}(3,1)$, and with commutators given by

$$[J_a, J_b] = \epsilon_{abc} J_c, \qquad [J_a, K_b] = \epsilon_{abc} K_c, \quad [K_a, K_b] = -\epsilon_{abc} J_c. \tag{2.75}$$

The second one, let us call it $\mathfrak{h}^{3+1}$, is the Lie subalgebra containing the time translation P_0 and the rotations J_a, with Lie brackets

$$[P_0, J_a] = 0, \qquad [J_a, J_b] = \epsilon_{abc} J_c. \tag{2.76}$$

This Lie subalgebra, which again is Λ-independent, is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}$. We have that the intersection of $\mathfrak{l}^{3+1}$ and $\mathfrak{h}^{3+1}$ is again a Lie subalgebra, the rotation subalgebra $\mathfrak{so}(3)$.

Consider the faithful representation of $\mathfrak{g}_\Lambda^{3+1}$ given by $\rho : \mathfrak{g}_\Lambda^{3+1} \to GL(5,\mathbb{R})$ and let us identify $\mathfrak{g}_\Lambda^{3+1}$ with the image under this representation $\rho(\mathfrak{g}_\Lambda^{3+1})$. We can write a generic element Q of this Lie algebra as

$$Q_\Lambda = x^\mu P_\mu + \xi^a K_a + \theta^a J_a = \begin{pmatrix} 0 & \Lambda x^0 & -\Lambda x^1 & -\Lambda x^2 & -\Lambda x^3 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \tag{2.77}$$

For latter purposes we also write down the generic element in the particular case $\Lambda = 0$, for which the first raw vanishes

$$Q_0 = x^\mu P_\mu + \xi^a K_a + \theta^a J_a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \tag{2.78}$$

Note that this representation is constructed in such a way that the projection to (2+1) (or (1+1)) dimensions is obtained just setting the relevant coordinates to zero, or equivalently, taking away the fifth (or forth and fifth) row(s) and column(s).

2.2.2 (2+1) dimensional Lorentzian algebras

From these Lie algebras it is straightforward to write down the corresponding one for $(2+1)$ dimensions. We write $\mathfrak{g}_\Lambda^{2+1}$ and we have the following commutation relations

$$\begin{aligned} &[J, P_a] = \epsilon_{ab} P_b, && [J, K_a] = \epsilon_{ab} K_b, && [J, P_0] = 0, \\ &[K_a, P_b] = \delta_{ab} P_0, && [K_a, P_0] = P_a, && [K_1, K_2] = -J, \\ &[P_0, P_a] = -\Lambda K_a, && [P_1, P_2] = \Lambda J, && \end{aligned} \tag{2.79}$$

where we have written $J = J_3$ for the generator of the unique spatial rotation in $(2+1)$ dimensions. For $\Lambda < 0$ we have that $\mathfrak{g}_\Lambda^{2+1} \simeq \mathfrak{so}(2,2)$, for $\Lambda = 0$ it is $\mathfrak{g}_\Lambda^{2+1} \simeq \mathfrak{iso}(2,1)$, while for $\Lambda > 0$ we have that $\mathfrak{g}_\Lambda^{2+1} \simeq \mathfrak{so}(3,1)$.

The structure of the centre of $\mathfrak{g}_\Lambda^{2+1}$ is modified with this dimensional reduction, since now we have two quadratic Casimir elements, the first one being formally similar to (2.73) and given by

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda(J^2 - \mathbf{K}^2), \tag{2.80}$$

while the second one

$$\mathcal{W} = -JP_0 + K_1 P_2 - K_2 P_1 \tag{2.81}$$

is the so-called Pauli-Lubanski vector. This second quadratic Casimir is obtained from (2.74) where the only remaining component is W_3. The Lorentz algebra $\mathfrak{l}^{2+1}$, isomorphic to $\mathfrak{so}(2,1)$, is structurally similar to (2.75), with explicit commutators

$$[J, K_a] = \epsilon_{ab} K_b, \qquad [K_1, K_2] = J. \tag{2.82}$$

The second subalgebra $\mathfrak{h}^{2+1} \simeq \mathbb{R}^2$ is now generated by the time translation and the rotation generator, so it is abelian.

2.2.3 (1+1) dimensional Lorentzian algebras

In $(1+1)$ dimensions we have $\mathfrak{g}_\Lambda^{1+1}$ with commutators

$$[K, P_0] = P_1, \qquad [K, P_1] = P_0, \qquad [P_0, P_1] = -\Lambda K \tag{2.83}$$

where we have written $K = K_1$ for the unique remaining boost. Note that in this case, for $\Lambda \neq 0$, the automorphism $P_\mu \to \sqrt{\Lambda} P'_\mu$ transforms the last bracket in $[P'_0, P'_1] = -K$, so we have that in fact the Lie algebras of the de Sitter and anti-de Sitter groups in $(1+1)$ dimensions are isomorphic. So for both $\Lambda < 0$ and $\Lambda > 0$ we have that $\mathfrak{g}_\Lambda^{1+1} \simeq \mathfrak{so}(2,1)$ and for $\Lambda = 0$ it is $\mathfrak{g}_\Lambda^{1+1} \simeq \mathfrak{iso}(1,1)$. In this dimension only the first quadratic Casimir survives, giving

$$\mathcal{C} = P_0^2 - P_1^2 + \Lambda K^2. \tag{2.84}$$

In this case $\mathfrak{l}^{1+1}$ is generated by the boost and $\mathfrak{h}^{1+1}$ by the time translation, being both of them unidimensional.

2.3 Geometry of Lorentzian groups and spacetimes

At this moment, we are interested in describing the Lorentzian spacetimes (Minkowski and (anti)-de Sitter) as coset spaces of the corresponding Lorentzian groups G divided by the corresponding isotropy group L, the Lorentz group[2].

Note that if n represents the spatial dimension of the maximally symmetric spacetime of constant curvature $M = G/L$ under consideration, then its isometry group G has dimension $(n+1)(n+2)/2$. So we have that $\dim M = n+1$, $\dim G = (n+1)(n+2)/2$ and $\dim L = \dim G - \dim L = n(n+1)/2$.

For later purposes, we will introduce coordinates in both the groups and the homogeneous spaces in such a way that most computations on the group can be straightforwardly translated to the coset space. In order to do that, suppose that

$$
\begin{aligned}
(x^\alpha, \xi^a, \theta^a) : G \to \mathbb{R}^{(n+1)(n+2)/2},& \\
g \to (x^\alpha(g), \xi^a(g), \theta^a(g))&
\end{aligned}
\tag{2.85}
$$

are a set of local coordinates on the group G and $p : G \to G/L, g \to gL$ is the canonical projection. The set of functions on the coset space G/L is precisely the set of L-invariant functions on the group G, see (2.15), i.e. $\mathcal{C}^\infty(G/L) = \mathcal{C}^\infty(G)^L$. So in order to define coordinate functions on G/L it is sufficient to find $n+1$ independent L-invariant functions on G.

However, as it will become clear later, we are interested in defining coordinate functions such that the following diagram commutes

$$
\begin{tikzcd}
G \arrow[r, "p"] \arrow[d, "{(x^\alpha,\xi^a,\theta^a)}"'] & G/L \arrow[d, "\tilde{x}^\alpha"] \\
\mathbb{R}^{(n+1)(n+2)/2} \arrow[r, "\tilde{p}"'] & \mathbb{R}^{n+1}
\end{tikzcd}
\tag{2.86}
$$

and so

$$
\tilde{x}^\alpha(gL) = \tilde{x}^\alpha \circ p(g) = \tilde{p} \circ (x^\alpha, \xi^a, \theta^a)(g) = x^\alpha(g). \tag{2.87}
$$

In this way we can identify $\tilde{x}^\alpha = x^\alpha$, and in fact we will do that whenever confusion is not possible. The so-called *exponential coordinates of the second kind*, provided that the exponentiation ordering is the appropriate one, fulfil all of our requirements. In the rest of this Section we describe the explicit form of these coordinates for the three Lorentzian groups.

2.3.1 (3+1) dimensional Lorentzian spacetimes

In order to construct the (3+1)-dimensional (anti-)de Sitter and Minkowski spacetimes (which we call generally M_Λ^{3+1}) as coset spaces, we parametrize an element of the (anti-)de

[2] In Chapter 3 we will consider a different coset space, the space of time-like geodesics for these Lorentzian spacetime, where the isometry subgroup H has $\mathfrak{h}$ as its Lie algebra.

Sitter group G_Λ^{3+1} in the form

$$\begin{aligned} G_\Lambda^{3+1} &= \exp x^0\rho(P_0)\exp x^1\rho(P_1)\exp x^2\rho(P_2)\exp x^3\rho(P_3) \\ &\quad\times \exp\xi^1\rho(K_1)\exp\xi^2\rho(K_2)\exp\xi^3\rho(K_3)\exp\theta^1\rho(J_1)\exp\theta^2\rho(J_2)\exp\theta^3\rho(J_3)\,, \end{aligned} \tag{2.88}$$

where the Lorentz subgroup L^{3+1} is parametrized by

$$L^{3+1} = \exp\xi^1\rho(K_1)\exp\xi^2\rho(K_2)\exp\xi^3\rho(K_3)\exp\theta^1\rho(J_1)\exp\theta^2\rho(J_2)\exp\theta^3\rho(J_3). \tag{2.89}$$

In this way we have that $M_\Lambda^{3+1} = G_\Lambda^{3+1}/L^{3+1}$ and so we identify the three maximally symmetric Lorentzian spacetimes of constant curvature with coset spaces of their respective group of isometries divided by the Lorentz subgroup. These cosets are different depending on the cosmological constant, and we have that

- $\Lambda < 0$: Anti-de Sitter spacetime $\mathbf{AdS}^{3+1} = SO(3,2)/SO(3,1)$.
- $\Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} = ISO(3,1)/SO(3,1)$.
- $\Lambda > 0$: de Sitter spacetime $\mathbf{dS}^{3+1} = SO(4,1)/SO(3,1)$.

For each of these homogeneous spaces, which are indeed symmetric, we can identify their tangent space at every point $m = gL \in M_\Lambda$ with the translation sector, i.e.

$$T_m(M_\Lambda^{3+1}) = T_{gL^{3+1}}(G_\Lambda^{3+1}/L^{3+1}) \simeq \mathfrak{g}_\Lambda^{3+1}/\mathfrak{l}^{3+1} \simeq \mathfrak{t}^{3+1} = \mathrm{span}\,\{P_0, \mathbf{P}\}, \tag{2.90}$$

as explained in §2.1.6.

Although these three spacetimes admit a common description as coset spaces they are quite different. For example, from the topological point of view both $\mathbf{M}^{3+1}$ and $\mathbf{dS}^{3+1}$ are simply connected, while $\pi_1(\mathbf{AdS}^{3+1}) = \mathbb{Z}$. The generator of $\pi_1(\mathbf{AdS}^{3+1})$ can be taken in the time dimension, or in other words, $\mathbf{AdS}^{3+1}$ has time-like closed geodesics and thus it is not straightforward to consider it as a physically realistic spacetime model. However, one can consider its double cover, which is simply-connected, and this problem disappears. Even if $\mathbf{AdS}^{3+1}$ is not in principle a physically realistic spacetime model, due to the famous AdS/CFT conjecture [188] the space $\mathbf{AdS}^{3+1}$, together with its analogues in other dimensions, has attracted much attention. In fact, as topological spaces we have that $\mathbf{M}^{3+1} \simeq \mathbb{R}^4$, $\mathbf{dS}^{3+1} \simeq \mathbb{R}\times S^3$ and $\mathbf{AdS}^{3+1} \simeq S^1\times\mathbb{R}^3$ ($\simeq$ denotes here homeomorphic). Some of the geometrical properties of these spaces will be discussed below.

In the coordinates defined by the inverse map of (2.88) a group element is given by a matrix element of the form

$$G_\Lambda^{3+1} = \begin{pmatrix} s^4 & \bar{A} \\ \bar{s}^T & \mathbf{B} \end{pmatrix}, \tag{2.91}$$

where $\mathbf{B}$ is a 4×4 matrix and we recall that $\bar{s} = (s^0, s^1, s^2, s^3)$ and $\bar{s}^T$ denotes the transpose of $\bar{s}$. When the cosmological constant vanishes $\Lambda = 0$, G_0^{3+1} is just the Poincaré group $\mathbb{R}^4 \ltimes SO(3,1)$, and the matrix element (2.91) takes the simpler form

$$G_0^{3+1} = \begin{pmatrix} 1 & \bar{0} \\ \bar{x}^T & \mathbf{L} \end{pmatrix}, \tag{2.92}$$

where $\mathbf{L}$ is the 4×4 matrix representation of an element of the Lorentz subgroup. So we have that

$$\lim_{\Lambda\to 0} G_\Lambda^{3+1} = G_0^{3+1}, \qquad \lim_{\Lambda\to 0} s^4 = 1, \qquad \lim_{\Lambda\to 0} s^\alpha = x^\alpha, \qquad \lim_{\Lambda\to 0} A^\alpha = 0. \tag{2.93}$$

The explicit form of the element (2.91) is quite complicated, so we omit it for the sake of brevity, but we do write down explicitly the one-parameter subgroups of G_Λ, which are obtained by exponentiation of the associated Lie algebra element. These are given by

$$\begin{gathered}
\mathrm{e}^{x^0 P_0} = \begin{pmatrix} \cos\eta x^0 & -\eta\sin\eta x^0 & 0 & 0 & 0 \\ \frac{1}{\eta}\sin\eta x^0 & \cos\eta x^0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{e}^{x^1 P_1} = \begin{pmatrix} \cosh\eta x^1 & 0 & \eta\sinh\eta x^1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\eta}\sinh\eta x^1 & 0 & \cosh\eta x^1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{e}^{x^2 P_2} = \begin{pmatrix} \cosh\eta x^2 & 0 & 0 & \eta\sinh\eta x^2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{\eta}\sinh\eta x^2 & 0 & 0 & \cosh\eta x^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{e}^{x^3 P_3} = \begin{pmatrix} \cosh\eta x^3 & 0 & 0 & 0 & \eta\sinh\eta x^3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\eta}\sinh\eta x^3 & 0 & 0 & 0 & \cosh\eta x^3 \end{pmatrix}, \\
\mathrm{e}^{\theta^1 J_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta^1 & -\sin\theta^1 \\ 0 & 0 & 0 & \sin\theta^1 & \cos\theta^1 \end{pmatrix}, \quad \mathrm{e}^{\xi^1 K_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cosh\xi^1 & \sinh\xi^1 & 0 & 0 \\ 0 & \sinh\xi^1 & \cosh\xi^1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{e}^{\theta^2 J_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta^2 & 0 & \sin\theta^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sin\theta^2 & 0 & \cos\theta^2 \end{pmatrix}, \quad \mathrm{e}^{\xi^2 K_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cosh\xi^2 & 0 & \sinh\xi^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \sinh\xi^2 & 0 & \cosh\xi^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{e}^{\theta^3 J_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta^3 & -\sin\theta^3 & 0 \\ 0 & 0 & \sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{e}^{\xi^3 K_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cosh\xi^3 & 0 & 0 & \sinh\xi^3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \sinh\xi^3 & 0 & 0 & \cosh\xi^3 \end{pmatrix},
\end{gathered} \tag{2.94}$$

where as mentioned before the parameter η is related to the cosmological constant by $\eta^2 = -\Lambda$. This means that η is either a real number ($\eta = 1/\tau$) for $\mathbf{AdS}^{3+1}$ or a purely imaginary one ($\eta = \mathrm{i}/\tau$) for $\mathbf{dS}^{3+1}$, where τ is the radius of the universe.

This matrix representation of the isometry groups G_Λ^{3+1} and their Lie algebras $\mathfrak{g}_\Lambda^{3+1}$ can be characterized in terms of the bilinear form represented by the matrix

$$\mathbf{I}_\Lambda = \mathrm{diag}(+1, -\Lambda, \Lambda, \Lambda, \Lambda), \tag{2.95}$$

which identifies them with the isometry groups of the five-dimensional linear space $(\mathbb{R}^5, \mathbf{I}_\Lambda)$ with *ambient* coordinates $(s^4, s^0, s^1, s^2, s^3) \equiv (s^4, s^0, \mathbf{s})$. This alternative description is given by

$$\begin{aligned}
G_\Lambda^{3+1} &= \left\{ M \in \mathrm{GL}(5,\mathbb{R}) : \ M^T \mathbf{I}_\Lambda M = \mathbf{I}_\Lambda \right\}, \\
\mathfrak{g}_\Lambda^{3+1} &= \left\{ M \in \mathrm{Mat}(5,\mathbb{R}) : \ M^T \mathbf{I}_\Lambda + \mathbf{I}_\Lambda M = 0 \right\},
\end{aligned}$$

where M^T denotes the transpose of M (note that these expressions are just the ones defining the indefinite orthogonal groups in Example 2.1).

Here, the origin of the spacetime has ambient coordinates $O = (1, 0, 0, 0, 0)$ and is invariant under the Lorentz subgroup $L < G_\Lambda^{3+1}$ given by (2.89). The orbit passing through O corresponds to the (3+1) homogeneous spacetime which is contained in the pseudosphere (see Theorem 2.2)

$$\Sigma_\Lambda \equiv (s^4)^2 - \Lambda\left((s^0)^2 - \mathbf{s}^2\right) = 1, \tag{2.96}$$

determined by $\mathbf{I}_\Lambda$ (2.95). Note that the in the limit $\Lambda \to 0$ ($\tau \to \infty$), which corresponds to the contraction to the Minkowski space, the pseudosphere Σ_Λ gives rise to two hyperplanes, which are characterised by the condition $s^4 = \pm 1$. From now on we will identify the Minkowski space with the hyperplane given by $s^4 = +1$, thus containing the origin O.

The metric on the homogeneous spacetime is obtained from the flat ambient metric (given by $\mathbf{I}_\Lambda$) and dividing it by the sectional curvature (which equals $-\Lambda$, see below) and restricting the resulting metric to the pseudosphere Σ_Λ:

$$\begin{aligned} \mathrm{d}\sigma^2 &= -\frac{1}{\Lambda}\left((\mathrm{d}s^4)^2 - \Lambda\big((\mathrm{d}s^0)^2 - (\mathrm{d}s^1)^2 - (\mathrm{d}s^2)^2 - (\mathrm{d}s^3)^2\big)\right)\Big|_{\Sigma_\Lambda} \\ &= (\mathrm{d}s^0)^2 - (\mathrm{d}s^1)^2 - (\mathrm{d}s^2)^2 - (\mathrm{d}s^3)^2 - \Lambda\,\frac{(s^0\mathrm{d}s^0 - s^1\mathrm{d}s^1 - s^2\mathrm{d}s^2 - s^3\mathrm{d}s^3)^2}{1 + \Lambda((s^0)^2 - \mathbf{s}^2)}. \end{aligned} \tag{2.97}$$

Now let us introduce four *intrinsic* spacetime coordinates that will be helpful in the sequel: these are the so-called *geodesic parallel coordinates* (x^0, x^1, x^2, x^3) [189], which can be regarded as a generalization of the flat Cartesian coordinates to non-vanishing curvature. They are defined in terms of the action of the one-parameter subgroups (2.94) for P_0, $\mathbf{P}$ on the origin $O = (1, 0, 0, 0, 0)$ of the spacetime:

$$(s^4, s^0, \mathbf{s})^T = \exp(x^0 P_0)\exp(x^1 P_1)\exp(x^2 P_2)\exp(x^3 P_3) O^T,$$

yielding

$$\begin{aligned} s^4 &= \cos\eta x^0 \cosh\eta x^1 \cosh\eta x^2 \cosh\eta x^3, \\ s^0 &= \frac{\sin\eta x^0}{\eta}\cosh\eta x^1 \cosh\eta x^2 \cosh\eta x^3, \\ s^1 &= \frac{\sinh\eta x^1}{\eta}\cosh\eta x^2 \cosh\eta x^3, \\ s^2 &= \frac{\sinh\eta x^2}{\eta}\cosh\eta x^3, \\ s^3 &= \frac{\sinh\eta x^3}{\eta}. \end{aligned} \tag{2.98}$$

The geometrical meaning of the coordinates $(x_0, \mathbf{x})$ that parametrize a generic point Q in the spacetime via (2.98) is as follows. Let l_0 a time-like geodesic and l_1, l_2, l_3 three space-like geodesics such that these four basis geodesics are orthogonal at O. Then x^0 is the geodesic distance from O up to a point Q_1 measured along the time-like geodesic l_0;

x^1 is the geodesic distance between Q_1 and another point Q_2 along a space-like geodesic l_1' orthogonal to l_0 through Q_1 and parallel to l_1; x^2 is the geodesic distance between Q_2 and another point Q_3 along a space-like geodesic l_2' orthogonal to l_1' through Q_2 and parallel to l_2; and x^2 is the geodesic distance between Q_3 and Q along a space-like geodesic l_3' orthogonal to l_2' through Q_3 and parallel to l_3.

Recall that time-like geodesics (as l_0) are compact in $\mathbf{AdS}^{3+1}$ and non-compact in $\mathbf{dS}^{3+1}$, while space-like ones (as l_i, l_i'; $i = 1, 2, 3$) are compact in $\mathbf{dS}^{3+1}$ but non-compact in $\mathbf{AdS}^{3+1}$. Thus the trigonometric functions depending on x^0 are circular in $\mathbf{AdS}^{3+1}$ ($\eta = 1/\tau$) and hyperbolic in $\mathbf{dS}^{3+1}$ ($\eta = \mathrm{i}/\tau$) and, conversely, those depending on x^a are circular in $\mathbf{dS}^{3+1}$ but hyperbolic in $\mathbf{AdS}^{3+1}$. By inserting the parametrisation (2.98) into the metric (2.97) we obtain the corresponding expression in terms of geodesic parallel coordinates

$$\begin{aligned} \mathrm{d}\sigma^2 =& \cosh^2(\eta x^1)\cosh^2(\eta x^2)\cosh^2(\eta x^3)(\mathrm{d}x^0)^2 \\ & - \cosh^2(\eta x^2)\cosh^2(\eta x^3)(\mathrm{d}x^1)^2 - \cosh^2(\eta x^3)(\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\,. \end{aligned} \tag{2.99}$$

For $\Lambda \in \{\pm 1, 0\}$ this expression reduces to

$$\begin{aligned} \mathbf{AdS}^{3+1} : \mathrm{d}\sigma^2 =& \cosh^2 x^1 \cosh^2 x^2 \cosh^2 x^3\,(\mathrm{d}x^0)^2 - \cosh^2 x^2 \cosh^2 x^3\,(\mathrm{d}x^1)^2 \\ & - \cosh^2 x^3\,(\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\,. \\ \mathbf{M}^{3+1} : \mathrm{d}\sigma^2 =& (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\,. \\ \mathbf{dS}^{3+1} : \mathrm{d}\sigma^2 =& \cos^2 x^1 \cos^2 x^2 \cos^2 x^3\,(\mathrm{d}x^0)^2 - \cos^2 x^2 \cos^2 x^3\,(\mathrm{d}x^1)^2 \\ & - \cos^2 x^3\,(\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\,. \end{aligned} \tag{2.100}$$

This approach is such that the projection to lower dimensions is straightforward, just by setting the relevant coordinates to zero. Also, it should be stressed that the limit of vanishing cosmological constant $\Lambda \to 0$ is always well-defined and allows us to recover the results for the Poincaré group and Minkowski spacetime.

2.3.2 Geometric properties of Lorentzian spaces

As we have previously stated, $\mathbf{M}^{3+1}$, $\mathbf{dS}^{3+1}$ and $\mathbf{AdS}^{3+1}$ are homogeneous symmetric spaces, so we can apply the results from §2.1.7 to these particular examples.

First of all, let us study the canonical connection (see Theorem 2.7) on these spaces. We know that, in addition to be torsion free, it satisfies that $\nabla R = 0$. Moreover it is complete and its geodesics are one-parameter subgroups (2.94) (and Lie group translates of them). Its Riemann tensor can be computed at the identity by v) of Theorem 2.7 (it is just given by $R(P_\alpha, P_\beta)P_\gamma = -[[P_\alpha, P_\beta], P_\gamma]$), and we have that

$$\begin{aligned} R(P_0, P_a)P_0 &= -[[P_0, P_a], P_0] = \Lambda P_a, \\ R(P_0, P_a)P_b &= -[[P_0, P_a], P_b] = \Lambda \delta_{ab} P_0, \\ R(P_a, P_b)P_0 &= -[[P_a, P_b], P_0] = 0, \\ R(P_a, P_b)P_c &= -[[P_a, P_b], P_c] = -\Lambda \epsilon_{abd}\epsilon_{dce} P_e. \end{aligned} \tag{2.101}$$

Note that the Riemann tensor is identically zero in the vanishing-constant (flat) case $\Lambda = 0$.

In order to study the metric properties of these spacetimes, let us now consider the symmetric bilinear form induced by the Killing-Cartan form on $\mathfrak{g}$ in the kinematical basis (2.72), whose non-zero components are given by

$$\begin{aligned} \langle P_0, P_0 \rangle &= 1, \qquad \langle P_a, P_a \rangle = -1, \\ \langle J_a, J_a \rangle &= -\Lambda, \qquad \langle K_a, K_a \rangle = \Lambda. \end{aligned} \tag{2.102}$$

Note that this is just the Casimir (2.73) seen as a bilinear form on $\mathfrak{g}$, induced by the identification $\mathfrak{g} \simeq \mathfrak{g}^*$. By Theorem 2.9 this bilinear metric defines a Lorentzian metric on $\mathbf{dS}^{3+1}$ and $\mathbf{AdS}^{3+1}$. Moreover, although the Poincaré group G_0^{3+1} is not semi-simple and the result of the theorem cannot directly be used, since $\langle \cdot, \cdot \rangle$ is Ad_L-invariant and its restriction $\langle \cdot, \cdot \rangle_{\mathfrak{t}}$ is non-degenerate, then it also defines a Lorentzian metric on $\mathbf{M}^{3+1}$ by Proposition 2.4. This bilinear form is the one defining the metric g (2.99). We recall that the explicit construction of metrics induced by the Killing-Cartan form, and their relations with Casmirs of their respective Lie algebras, is studied for the complete family of Cayley-Klein spaces in [189, 187, 190], by following a contraction procedure.

On the other hand, by Theorem 2.8 the metric g associated to (2.102) induces the canonical connection on M_Λ^{3+1}, in the sense that the canonical connection is the only metric compatible connection for g (the so-called Levi-Civita connection). Let us now compute all these geometric properties in the geodesic parallel local coordinates (x^0, x^1, x^2, x^3) previously defined. In order to do that, we need to introduce some notation. The *Christoffel symbols of the second kind* $\Gamma^\alpha{}_{\beta\gamma}$ are defined as the unique coefficients satisfying

$$\nabla_\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \sum_{\gamma=0}^{3} \Gamma^\gamma{}_{\alpha\beta} \left(\frac{\partial}{\partial x^\gamma} \right). \tag{2.103}$$

If we write the metric g in terms of these local coordinates, set $g = g_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$ and denote by $g^{\alpha\beta}$ the inverse of the metric components, then we have that $\Gamma^\alpha{}_{\beta\gamma}$ are given by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} \sum_{\delta=0}^{3} g^{\alpha\delta} \left(\frac{\partial g_{\beta\delta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right). \tag{2.104}$$

In terms of the Christoffel symbols of the second kind $\Gamma^\alpha{}_{\beta\gamma}$, the components of the Riemann tensor are given by

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\delta\beta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\gamma\beta}}{\partial x^\delta} + \sum_{\epsilon=0}^{3} \left(\Gamma^\epsilon{}_{\delta\beta} \Gamma^\alpha{}_{\gamma\epsilon} - \Gamma^\epsilon{}_{\gamma\beta} \Gamma^\alpha{}_{\delta\epsilon} \right). \tag{2.105}$$

Note that the components of the Riemann tensor are related to the corresponding basis of $\mathfrak{t} = \operatorname{span}\{P_0, \mathbf{P}\}$ through

$$\sum_{\alpha=0}^{3} R^\alpha{}_{\beta\gamma\delta} P_\alpha = R(P_\gamma, P_\delta) P_\beta. \tag{2.106}$$

We also recall that the components of the *Riemann tensor* have the following symmetries

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} &= -R^{\alpha}{}_{\beta\delta\gamma}, \\ R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\delta\beta\gamma} &+ R^{\alpha}{}_{\gamma\delta\beta} = 0. \end{aligned} \tag{2.107}$$

We also define $R_{\ \alpha\beta\gamma\delta} = \sum_{\epsilon=0}^{3} g_{\alpha\epsilon} R^{\epsilon}{}_{\beta\gamma\delta}$. The components of the *Ricci tensor* $R_{\alpha\beta}$ are given by

$$R_{\alpha\beta} = \sum_{\gamma=0}^{3} R^{\gamma}{}_{\alpha\gamma\beta}, \tag{2.108}$$

while the *scalar curvature* R is defined as

$$R = \sum_{\alpha,\beta=0}^{3} g^{\alpha\beta} R_{\alpha\beta}. \tag{2.109}$$

Also, the sectional curvature of the coordinated plane $\alpha\beta$ is given by

$$K(\alpha\beta) = \frac{\sum_{\gamma=0}^{3} g_{\alpha\gamma} R^{\gamma}{}_{\beta\alpha\beta}}{g_{\alpha\alpha} g_{\beta\beta} - g_{\alpha\beta} g_{\beta\alpha}}. \tag{2.110}$$

In the local coordinates (x^0, x^1, x^2, x^3), the non-vanishing Christoffel symbols [191] of the second kind (2.104) are given by

$$\begin{aligned}
&\Gamma^{0}{}_{01} = \eta \tanh(\eta x^1), \\
&\Gamma^{0}{}_{02} = \Gamma^{1}{}_{12} = \eta \tanh(\eta x^2), \\
&\Gamma^{0}{}_{03} = \Gamma^{1}{}_{13} = \Gamma^{2}{}_{23} = \eta \tanh(\eta x^3), \\
&\Gamma^{1}{}_{00} = \eta \tanh(\eta x^1) \cosh^2(\eta x^1), \\
&\Gamma^{2}{}_{00} = \eta \tanh(\eta x^2) \cosh^2(\eta x^1) \cosh^2(\eta x^2), \\
&\Gamma^{3}{}_{00} = \eta \tanh(\eta x^3) \cosh^2(\eta x^1) \cosh^2(\eta x^2) \cosh^2(\eta x^3), \\
&\Gamma^{2}{}_{11} = -\eta \tanh(\eta x^2) \cosh^2(\eta x^3), \\
&\Gamma^{3}{}_{11} = -\eta \tanh(\eta x^3) \cosh^2(\eta x^2) \cosh^2(\eta x^3), \\
&\Gamma^{3}{}_{22} = -\eta \tanh(\eta x^3) \cosh^2(\eta x^3),
\end{aligned} \tag{2.111}$$

while the non-vanishing components of the Riemann tensor (2.105) read

$$\begin{aligned}
&R^{1}{}_{010} = R^{2}{}_{020} = R^{3}{}_{030} = \eta^2 \cosh^2(\eta x^1) \cosh^2(\eta x^2) \cosh^2(\eta x^3), \\
&R^{2}{}_{121} = R^{3}{}_{131} = -\eta^2 \cosh^2(\eta x^2) \cosh^2(\eta x^3), \\
&R^{3}{}_{232} = -\eta^2 \cosh^2(\eta x^3), \\
&R^{0}{}_{101} = R^{0}{}_{202} = -\eta^2 \cosh^2(\eta x^2) \cosh^2(\eta x^3), \\
&R^{1}{}_{212} = -\eta^2 \cosh^2(\eta x^3), \\
&R^{0}{}_{303} = R^{1}{}_{313} = R^{2}{}_{323} = -\eta^2.
\end{aligned} \tag{2.112}$$

Note that evaluating these components at the origin of M_Λ, we recover (2.101). The Ricci tensor is diagonal, with components

$$\begin{aligned}
R_{00} &= 3\eta^2\cosh^2(\eta x^1)\cosh^2(\eta x^2)\cosh^2(\eta x^3),\\
R_{11} &= -3\eta^2\cosh^2(\eta x^2)\cosh^2(\eta x^3),\\
R_{22} &= -3\eta^2\cosh^2(\eta x^3),\\
R_{33} &= -3\eta^2,
\end{aligned} \tag{2.113}$$

while the scalar and sectional curvatures of the (3+1) spacetime are given by $R = 12\eta^2 = -12\Lambda$ and $K(ab) = \eta^2 = -\Lambda$, respectively.

2.3.3 (2+1) dimensional Lorentzian spaces

In $(2+1)$ dimensions, the explicit cosets are:

- $\Lambda < 0:$ Anti-de Sitter spacetime $\mathbf{AdS}^{2+1} = SO(2,2)/SO(2,1)$.
- $\Lambda = 0:$ Minkowski spacetime $\mathbf{M}^{2+1} = ISO(2,1)/SO(2,1)$.
- $\Lambda > 0:$ de Sitter spacetime $\mathbf{dS}^{2+1} = SO(3,1)/SO(2,1)$.

In this dimension, we have the following homeomorphisms: $\mathbf{AdS}^{2+1} \simeq S^1\times\mathbb{R}^2$, $\mathbf{M}^{2+1}\simeq\mathbb{R}^3$ and $\mathbf{dS}^{2+1}\simeq\mathbb{R}\times S^2$, which clearly show how these three spaces are different even from a purely topological point of view. The picture is qualitatively similar to the $(3+1)$-dimensional case, so we do not comment further on this.

In the (2+1) dimensional case the matrices G^{2+1}_Λ and G^{2+1}_0 are obtained from (2.91) and (2.92), respectively, just by setting $x^3=0$, $\xi^3=0$, $\theta^1=0$, $\theta^2=0$ and $\theta^3=\theta$. It is easy to see that the last column and last row vanish. Explicitly we have

$$G^{2+1}_\Lambda = \begin{pmatrix} s^4 & A^4_0 & A^4_1 & A^4_2\\ s^0 & B^0_0 & B^0_1 & B^0_2\\ s^1 & B^1_0 & B^1_1 & B^1_2\\ s^2 & B^2_0 & B^2_1 & B^2_2 \end{pmatrix} \tag{2.114}$$

where

$$\begin{aligned}
s^4 &= \cos\eta x^0\cosh\eta x^1\cosh\eta x^2,\\
s^0 &= \frac{\sin\eta x^0}{\eta}\cosh\eta x^1\cosh\eta x^2,\\
s^1 &= \frac{\sinh\eta x^1}{\eta}\cosh\eta x^2,\\
s^2 &= \frac{\sinh\eta x^2}{\eta},
\end{aligned} \tag{2.115}$$

$$
\begin{aligned}
A_0^4 =& \cos(\eta x^0)(\sinh\xi^1\cosh\xi^2\sinh(\eta x^1)+\sinh\xi^2\cosh(\eta x^1)\sinh(\eta x^2))-\cosh\xi^1\cosh\xi^2\sin(\eta x^0),\\
A_1^4 =& -\cos\theta\sinh\xi^1\sin(\eta x^0)+\cosh\xi^1(\cos\theta\cos(\eta x^0)\sinh(\eta x^1)-\sin\theta\sinh(\xi^2)\sin(\eta x^0))\\
&+\sin\theta\cos(\eta x^0)(\sinh\xi^1\sinh\xi^2\sinh(\eta x^1)+\cosh\xi^2\cosh(\eta x^1)\sinh(\eta x^2)),\\
A_2^4 =& \sin\theta\sinh\xi^1\sin(\eta x^0)-\cosh\xi^1(\cos\theta\sinh\xi^2\sin(\eta x^0)+\sin\theta\cos(\eta x^0)\sinh(\eta x^1))\\
&+\cos\theta\cos(\eta x^0)(\sinh\xi^1\sinh\xi^2\sinh(\eta x^1)+\cosh\xi^2\cosh(\eta x^1)\sinh(\eta x^2)),
\end{aligned}
\tag{2.116}
$$

$$
\begin{aligned}
B_0^0 =& \cosh\xi^1\cosh\xi^2\cos(\eta x^0)+\sin(\eta x^0)(\sinh\xi^1\cosh\xi^2\sinh(\eta x^1)+\sinh\xi^2\cosh(\eta x^1)\sinh(\eta x^2)),\\
B_1^0 =& \cos\theta\sinh\xi^1\cos(\eta x^0)+\cosh\xi^1(\sin\theta\sinh\xi^2\cos(\eta x^0)+\cos\theta\sin(\eta x^0)\sinh(\eta x^1)),\\
&+\sin\theta\sin(\eta x^0)(\sinh\xi^1\sinh\xi^2\sinh(\eta x^1)+\cosh\xi^2\cosh(\eta x^1)\sinh(\eta x^2)),\\
B_2^0 =& \cos\theta(\cosh\xi^1\sinh\xi^2\cos(\eta x^0)+\sin(\eta x^0)(\sinh\xi^1\sinh\xi^2\sinh(\eta x^1)+\cosh\xi^2\cosh(\eta x^1)\sinh(\eta x^2))),\\
&-\sin\theta(\sinh\xi^1\cos(\eta x^0)+\cosh\xi^1\sin(\eta x^0)\sinh(\eta x^1)),\\
B_0^1 =& \sinh\xi^1\cosh\xi^2\cosh(\eta x^1)+\sinh\xi^2\sinh(\eta x^1)\sinh(\eta x^2),\\
B_1^1 =& \cosh(\eta x^1)(\sin\theta\sinh\xi^1\sinh\xi^2+\cos\theta\cosh\xi^1)+\sin\theta\cosh\xi^2\sinh(\eta x^1)\sinh(\eta x^2),\\
B_2^1 =& \cosh(\eta x^1)(\cos\theta\sinh\xi^1\sinh\xi^2-\sin\theta\cosh\xi^1)+\cos\theta\cosh\xi^2\sinh(\eta x^1)\sinh(\eta x^2),\\
B_0^2 =& \sinh\xi^2\cosh(\eta x^2),\\
B_1^2 =& \sin\theta\cosh\xi^2\cosh(\eta x^2),\\
B_2^2 =& \cos\theta\cosh\xi^2\cosh(\eta x^2).
\end{aligned}
\tag{2.117}
$$

The metric in these coordinates reads

$$
\mathrm{d}\sigma^2=\cosh^2(\eta x^1)\cosh^2(\eta x^2)(\mathrm{d}x^0)^2-\cosh^2(\eta x^2)(\mathrm{d}x^1)^2-(\mathrm{d}x^2)^2, \tag{2.118}
$$

which is just (2.99) with the relevant coordinates set equal to zero. The left- and right-invariant vector fields are given in Table 2.1.

For the Poincaré group G_0^{2+1} these expressions are much simpler, since the matrix element reads

$$
G_0^{2+1}=\begin{pmatrix}
1 & 0 & 0 & 0\\
x^0 & \cosh\xi^1\cosh\xi^2 & \sinh\xi^1\cos\theta+\cosh\xi^1\sinh\xi^2\sin\theta & -\sinh\xi^1\sin\theta+\cosh\xi^1\sinh\xi^2\cos\theta\\
x^1 & \sinh\xi^1\cosh\xi^2 & \cosh\xi^1\cos\theta+\sinh\xi^1\sinh\xi^2\sin\theta & -\cosh\xi^1\sin\theta+\sinh\xi^1\sinh\xi^2\cos\theta\\
x^2 & \sinh\xi^2 & \cosh\xi^2\sin\theta & \cosh\xi^2\cos\theta
\end{pmatrix}. \tag{2.119}
$$

and the left- and right-invariant vector fields are given in Table 2.2. The Lorentz subgroup reads

$$
L^{2+1}=\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & \cosh\xi^1\cosh\xi^2 & \sinh\xi^1\cos\theta+\cosh\xi^1\sinh\xi^2\sin\theta & -\sinh\xi^1\sin\theta+\cosh\xi^1\sinh\xi^2\cos\theta\\
0 & \sinh\xi^1\cosh\xi^2 & \cosh\xi^1\cos\theta+\sinh\xi^1\sinh\xi^2\sin\theta & -\cosh\xi^1\sin\theta+\sinh\xi^1\sinh\xi^2\cos\theta\\
0 & \sinh\xi^2 & \cosh\xi^2\sin\theta & \cosh\xi^2\cos\theta
\end{pmatrix}. \tag{2.120}
$$

2.3.4 (1+1) dimensional Lorentzian spaces

Finally, in $(1+1)$ dimensions we have the following spaces, constructed as cosets:

Table 2.1: [148] Left- and right-invariant vector fields for the isometry groups of the (2+1)-dimensional de Sitter ($\Lambda > 0$), anti-de Sitter ($\Lambda < 0$) and Minkowski ($\Lambda = 0$) spaces in terms of $\eta = \sqrt{-\Lambda}$.

$$X^L_{P_0} = \frac{\cosh\xi^1\cosh\xi^2}{\cosh(\eta x^1)\cosh(\eta x^2)}\left(\partial_{x^0} - \eta\sinh(\eta x^1)\partial_{\xi^1}\right) + \frac{\sinh\xi^1\cosh\xi^2}{\cosh(\eta x^2)}\,\partial_{x^1} + \sinh\xi^2\,\partial_{x^2} - \eta\tanh(\eta x^2)\cosh\xi^2\,\partial_{\xi^2}$$

$$\begin{aligned} X^L_{P_1} = {} & \left(\frac{\sinh\xi^1\cos\theta + \cosh\xi^1\sinh\xi^2\sin\theta}{\cosh(\eta x^1)\cosh(\eta x^2)}\right)\left(\partial_{x^0} - \eta\sinh(\eta x^1)\partial_{\xi^1}\right) + \left(\frac{\cosh\xi^1\cos\theta + \sinh\xi^1\sinh\xi^2\sin\theta}{\cosh(\eta x^2)}\right)\partial_{x^1} \\ & + \cosh\xi^2\sin\theta\,\partial_{x^2} - \eta\tanh(\eta x^2)\left(\tanh\xi^2\cos\theta\,\partial_{\xi^1} + \sinh\xi^2\sin\theta\,\partial_{\xi^2} - \frac{\cos\theta}{\cosh\xi^2}\,\partial_\theta\right) \end{aligned}$$

$$\begin{aligned} X^L_{P_2} = {} & \left(\frac{\cosh\xi^1\sinh\xi^2\cos\theta - \sinh\xi^1\sin\theta}{\cosh(\eta x^1)\cosh(\eta x^2)}\right)\left(\partial_{x^0} - \eta\sinh(\eta x^1)\partial_{\xi^1}\right) + \left(\frac{\sinh\xi^1\sinh\xi^2\cos\theta - \cosh\xi^1\sin\theta}{\cosh(\eta x^2)}\right)\partial_{x^1} \\ & + \cosh\xi^2\cos\theta\,\partial_{x^2} + \eta\tanh(\eta x^2)\left(\tanh\xi^2\sin\theta\,\partial_{\xi^1} - \sinh\xi^2\cos\theta\,\partial_{\xi^2} - \frac{\sin\theta}{\cosh\xi^2}\,\partial_\theta\right) \end{aligned}$$

$$X^L_{K_1} = \frac{\cos\theta}{\cosh\xi^2}\,\partial_{\xi^1} + \sin\theta\,\partial_{\xi^2} + \tanh\xi^2\cos\theta\,\partial_\theta$$

$$X^L_{K_2} = -\frac{\sin\theta}{\cosh\xi^2}\,\partial_{\xi^1} + \cos\theta\,\partial_{\xi^2} - \tanh\xi^2\sin\theta\,\partial_\theta$$

$$X^L_J = \partial_\theta$$

$$X^R_{P_0} = \partial_{x^0}$$

$$X^R_{P_1} = -\sin(\eta x^0)\tanh(\eta x^1)\,\partial_{x^0} + \cos(\eta x^0)\,\partial_{x^1} - \eta\,\frac{\sin(\eta x^0)}{\cosh(\eta x^1)}\,\partial_{\xi^1}$$

$$\begin{aligned} X^R_{P_2} = {} & -\frac{\sin(\eta x^0)\tanh(\eta x^2)}{\cosh(\eta x^1)}\left(\partial_{x^0} - \eta\sinh(\eta x^1)\,\partial_{\xi^1}\right) - \cos(\eta x^0)\sinh(\eta x^1)\tanh(\eta x^2)\,\partial_{x^1} + \cos(\eta x^0)\cosh(\eta x^1)\,\partial_{x^2} \\ & + \eta\left(\frac{\cos(\eta x^0)\sinh(\eta x^1)\sinh\xi^1 - \sin(\eta x^0)\cosh\xi^1}{\cosh(\eta x^2)}\right)\partial_{\xi^2} \\ & + \eta\left(\frac{\cos(\eta x^0)\sinh(\eta x^1)\cosh\xi^1 - \sin(\eta x^0)\sinh\xi^1}{\cosh(\eta x^2)\cosh\xi^2}\right)\left(\partial_\theta - \sinh\xi^2\partial_{\xi^1}\right) \end{aligned}$$

$$X^R_{K_1} = \frac{\cos(\eta x^0)\tanh(\eta x^1)}{\eta}\,\partial_{x^0} + \frac{\sin(\eta x^0)}{\eta}\,\partial_{x^1} + \frac{\cos(\eta x^0)}{\cosh(\eta x^1)}\,\partial_{\xi^1}$$

$$\begin{aligned} X^R_{K_2} = {} & \frac{\cos(\eta x^0)\tanh(\eta x^2)}{\eta\cosh(\eta x^1)}\left(\partial_{x^0} - \eta\sinh(\eta x^1)\,\partial_{\xi^1}\right) - \frac{\sin(\eta x^0)\sinh(\eta x^1)\tanh(\eta x^2)}{\eta}\,\partial_{x^1} + \frac{\sin(\eta x^0)\cosh(\eta x^1)}{\eta}\,\partial_{x^2} \\ & + \left(\frac{\sin(\eta x^0)\sinh(\eta x^1)\sinh\xi^1 + \cos(\eta x^0)\cosh\xi^1}{\cosh(\eta x^2)}\right)\partial_{\xi^2} \\ & + \left(\frac{\sin(\eta x^0)\sinh(\eta x^1)\cosh\xi^1 + \cos(\eta x^0)\sinh\xi^1}{\cosh(\eta x^2)\cosh\xi^2}\right)\left(\partial_\theta - \sinh\xi^2\partial_{\xi^1}\right) \end{aligned}$$

$$X^R_J = -\frac{\cosh(\eta x^1)\tanh(\eta x^2)}{\eta}\,\partial_{x^1} + \frac{\sinh(\eta x^1)}{\eta}\,\partial_{x^2} - \frac{\cosh(\eta x^1)}{\cosh(\eta x^2)}\left(\cosh\xi^1\tanh\xi^2\,\partial_{\xi^1} - \sinh\xi^1\,\partial_{\xi^2} - \frac{\cosh\xi^1}{\cosh\xi^2}\,\partial_\theta\right)$$

Table 2.2: Left- and right-invariant vector fields for the (2+1) Poincaré group G_0^{2+1}.

$$X^L_{P_0} = \cosh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \sinh\xi^2\partial_{x^2}$$

$$X^L_{P_1} = \cos\theta\left(\sinh\xi^1\partial_{x^0} + \cosh\xi^1\partial_{x^1}\right) + \sin\theta\left(\sinh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \cosh\xi^2\partial_{x^2}\right)$$

$$X^L_{P_2} = -\sin\theta\left(\sinh\xi^1\partial_{x^0} + \cosh\xi^1\partial_{x^1}\right) + \cos\theta\left(\sinh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \cosh\xi^2\partial_{x^2}\right)$$

$$X^L_{K_1} = \frac{\cos\theta}{\cosh\xi^2}\left(\partial_{\xi^1} + \sinh\xi^2\,\partial_\theta\right) + \sin\theta\,\partial_{\xi^2}$$

$$X^L_{K_2} = -\frac{\sin\theta}{\cosh\xi^2}\left(\partial_{\xi^1} + \sinh\xi^2\,\partial_\theta\right) + \cos\theta\,\partial_{\xi^2}$$

$$X^L_J = \partial_\theta$$

$$X^R_{P_0} = \partial_{x^0}$$

$$X^R_{P_1} = \partial_{x^1}$$

$$X^R_{P_2} = \partial_{x^2}$$

$$X^R_{K_1} = x^1\,\partial_{x^0} + x^0\,\partial_{x^1} + \partial_{\xi^1}$$

$$X^R_{K_2} = x^2\partial_{x^0} + x^0\partial_{x^2} + \frac{\sinh\xi^1}{\cosh\xi^2}\left(-\sinh\xi^2\partial_{\xi^1} + \partial_\theta\right) + \cosh\xi^1\partial_{\xi^2}$$

$$X^R_J = -x^2\partial_{x^1} + x^1\partial_{x^2} + \frac{\cosh\xi^1}{\cosh\xi^2}\left(\partial_\theta - \sinh\xi^2\partial_{\xi^1}\right) + \sinh\xi^1\partial_{\xi^2}$$

- $\Lambda < 0$: Anti-de Sitter spacetime $\mathbf{AdS}^{1+1} = SO(2,1)/SO(1,1)$.
- $\Lambda = 0$: Minkowski spacetime $\mathbf{M}^{1+1} = ISO(1,1)/SO(1,1)$.
- $\Lambda > 0$: de Sitter spacetime $\mathbf{dS}^{1+1} = SO(2,1)/SO(1,1)$.

In this low dimensional case $\mathbf{AdS}^{1+1}$ is diffeomorphic to $\mathbf{dS}^{1+1}$, and neither of them is simply-connected, in fact $\pi_1(\mathbf{dS}^{1+1}) = \pi_1(\mathbf{AdS}^{1+1}) \simeq \mathbb{Z}$. However, the causal structure is different, because for $\mathbf{dS}^{1+1}$ the generator of π_1 is space-related while for $\mathbf{AdS}^{1+1}$ is time-related. Topologically we have that $\mathbf{AdS}^{1+1} \simeq \mathbf{dS}^{1+1} \simeq \mathbb{R} \times S$, while $\mathbf{M}^{1+1} \simeq \mathbb{R}^2$. The group element now reads

$$G_\Lambda^{2+1} = \begin{pmatrix} s^4 & A_0^4 & A_1^4 \\ s^0 & B_0^0 & B_1^0 \\ s^1 & B_0^1 & B_1^1 \end{pmatrix} \tag{2.121}$$

where

$$\begin{aligned} s^4 &= \cos\eta x^0 \cosh\eta x^1, \\ s^0 &= \frac{\sin\eta x^0}{\eta}\cosh\eta x^1, \\ s^1 &= \frac{\sinh\eta x^1}{\eta}, \end{aligned} \tag{2.122}$$

$$\begin{aligned} A_0^4 =& \eta\left(-\sin(\eta x^0)\cosh\xi + \cos\eta x^0 \sinh\eta x^1 \sinh\xi\right), \\ A_1^4 =& \eta\left(-\sin(\eta x^0)\sinh\xi + \cos\eta x^0 \sinh\eta x^1 \cosh\xi\right), \end{aligned} \tag{2.123}$$

$$\begin{aligned} B_0^0 &= \cos\eta x^0 \cosh\xi + \sin\eta x^0 \sinh\eta x^1 \sinh\xi, \\ B_1^0 &= \cos\eta x^0 \sinh\xi + \sin\eta x^0 \sinh\eta x^1 \cosh\xi, \\ B_0^1 &= \cosh\eta x^1 \sinh\xi, \\ B_1^1 &= \cosh\eta x^1 \cosh\xi. \end{aligned} \tag{2.124}$$

Left- and right-invariant vector fields in these coordinates are obtained from (2.121) and are given in Table 2.3.

For the Poincaré group G_0^{1+1} the group element takes the simple form

$$G_0^{1+1} = \begin{pmatrix} 1 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi \\ x^1 & \sinh\xi & \cosh\xi \end{pmatrix}, \tag{2.125}$$

and the explicit left- and right-invariant vector fields are given in Table 2.4. The Lorentz subgroup is just

$$L^{1+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\xi & \sinh\xi \\ 0 & \sinh\xi & \cosh\xi \end{pmatrix}. \tag{2.126}$$

Table 2.3: [148] Left- and right-invariant vector fields for the isometry groups of the (1+1)-dimensional de Sitter ($\Lambda > 0$), anti-de Sitter ($\Lambda < 0$) and Minkowski ($\Lambda = 0$) spaces in terms of $\eta = \sqrt{-\Lambda}$.

$$X^L_{P_0} = \frac{1}{\cosh(\eta x^1)}\Big(\cosh\xi\,\partial_{x^0} + \cosh(\eta x^1)\sinh\xi\,\partial_{x^1} - \eta\sinh(\eta x^1)\cosh\xi\,\partial_\xi\Big)$$

$$X^L_{P_1} = \frac{1}{\cosh(\eta x^1)}\Big(\sinh\xi\,\partial_{x^0} + \cosh(\eta x^1)\cosh\xi\,\partial_{x^1} - \eta\sinh(\eta x^1)\sinh\xi\,\partial_\xi\Big)$$

$$X^L_K = \partial_\xi$$

$$X^R_{P_0} = \partial_{x^0}$$

$$X^R_{P_1} = \frac{1}{\cosh(\eta x^1)}\Big(-\sin(\eta x^0)\sinh(\eta x^1)\,\partial_{x^0} + \cos(\eta x^0)\cosh(\eta x^1)\,\partial_{x^1} - \eta\sin(\eta x^0)\,\partial_\xi\Big)$$

$$X^R_K = \frac{1}{\cosh(\eta x^1)}\left(\frac{\cos(\eta x^0)\sinh(\eta x^1)}{\eta}\,\partial_{x^0} + \frac{\sin(\eta x^0)\cosh(\eta x^1)}{\eta}\,\partial_{x^1} + \cos(\eta x^0)\,\partial_\xi\right)$$

We have at hand all the background and explicit expressions needed for the rest of this Thesis, as far as the definition and geometric properties of the three maximally symmetric Lorentzian spacetimes is concerned. In the next Chapter this background will be extended to the Poisson homogeneous space framework.

Table 2.4: Left- and right-invariant vector fields for the (1+1) Poincaré group G_0^{1+1}.

$X^L_{P_0} = \cosh\xi\,\partial_{x^0} + \sinh\xi\,\partial_{x^1}$
$X^L_{P_1} = \sinh\xi\,\partial_{x^0} + \cosh\xi\,\partial_{x^1}$
$X^L_K = \partial_\xi$
$X^R_{P_0} = \partial_{x^0}$
$X^R_{P_1} = \partial_{x^1}$
$X^R_K = x^1\partial_{x^0} + x^0\partial_{x^1} + \partial_\xi$

Chapter 3

Poisson-Lie groups and Poisson homogeneous spaces

The main aim of this Chapter is to introduce the relevant tools from Poisson geometry that will be needed throughout this Thesis. As a general idea, we could say that we switch to the realm of Poisson geometry and we essentially introduce Poisson structures on the geometric objects considered in the first Chapter. As we will see this 'Poisson version' becomes in many cases richer due to the fact that a given Lie group or homogeneous space admit, in general, several different Poisson structures.

In §3.1 we introduce some basics on Poisson geometry. In §3.2 we consider Poisson-Lie groups and Poisson homogeneous spaces, which can be thought of as the 'Poisson version' of Lie groups and homogeneous spaces, respectively. In §3.3 we introduce the notion of Lie bialgebra as the tangent counterpart of a Poisson-Lie group, and the relationship between Lie bialgebras and the (modified) classical Yang-Baxter equation is studied. In particular, we see how solutions of this equation define the so-called coboundary Poisson-Lie groups, which have canonically defined a Poisson structure, the so-called Sklyanin bracket, on the Lie group. This has the great advantage that, endowed with such a Poisson structure, the Lie group becomes a Poisson-Lie group. In §3.4 classical Drinfel'd doubles are described with some detail, since they are directly connected to Lie bialgebras and play a remarkable role in this Thesis. Once all these concepts have been introduced, we have all the ingredients to study Poisson homogeneous spaces, which are the main topic of this Thesis, and are introduced in §3.5. In particular we present the result by Drinfel'd relating Poisson homogeneous spaces to Lagrangian Lie subalgebras of the classical Drinfel'd double. The main references for Sections §3.1 to §3.5 are [192, 35, 193, 194].

In §3.6, the essentially geometric language employed so far is switched to a more algebraic one, having in mind the idea of quantizing the Poisson structures previously considered. Here we introduce the notions of an algebra (essentially a vector space with a product) and its dual, a coalgebra (in which the product is replaced by a coproduct), together with their natural morphisms. When both algebra and coalgebra structures are defined on the same vector space, we have an algebraic structure called a bialgebra. Then we introduce the notion of Hopf algebra, which is just a bialgebra with some extra structure. We also introduce the notion of (co)action, which is the algebraic concept underlying homogeneous spaces. When these (co)actions respect some predefined Poisson structure, we arrive to the notion of Poisson (co)actions, which can be thought of as the algebraic version of Poisson homogeneous spaces. Section §3.7 the notion of quantization for the algebraic structures introduced in §3.6 is properly defined. For these two Sections, we have followed [195, 35, 34, 196, 197], which contain the background material on these algebraic structures and their quantization.

The last Section §3.8 contains an explicit example of some of the concepts previously introduced, which will hopefully serve as a preliminary illustration of some of the most relevant constructions included within this Thesis. Here we work out the example of the (1+1)-dimensional κ-Minkowski spacetime seen as a Poisson homogeneous space for the Poincaré group, and we compute explicitly the Poisson-Hopf structure on its algebra of functions. Then we show how the coalgebra structure for the associated quantum universal enveloping algebra can be computed by applying the quantum duality principle, which allows us to derive explicitly the unique Poisson-Hopf algebra on the dual group whose linearization is the Lie-Poisson structure on the dual vector space of the original Poincaré Lie algebra. This will be essentially the framework that we will follow in the rest of this Thesis in order to construct (2+1) and (3+1) dimensional Lorentzian Poisson homogeneous spaces and their quantization.

3.1 Poisson geometry

Let M be a real smooth manifold and let us denote its algebra of smooth functions by $\mathcal{C}^\infty(M)$.

Definition 3.1. A *Poisson bracket* on M is an $\mathbb{R}$-bilinear map $\{\,,\,\}: \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ that satisfies the following three properties:

(P1) $\{f_1,f_2\} = -\{f_2,f_1\}$ *(Antisymmetry)*

(P2) $\{f_1,\{f_2,f_3\}\}+\{f_3,\{f_1,f_2\}\}+\{f_2,\{f_3,f_1\}\}=0$ *(Jacobi identity)*

(P3) $\{f_1f_2,f_3\} = f_1\{f_2,f_3\}+\{f_1,f_3\}f_2$ *(Leibniz rule)*

for all $f_1,f_2,f_3\in\mathcal{C}^\infty(M)$. A *Poisson manifold* is a pair $(M,\{\,,\,\})$.

A Poisson bracket can be equivalently defined by a bivector $\pi\in\Gamma(\bigwedge^2 TM)\subset TM\otimes TM$, by means of $\{f_1,f_2\}=(\mathrm{d}f_1\otimes\mathrm{d}f_2)(\pi)$, called the *Poisson bivector*. Therefore, the Poisson manifold $(M,\{\,,\,\})$ is also denoted by (M,π). Moreover, a Poisson manifold can also be defined as a pair (M,π) satisfying that $\pi\in\Gamma(\bigwedge^2 TM)$ and $[\pi,\pi]=0$ (where here $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket extending the Lie bracket of vector fields). The Jacobi identity (P2) is then equivalent to the condition

$$\sum_k\left(\frac{\partial\pi^{ij}}{\partial x^k}\pi^{kl}+\frac{\partial\pi^{jl}}{\partial x^k}\pi^{ki}+\frac{\partial\pi^{li}}{\partial x^k}\pi^{kj}\right)=0 \tag{3.1}$$

for the Poisson bivector π. We say that a *bivector is Poisson* if it satisfies this condition.

Definition 3.2. Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be twoPoisson manifolds. A *Poisson map* is a smooth map $\phi : M \to N$ such that

$$\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi \tag{3.2}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(N)$. An *isomorphism* of Poisson manifolds is a Poisson map that is also a diffeomorphism.

Let (M, π) be a Poisson manifold and let us define some important concepts. A *Poisson submanifold* (N, π_N) is a submanifold $N \subset M$ such that the inclusion mapping $i : N \hookrightarrow M$ is a Poisson mapping for the Poisson structure π on M and π_N on N.

We call $C \in \mathcal{C}^\infty(M)$ a *Casimir function* of a given Poisson bracket if it Poisson-commutes with every function $f \in \mathcal{C}^\infty(M)$, i.e. $\{C, f\} = 0$ for all $f \in \mathcal{C}^\infty(M)$.

For every function $H \in \mathcal{C}^\infty(M)$, we define the *Hamiltonian vector field* X_H associated to H as the unique vector field satisfying $X_H f = \{f, H\}$.

If $x = (x^1, \ldots, x^n)$ are local coordinates on M we have that

$$\{f_1, f_2\} = \sum_{i,j=1}^{n} \pi^{ij}(x) \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j}, \tag{3.3}$$

where $\pi^{ij}(x)$ are the components of the Poisson bivector

$$\pi_x = \sum_{i,j=1}^{n} \pi^{ij}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{3.4}$$

Let (M_1, π_1) and (M_2, π_2) Poisson manifolds. Then $M_1 \times M_2$ is a Poisson manifold with the *product Poisson structure* given by

$$\{f_1, f_2\}_{M_1 \times M_2}(m_1, m_2) = \{f_1(-, m_2), f_2(-, m_2)\}_{M_1}(m_1) + \{f_1(m_1, -), f_2(m_1, -)\}_{M_2}(m_2) \tag{3.5}$$

for all $m_1 \in M_1$ and $m_2 \in M_2$, and where for any function $f : M_1 \times M_2 \to \mathbb{R}$ we define

$$\begin{aligned} f(-, m_2) : M_1 &\to \mathbb{R} \\ m_1 &\to f(m_1, m_2) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f(m_1, -) : M_2 &\to \mathbb{R} \\ m_2 &\to f(m_1, m_2). \end{aligned} \tag{3.7}$$

In terms of the Poisson bivector the product Poisson structure is simply written as $\pi_{M_1 \times M_2} = \pi_{M_1} \oplus \pi_{M_2} \in \Gamma(\bigwedge^2 T(M_1 \times M_2))$.

A Poisson bivector π on M induces a bundle map

$$\sharp_\pi : T^*M \to TM, \tag{3.8}$$

and, by a slight abuse of notation, we also denote by $\sharp_\pi$ the associated map on sections

$$\begin{aligned} \sharp_\pi : \Omega^1(M) &\to \mathfrak{X}(M) \\ \alpha &\to i_\alpha \pi. \end{aligned} \tag{3.9}$$

We call the *rank of a Poisson structure at* $m \in M$ the rank of the linear map $\sharp_{\pi,m} : T^*_m M \to T_m M$.

The following example introduces an important Poisson structure on the dual space of a Lie algebra, which is called a *Lie-Poisson structure*. Since it will be extensively employed in the rest of this Thesis, we shall describe it in detail.

Example 3.1. Let $\mathfrak{g} = (V, [\cdot,\cdot])$ be a real finite dimensional Lie algebra and call $\mathfrak{g}^* = V^*$ the dual vector space. If we consider functions $f : \mathfrak{g}^* \to \mathbb{R}$, we have that for each $x \in \mathfrak{g}^*$ their differentials f_* define maps $f_{*,x} : T_x\mathfrak{g}^* \simeq \mathfrak{g}^* \to T_{f(x)}\mathbb{R} \simeq \mathbb{R}$. Now, if we denote by $\langle \cdot,\cdot \rangle$ the canonical identification between $\mathfrak{g}$ and $\mathfrak{g}^*$, we can define

$$\{f_1, f_2\}(x) = \langle [(f_1)_{*,x}, (f_2)_{*,x}], x \rangle \tag{3.10}$$

for each $x \in \mathfrak{g}^*$. Since $\mathfrak{g}$ is a vector space, if $\{T_1, \ldots, T_n\}$ is a basis of $\mathfrak{g}$, we can also think of T_i as global coordinates on $\mathfrak{g}^*$. If we set $[T_i, T_j] = c^k_{ij} T_k$, then we have that

$$\{f_1, f_2\}(x) = \sum_{i,j,k} c^k_{ij} \langle x, T_k \rangle \frac{\partial f_1}{\partial T_i} \frac{\partial f_2}{\partial T_j} = \sum_{i,j} \pi^{ij}(x) \frac{\partial f_1}{\partial T_i} \frac{\partial f_2}{\partial T_j}. \tag{3.11}$$

The fundamental brackets read

$$\{T_i, T_j\}(x) = \sum_k c^k_{ij} \langle x, T_k \rangle = \sum_k c^k_{ij} T_k(x) \tag{3.12}$$

for all $x \in \mathfrak{g}^*$. Thus, we have that for every real finite dimensional Lie algebra $\mathfrak{g}$ we can construct a Poisson structure on its dual just by replacing the Lie brackets by Poisson brackets. This justifies the notation we employ for this Lie-Poisson structure in $\mathfrak{g}^*$, which we call the *Poisson version of* $\mathfrak{g}$ and we denote by $\mathcal{P}(\mathfrak{g})$.

$\diamondsuit$

Definition 3.3. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a non-degenerate antisymmetric 2-form on M.

It is clear that a symplectic structure ω on M induces a linear isomorphism

$$\begin{aligned} \flat_\omega : \Gamma(TM) &\to \Gamma(T^*M) \\ X &\to i_X \omega. \end{aligned} \tag{3.13}$$

Note that every symplectic manifold is a Poisson manifold with the Poisson structure defined by

$$\{f_1, f_2\} = X_{f_1} f_2 \tag{3.14}$$

where X_{f_1} is the unique vector field defined by $X_{f_1} = i_{X_{f_1}} \omega$. The following theorem completely characterizes symplectic manifolds *locally*.

Theorem 3.1. *[198] **(Darboux theorem)** Let (M,ω) be a $2n$-dimensional symplectic manifold. Then, for all $m \in M$ there exists a neighborhood U of m and local coordinates $(q^1,\dots,q^n,p^1,\dots,p^n)$ on U such that*

$$\omega|_U = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p^i. \tag{3.15}$$

Corollary 3.1. *Symplectic manifolds are even dimensional.*

If the rank of a Poisson manifold equals the dimension of the manifold for every $m \in M$ then it is called *non-degenerate* or *symplectic*, and in fact it is a symplectic manifold with symplectic structure given by $\omega(X_1,X_2) = \pi(\sharp_\pi^{-1}X_1, \sharp_\pi^{-1}X_2)$ for all $X_1,X_2 \in \mathfrak{X}(M)$ (note that this is well-defined since in the case of constant rank the map $\sharp_\pi$ (3.8) is a bundle isomorphism). In the remaining of this section we will describe the local structure of Poisson manifolds. As proved by Weinstein in [199], any Poisson manifold can be seen locally as a product of a symplectic manifold and a degenerate Poisson manifold. Before that, let us introduce the so-called symplectic foliation of a Poisson manifold.

Proposition 3.1. *[35] Let (M,π) be a Poisson manifold, and for every two points $m,n \in M$ consider the relation $m \sim n$ if and only if n can be reached from m by a piecewise smooth curve, each of its segments being the integral curve of a Hamiltonian vector field. Then $\sim$ is an equivalence relation and the equivalence classes $M/\sim$ of $\sim$ are Poisson submanifolds of M. The dimension of each of these submanifods N equals the rank of the Poisson structure at each point $n \in N$.*

The Poisson submanifolds defined by the previous Proposition are called the *symplectic leaves of M*. Clearly we have that Casimir functions are constant along symplectic leaves, so the symplectic leaves of a Poisson manifold are contained on the level sets of Casimir functions. In fact, if the rank of the Poisson structure is constant inside an open subset $U \subset M$, then the symplectic leaves on U are exactly the intersection of the level sets for all the Casimir functions of M.

Theorem 3.2. *[199] **(Splitting theorem)** Let $m \in M$ be any point in a Poisson manifold M. Then there are a neighborhood U of m in M and an isomorphism $\phi = (\phi_S,\phi_N)$ from U to a product manifold $S \times N$ such that S is symplectic and the rank of the Poisson structure of N at $\phi(m)$ is zero. The factors S and N are unique up to local isomorphism.*

A corollary to this theorem is a Poisson version of the existence of Darboux coordinates.

Corollary 3.2. *Let (M,π) be an n-dimensional Poisson manifold with constant rank k in a neighborhood U of m in M. Then there exists a set of local coordinates $\{q^i,p^i,y^j\}$ (where $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n-2k\}$) on U such that $\{q^i,q^j\} = \{p^i,p^j\} = \{q^i,y^j\} = \{p^i,y^j\} = \{y^i,y^j\} = 0, \{q^i,p^j\} = \delta_{ij}$. In terms of these coordinates the Poisson bivector reads*

$$\pi_u = \sum_{i=1}^{k} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} + \sum_{i<j}^{n-2k} P(y^1(u),\dots,y^{n-2k}(u)) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \tag{3.16}$$

for all $u \in U$, where $P : M \to \mathbb{R}$ is a function satisfying that $P(y^1(m),\dots,y^{n-2k}(m)) = 0$.

3.2 Poisson-Lie groups and Poisson homogeneous spaces

Definition 3.4. A *Poisson-Lie group* (G, π) is a Lie group G endowed with a Poisson structure π in such a way that the multiplication $\mu : G \times G \to G$ is a Poisson map with respect to π on G and the product Poisson structure $\pi_{G\times G} = \pi \oplus \pi$ (3.5) on $G \times G$.

Explicitly, the previous definition means that

$$\{f_1, f_2\}_G \circ \mu(g_1, g_2) = \{f_1 \circ \mu, f_2 \circ \mu\}_{G\times G}(g_1, g_2), \tag{3.17}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(G)$ and $g_1, g_2 \in G$.

This definition makes use of two Poisson structures, π on G and $\pi_{G\times G}$ on $G \times G$. In fact, it is equivalent [193] to the following condition for the Poisson structure on G

$$\{f_1, f_2\}_G(g_1, g_2) = \{f_1 \circ L_{g_1}, f_2 \circ L_{g_1}\}_G(g_2) + \{f_1 \circ R_{g_2}, f_2 \circ R_{g_2}\}_G(g_1) \tag{3.18}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(G), g_1, g_2 \in G$, and L_{g_1} and R_{g_2} are defined in (2.5) and (2.6), respectively. Equivalently, in terms of the corresponding Poisson bivectors we have that

$$\pi_{g_1 g_2} = L_{g_1,*}\pi_{g_2} + R_{g_2,*}\pi_{g_1}, \tag{3.19}$$

so we say that the Poisson bivector is multiplicative. From here it is clear that a Poisson-Lie group is never symplectic, because in particular $\pi_e = 0$.

In the same way that Lie group actions on smooth manifolds have been introduced in §2.1.1, giving rise to the notion of G-spaces, let us introduce now the notion of a Poisson G-space as a Poisson manifold (M, π) endowed with an action by a Lie group G.

Definition 3.5. A *Poisson G-space* is a Poisson manifold (M, π_M) endowed with a Lie group action $\alpha : (G \times M, \pi_G \oplus \pi_M) \to (M, \pi)$ that is a Poisson map.

Therefore, for a Poisson G-space, we have that

$$\{f_1, f_2\}_M \circ \alpha(g, m) = \{f_1 \circ \alpha, f_2 \circ \alpha\}_{G\times M}(g, m), \tag{3.20}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(M)$, all $g \in G$ and all $m \in M$, which is analogous to (3.17). This condition can be expressed in terms of the Poisson brackets on M and G as

$$\{f_1, f_2\}_M \circ \alpha(g, m) = \{f_1 \circ \alpha_g, f_2 \circ \alpha_g\}_M(m) + \{f_1 \circ \alpha_m, f_2 \circ \alpha_m\}_G(g) \tag{3.21}$$

where the notation here used is the same as the one for (2.11) and (2.12). Note that $f_i \circ \alpha_g : M \to M$ and $f_i \circ \alpha_m : G \to G$. In terms of the Poisson bivectors on M and G, the relation (3.21) can be rewritten as

$$\pi_M(\alpha(g, m)) = (\alpha_g)_*\pi_M(m) + (\alpha_m)_*\pi_G(g). \tag{3.22}$$

Either of the identities (3.20), (3.21) and (3.22), which are just equivalent ways of expressing the relation between Poisson structures on M and G and the action $\alpha : G \to \mathrm{Diff}\,(M)$, are frequently referred to as the *covariance condition.* So, given a Poisson G-space (M, π_M) we will say that it is covariant under the Poisson-Lie group (G, π_G) to emphasize that the action is compatible with the Poisson structures defined on the homogeneous space and on the Lie group.

3.3 Lie bialgebras

Definition 3.6. A *Lie bialgebra* $(\mathfrak{g}, \delta)$ is a Lie algebra $\mathfrak{g}$ endowed with a map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the *cocommutator*, fulfilling the following conditions:

(B1) $\delta(X) \in \bigwedge^2 \mathfrak{g} \quad \forall X \in \mathfrak{g}$ (skew-symmetry)

(B2) $\sum_{cycl} (\delta \otimes \mathrm{id}) \circ \delta(X) = 0 \quad \forall X \in \mathfrak{g}$ (Co-Jacobi condition)

(B3) $\delta([X, Y]) = ad_X \delta(Y) - ad_Y \delta(X), \quad \forall X, Y \in \mathfrak{g}$ (1-cocycle conditon)

If we take a basis of $\mathfrak{g}$ such that $[X_i, X_j] = c_{ij}^k X_k$ and $\delta(X_n) = f_n^{lm} X_l \otimes X_m$, then the skew-symmetry condition (B1) implies

$$f_n^{lm} = -f_n^{ml} \tag{3.23}$$

for all n, m, l. The co-Jacobi condition (B2) implies

$$\sum_l \left(f_n^{lm} f_l^{ij} + f_n^{lj} f_l^{mi} + f_n^{li} f_l^{jm} \right) = 0, \tag{3.24}$$

that can alternatively be written as

$$\sum_{\{i,j,m\} \in \Sigma_3} f_n^{lm} f_l^{ij} = 0, \tag{3.25}$$

where $\sum_{\{i,j,m\} \in \Sigma_3}$ represents a cyclic sum over indices i, j, m, while the 1-cocycle condition (B3) implies

$$\sum_k f_k^{lm} c_{ij}^k = f_i^{lk} c_{kj}^m + f_i^{km} c_{kj}^l + f_j^{lk} c_{ik}^m + f_j^{km} c_{ik}^l. \tag{3.26}$$

Note that conditions (B1) and (B2) can be restated by saying that the dual map ${}^t\delta$: $\mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket (in other words (B1) and (B2) are just (L1) and (L2) for the dual Lie algebra $\mathfrak{g}^*$ with structure constants f_i^{jk}). Writing ${}^t\delta = [,]_*$ we have that $[f, g]_* = (f \otimes g) \circ \delta$ for all $f, g \in \mathfrak{g}^*$.

Definition 3.7. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. We say that $(\mathfrak{h}, \delta|_{\mathfrak{h}})$ is a sub-Lie bialgebra of $(\mathfrak{g}, \delta)$ if $\delta(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{h}$.

Definition 3.8. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and (M, π) a Poisson manifold, such that $\mathfrak{g}$ acts on M by an infinitesimal action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$. We say that ρ is a Poisson action if

$$\mathcal{L}_{\rho(X)} \pi = (\rho \wedge \rho) \delta(X), \tag{3.27}$$

for all $X \in \mathfrak{g}$, where $\mathcal{L}$ represents the Lie derivative.

3.3.1 Lie bialgebras as tangent counterparts of Poisson-Lie groups

Let G be a Lie group, and let us consider the map

$$\begin{aligned} \eta : G &\to \bigwedge^2 \mathfrak{g} \\ g &\to (R_{g^{-1}})_* \pi(g), \end{aligned} \tag{3.28}$$

which is just the right translation (2.6) of the Poisson bivector π to the identity e. The fact that π is multiplicative (3.19) allows us to prove that η is in fact a cocycle of G with values in $\bigwedge^2 \mathfrak{g}$, i.e.

$$\eta(g_1 g_2) = \eta(g_1) + \mathrm{Ad}_{g_1} \eta(g_2), \tag{3.29}$$

for all $g_1, g_2 \in G$. The derivative at the identity e of η defines a map

$$\begin{aligned} \delta : \mathfrak{g} &\to \bigwedge^2 \mathfrak{g} \\ X &\to \left.\frac{d}{dt}\right|_{t=0} \eta(e^{tX}). \end{aligned} \tag{3.30}$$

It is easy to prove [193] that π being multiplicative (3.19) implies the 1-cocycle condition (B3) while π being Poisson (3.1) implies the co-Jacobi condition (B2). Skew-symmetry (B1) is trivially satisfied, so we have that $(\mathfrak{g}, \delta)$ is a Lie bialgebra.

Thus, we have proved that the tangent counterpart of a Poisson-Lie group is a Lie bialgebra. By this we mean that in the tangent space to a Poisson-Lie group a natural Lie bialgebra structure is defined. The converse is true in the same sense that for Lie groups, so we have the following remarkable result due to Drinfel'd

Theorem 3.3. *[19] There is a one-to-one correspondence between Lie bialgebras and connected and simply connected Poisson-Lie groups.*

3.3.2 Lie bialgebras and the classical Yang-Baxter equation

A great simplification in the problem of constructing (at least locally) Poisson-Lie structures is thus obtained by reducing this problem to the one of finding Lie bialgebra structures and then 'exponentiate' them. However, a further simplification arises by the fact that the cocommutator is a cocycle, and a particular case of cocycles are those which are cobondaries, i.e. those ones whose cocommutator can be written as

$$\begin{aligned} \delta : \mathfrak{g} &\to \mathfrak{g} \otimes \mathfrak{g} \\ X &\to \mathrm{ad}_X(r) \end{aligned} \tag{3.31}$$

for all X in $\mathfrak{g}$, where $r \in \mathfrak{g} \otimes \mathfrak{g}$. We say that a Lie bialgebra $(\mathfrak{g}, \delta)$ is a *coboundary Lie bialgebra* if its cocommutator is of the form (3.31). Before describing coboundary Lie algebras, we need to introduce some notation. If we write

$$r = \sum_i X_i \otimes Y_i \tag{3.32}$$

then we denote $r_{12} = r$ and $r_{21} = \sigma(r) = \sum_i Y_i \otimes X_i$.

Definition 3.9. *The algebraic Schouten bracket* is defined by

$$[[r, r]] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{3.33}$$

where

$$\begin{aligned} [r_{12}, r_{13}] &= \sum_{i,j} [X_i, X_j] \otimes Y_i \otimes Y_j, \\ [r_{12}, r_{23}] &= \sum_{i,j} X_i \otimes [Y_i, X_j] \otimes Y_j, \\ [r_{13}, r_{23}] &= \sum_{i,j} X_i \otimes X_j \otimes [Y_i, Y_j]. \end{aligned} \tag{3.34}$$

The following two equations, written in terms of the algebraic Schouten bracket, will play a prominent role in the rest of this work.

Definition 3.10. *The classical Yang-Baxter equation (CYBE)* is defined as

$$[[r, r]] = 0, \tag{3.35}$$

while *the modified classical Yang-Baxter equation (mCYBE)* is

$$\mathrm{ad}_X [[r, r]] = 0, \tag{3.36}$$

for all $X \in \mathfrak{g}$.

It is clear that solutions of the CYBE are also solutions of the mCYBE. However, solutions of the mCYBE which are not solutions of the CYBE play an important role, so we introduce the following notation:

Definition 3.11. Let $\mathfrak{g}$ be a Lie algebra. We say that an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a *classical r-matrix* if

(R1) $\mathrm{ad}_X (r_{12} + r_{21}) = 0 \qquad \forall X \in \mathfrak{g}$,

(R2) $[[r, r]] = 0 \qquad \forall X \in \mathfrak{g}$.

With this definition, we have the following result that answers positively the question about the existence of quantizations for r-matrices, in the sense of [200].

Theorem 3.4. *[200] Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ a classical r-matrix. Then there exists a quantum R-matrix $R \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})[[h]]$ such that $R = 1 + hr \,(\mathrm{mod}\, h^2)$.*

The parameter h is known as the deformation (or quantization) parameter. Note that quantizations play an important role in this work, and they will be treated in detail in §3.6. Now we have the following result, which identifies solutions of the mCYBE with Lie bialgebra structures on $\mathfrak{g}$.

Proposition 3.2. *[35] Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. The map δ defined by (3.31) is the cocommutator of a Lie bialgebra structure on $\mathfrak{g}$ if and only if the following conditions are satisfied:*

i) $\mathrm{ad}_X(r_{12} + r_{21}) = 0 \qquad \forall X \in \mathfrak{g}$,

ii) $\mathrm{ad}_X[[r, r]] = 0 \qquad \forall X \in \mathfrak{g}$.

It is clear that every classical r-matrix defines a coboundary Lie bialgebra, but not every Lie bialgebra is defined by a classical r-matrix. The following definition identifies the four different types of Lie bielgebras.

Definition 3.12. We call a Lie bialgebra $(\mathfrak{g}, \delta)$:

i) *Non coboundary* if its cocommutator cannot be written as $\delta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, X \to \mathrm{ad}_X r$ for any $r \in \mathfrak{g} \otimes \mathfrak{g}$.

ii) *Coboundary* if its cocommutator can be written as $\delta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, X \to \mathrm{ad}_X r$ with a (skew-symmetric) element $r \in \mathfrak{g} \otimes \mathfrak{g}$.

iii) *Quasitriangular* if its cocommutator can be written as $\delta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, X \to \mathrm{ad}_X r$ with a classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.

iv) *Triangular* if its cocommutator can be written as $\delta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, X \to \mathrm{ad}_X r$ with a skew-symmetric classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$.

In this way, every triangular Lie bialgebra is quasitriangular, and both of them are particular cases of coboundary Lie bialgebras. In fact, the definition of Lie bialgebras by skew-symmetric solutions of the mCYBE will be a key part of this work, so we state as a Proposition the following straightforward result.

Proposition 3.3. *Coboundary Lie bialgebras are in a one-to-one correspondence with skew-symmetric solutions of the mCYBE.*

Proof. We previously need the following result

Lemma 3.1. *[194] Let $\mathfrak{g}$ be a Lie algebra. Then:*

i) If $r = r' + t \in \mathfrak{g} \otimes \mathfrak{g}$ with $r' = -r'_{21}$ and $\mathrm{ad}_X t = 0$ for all $X \in \mathfrak{g}$, then

$$[[r, r]] = [[r', r']] + [[t, t]]. \tag{3.37}$$

ii) If $\mathrm{ad}_X t = 0$ for all $X \in \mathfrak{g}$, then $\mathrm{ad}_X[[t, t]] = 0$ for all $X \in \mathfrak{g}$.

Now, let us assume that $(\mathfrak{g}, \delta)$ is coboundary, i.e. $\delta(X) = \mathrm{ad}_X r \in \bigwedge^2 \mathfrak{g}$ for all $X \in \mathfrak{g}$. Then, from $i)$ of Proposition 3.2, the only possibility is that $r = r' + t$, where $r' \in \bigwedge^2 \mathfrak{g}$ and $\mathrm{ad}_X t = 0$ for all $X \in \mathfrak{g}$, for if it is not, then the cocommutator will not be skew-symmetric. Now, using $i)$ from the previous Lemma,

$$\mathrm{ad}_X[[r, r]] = \mathrm{ad}_X[[r', r']] + \mathrm{ad}_X[[t, t]] = 0, \tag{3.38}$$

but $\mathrm{ad}_X[[t, t]] = 0$ by $ii)$ from Lemma 3.1, so $\mathrm{ad}_X[[r', r']] = 0$ and then r and r' define the same Lie bialgebra $(\mathfrak{g}, \delta)$. □

From now on we generally assume that any $r \in \mathfrak{g} \otimes \mathfrak{g}$ defining a Lie bialgebra structure is skew-symmetric, so in fact $r \in \bigwedge^2 \mathfrak{g}$. The only exception will be Chapter 6 where we consider Drinfel'd double structures and the associated quasi-triangular r-matrices. However, even in this case we can skew-symmetrize them and work with skew-symmetric solutions of the mCYBE, as we will discuss in detail in §3.4.

For certain Lie algebras, solutions of the CYBE and solutions of the mCYBE are closely related. In particular, consider a Lie algebra $\mathfrak{g}$ endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ which is $\mathrm{ad}_{\mathfrak{g}}$-invariant. Note that this last condition is equivalent to say that the bilinear form is associative, because

$$(\mathrm{ad}_Y X, Z) + (X, \mathrm{ad}_Y Z) = 0 \qquad (\mathrm{ad}_{\mathfrak{g}}\text{-invariance}) \tag{3.39}$$

directly implies

$$([X,Y],Z) = (X,[Y,Z]) \qquad \text{(Associativity)} \tag{3.40}$$

for all $X, Y, Z \in \mathfrak{g}$. Lie algebras endowed with such a bilinear form are called *metric Lie algebras* and it is worth noticing that in [201, 202, 203] kinematical metric Lie algebras have been recently studied.

Now, for any metric Lie algebra $\mathfrak{g}$ the element $\omega^* \in \bigwedge^3 \mathfrak{g}^*$ defined by $\omega^*(X_1, X_2, X_3) = ([X_1, X_2], X_3)$ for all $X_1, X_2, X_3 \in \mathfrak{g}$, is $\mathrm{ad}^*_{\mathfrak{g}}$-invariant, since

$$\begin{aligned}
&\langle \mathrm{ad}^*_Y \omega^*, X_1 \otimes X_2 \otimes X_3 \rangle = \langle \omega^*, -\mathrm{ad}_Y (X_1 \otimes X_2 \otimes X_3) \rangle = \\
&\langle \omega^*, [X_1, Y] \otimes X_2 \otimes X_3 + X_1 \otimes [X_2, Y] \otimes X_3 + X_1 \otimes X_2 \otimes [X_3, Y] \rangle = \\
&([[X_1, Y], X_2], X_3) + ([X_1, [X_2, Y]], X_3) + ([X_1, X_2], [X_3, Y]) = \\
&([[X_1, X_2], Y], X_3) + ([X_1, X_2], [X_3, Y]) = 0,
\end{aligned} \tag{3.41}$$

where $\langle \cdot,\cdot \rangle$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and we have used the Jacobi identity for $\mathfrak{g}$, as well as (2.35) and (2.37). The identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ allows us to define an $\mathrm{ad}_{\mathfrak{g}}$-invariant element $\omega \in \bigwedge^3 \mathfrak{g}$. Now, *ii*) of Proposition (3.2) will be satisfied if

$$[[r,r]] = -\omega \tag{3.42}$$

so r is a solution of the mCYBE. But we can see the bilinear form $(\cdot,\cdot)$ as a quadratic and symmetric element of $\mathfrak{g}^* \otimes \mathfrak{g}^*$, and by means of $\langle \cdot,\cdot \rangle$ as a quadratic and symmetric element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ which is $\mathrm{ad}_{\mathfrak{g}}$-invariant (we call this element *the quadratic tensorized Casimir*). From Lemma 3.1 and the proof of Proposition 3.3 it follows that

$$[[\Omega, \Omega]] = \omega, \tag{3.43}$$

and so we have proved the following

Proposition 3.4. *Let $\mathfrak{g}$ be a metric Lie algebra. Then r' is a skew-symmetric solution of the mCYBE if and only $r = r' + \Omega$ is a solution of the CYBE.*

Note that Ω in this Proposition is the same as t in Lemma 3.1.

3.3.3 Coboundary Poisson-Lie groups and other Poisson structures from the mCYBE

We have introduced the notion of coboundary Lie bialgebras as those which are defined by means of a solution of the mCYBE. Poisson-Lie groups associated to coboundary Lie bialgebras are called *coboundary Poisson-Lie groups* and its Poisson-Lie structure is defined in the following Proposition.

Proposition 3.5. *[35] Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider a skew-symmetric r-matrix $r \in \bigwedge^2 \mathfrak{g}$ defining a coboundary Lie bialgebra $(\mathfrak{g}, \delta)$ by $\delta(X) = \mathrm{ad}_X r$ for all $X \in \mathfrak{g}$. Then the only Poisson-Lie structure on G whose tangent space is $(\mathfrak{g}, \delta)$ is defined by the following Poisson bracket*

$$\{f_1, f_2\} = \sum_{i,j} r^{ij} \left(X_i^L f_1 X_j^L f_2 - X_i^R f_1 X_j^R f_2 \right) \tag{3.44}$$

or equivalently by the Poisson bivector

$$\Pi = \sum_{i,j} r^{ij} \left(X_i^L \otimes X_j^L - X_i^R \otimes X_j^R \right) \tag{3.45}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(G)$. X_i^L and X_i^R are left- and right-invariant vector fields on G defined by (2.23).

The Poisson bracket defined by (3.44) (Poisson bivector defined by (3.45)) is called a *Sklyanin bracket (bivector).*

Along this work, the Sklyanin bracket (and its induced structures on coset spaces) will play a prominent role. However, there exist other Poisson structures on G associated to skew-symmetric solutions of the mCYBE (although clearly neither of them define Poisson-Lie structures). For the sake of completeness, we mention here three of them.

Proposition 3.6. *[35] Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and consider a skew-symmetric solution of the CYBE $r \in \bigwedge^2 \mathfrak{g}$, i.e. a triangular r-matrix. Then*

1. $\{f_1, f_2\}^L = \sum_{i,j} r^{ij} X_i^L f_1 X_j^L f_2$
2. $\{f_1, f_2\}^R = \sum_{i,j} r^{ij} X_i^R f_1 X_j^R f_2$

are Poisson structures on G. Moreover $\{\cdot,\cdot\}^L$ $\left(\{\cdot,\cdot\}^R\right)$ is left-invariant (right-invariant).

Proposition 3.7. *[194] Let G be a coboundary Poisson-Lie group defined by a skew-symmetric solution of the mCYBE $r_{(s)}$, then the* Heisenberg double Poisson structure

$$\{f_1, f_2\}^H = \sum_{i,j} r_{(s)}^{ij} \left(X_i^L f_1 X_j^L f_2 + X_i^R f_1 X_j^R f_2 \right) \tag{3.46}$$

endows G with the structure of a Poisson $G \times G$-space with respect to left and right multiplications, given by (2.5) and (2.6) respectively.

Proposition 3.8. *[194] Let G be a Poisson-Lie group defined by a quasitriangular classical r-matrix, then the* dual Poisson structure

$$\{f_1, f_2\}^D = \sum_{i,j} r^{ij}_{(s)} \left(X^L_i f_1 X^L_j f_2 + X^R_i f_1 X^R_j f_2 \right) - r^{ij} \left(X^L_i f_1 X^R_j f_2 - X^R_i f_1 X^L_j f_2 \right) \quad (3.47)$$

endows G with the structure of a Poisson G-space with respect to the adjoint action (2.19).

3.4 Manin triples and classical Drinfel'd doubles

Given a Lie bialgebra $(\mathfrak{g}, \delta)$, we have seen that its dual vector space $\mathfrak{g}^*$ is endowed with a Lie algebra structure. We now consider the vector space $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ and we define on it the (not necessarily positive definite) scalar product (also called pairing) given by

$$\langle (X, x), (Y, y) \rangle = f(Y) + g(X) \quad (3.48)$$

for all $X, Y \in \mathfrak{g}$ and $x, y \in \mathfrak{g}^*$.

Theorem 3.5. *[195] Let be $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $D(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}^*$. There exists a unique Lie algebra structure on $D(\mathfrak{g})$ such that*

(i) Both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $D(\mathfrak{g})$.

(ii) The scalar product (3.48) is associative, i.e. $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle, \quad \forall X, Y, Z \in D(\mathfrak{g})$.

From *(ii)* of the previous theorem, it directly follows that if $X \in \mathfrak{g}$ and $x \in \mathfrak{g}^*$, then

$$[X, x] = -\mathrm{ad}^*_X x + (x \otimes 1) \circ \delta(X). \quad (3.49)$$

Definition 3.13. The vector space $D(\mathfrak{g})$ endowed with the Lie algebra defined above is called the *classical Drinfel'd double Lie algebra of* $\mathfrak{g}$. The triple $(\mathfrak{g}, \mathfrak{g}^*, D(\mathfrak{g}))$ is called *a Manin triple.*

If we choose a basis $Y_i, i \in 1, \dots n$ on $\mathfrak{g}$ and write

$$[Y_i, Y_j] = c^k_{ij} Y_k, \qquad \delta(Y_i) = f^{jk}_i Y_j \otimes Y_k, \quad (3.50)$$

then, in terms of the algebraic dual basis y^i on $\mathfrak{g}^*$, i.e. $y^j(Y_i) = \delta^j_i$, the explicit commutation relations in $D(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}^*$ read

$$[Y_i, Y_j] = c^k_{ij} Y_k, \qquad [y^i, y^j] = f^{ij}_k y^k, \qquad [Y_i, y^j] = f^{jk}_i Y_k - c^j_{ik} y^k. \quad (3.51)$$

A simple computation shows that

$$C = \frac{1}{2} \sum_{i=1}^{n} \left(y^i Y_i + Y_i y^i \right) \quad (3.52)$$

is a quadratic Casimir element for $D(\mathfrak{g})$, i.e. C is a quadratic element of the center of the universal enveloping algebra $\mathcal{U}(D(\mathfrak{g}))$.

Definition 3.14. Let $D(G)$ be the only connected and simply connected Lie group with Lie algebra $D(\mathfrak{g})$. Then we call $D(G)$ *the classical double Lie group of G*, where $\mathfrak{g} = \mathrm{Lie}\,(G)$.

3.4.1 Quasitriangular Lie bialgebra structure on $D(\mathfrak{g})$

Any Drinfel'd double Lie algebra is canonically endowed with a coboundary (and in fact quasitriangular) Lie bialgebra structure, defined by the canonical r-matrix

$$r = y^i \otimes Y_i \in \mathfrak{g}^* \otimes \mathfrak{g} \subset D(\mathfrak{g}) \otimes D(\mathfrak{g}), \tag{3.53}$$

which is a solution of the classical Yang-Baxter equation (CYBE). In terms of r we define the associated cocommutator $\delta_D(X) = \mathrm{ad}_X r$. Explicitly we have

$$\delta_D(Y_i) = -f_i^{jk} Y_j \otimes Y_k, \qquad \delta_D(y^i) = c_{jk}^i y^j \otimes y^k. \tag{3.54}$$

It is clear that $(\mathfrak{g}, \delta_D|_{\mathfrak{g}})$ and $(\mathfrak{g}^*, \delta_D|_{\mathfrak{g}^*})$ are sub-Lie bialgebras of $D(\mathfrak{g}, \delta_D)$. Let us call Ω the tensorized form of (3.52), i.e.

$$\Omega = \frac{1}{2} \sum_{i=1}^{n} \left(y^i \otimes Y_i + Y_i \otimes y^i \right) \tag{3.55}$$

and consider the skew-symmetric r-matrix obtained from (3.53) by subtracting Ω, which reads

$$r' = r - \Omega = \frac{1}{2} y^i \wedge Y_i. \tag{3.56}$$

A straightforward computation shows that $\mathrm{ad}_X \Omega = 0$ for all $X \in D(\mathfrak{g})$, so by linearity of the cocommutator we have that

$$\delta_D(X) = \mathrm{ad}_X r' = \mathrm{ad}_X (r - \Omega) = \mathrm{ad}_X r. \tag{3.57}$$

3.5 Poisson homogeneous spaces

Among Poisson G-spaces (see Definition 3.5), those which are equipped with a transitive Poisson action are specially interesting. In fact, they are the analogues of homogeneous spaces (see Definition 2.15) in the category of Poisson manifolds.

Definition 3.15. For any Poisson-Lie group (G, Π), a *Poisson homogeneous space* (PHS) over G is a Poisson G-space (M, π_M, α, G) such that the action $\alpha : G \times M \to M$ is transitive.

Recall from Definition 2.5 that an action α is transitive if for all $m, n \in M$, there exists $g \in G$ such that $\alpha(g)m = n$. In other words, a transitive action allows us to connect any two points of M by means of an element of G. To be Poisson just means that this action is compatible with the Poisson structures on M and G.

Definition 3.16. Let (M, π_M, α^M, G) and (N, π_N, α^N, G) be two PHS. A *homomorphism of Poisson homogeneous spaces* is a Poisson map $\phi : (M, \pi_M) \to (N, \pi_N)$ compatible with the group action, i.e. it satisfies $\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi$ and $\alpha_g^N \circ \phi = \phi \circ \alpha_g^M$ for all $f_1, f_2 \in \mathcal{C}^\infty(N), g \in G$.

Once we have introduced the notion of a PHS, we proceed to describe their tangent structure. With this aim we need the following

Definition 3.17. Let $\mathfrak{g}$ be a Lie algebra. A *Lagrangian Lie subalgebra of the Drinfel'd double* $D(\mathfrak{g})$ is a Lie subalgebra $\mathfrak{l} < D(\mathfrak{g})$ such that $\mathfrak{l}^\perp = \mathfrak{l}$ with respect to the canonical pairing on its classical Drinfel'd double algebra (3.48).

A Lagrangian Lie subalgebra $\mathfrak{l} < D(\mathfrak{g})$ is called *coisotropic* if

$$(\mathfrak{l}\cap\mathfrak{g})^\perp = \mathfrak{l}\cap\mathfrak{g}^*, \tag{3.58}$$

where

$$(\mathfrak{l}\cap\mathfrak{g})^\perp \equiv \{x\in\mathfrak{g}^* | \langle x, X\rangle = 0, \quad \forall X\in\mathfrak{g}\}. \tag{3.59}$$

PHS for a Lie group G are related to Lagrangian Lie subalgebras of its classical Drinfel'd double. We now describe this relation in detail, following closely the recent work [60].

Theorem 3.6. *[39]. (See also [60]). Let (G,Π) be a Poisson-Lie group.*

i) Every pair (M,m) given by a PHS M over G and a point $m\in M$ defines a Lagrangian Lie subalgebra $\mathfrak{l} < D(\mathfrak{g})$ with $\mathfrak{l}\cap\mathfrak{g} = \mathfrak{h}_m$, where $\mathfrak{h}_m$ is the Lie algebra of the stabilizer $H_m = \{g\in G\,|\,\alpha_g(m) = m\}$.

ii) Isomorphism classes of PHS over G correspond to orbits of pairs $(\mathfrak{l}, H)$, where $\mathfrak{l} < D(\mathfrak{g})$ is a Lagrangian Lie subalgebra and $H < G$ is a Lie subgroup such that $\mathrm{Lie}\,(H) = \mathfrak{h} = \mathfrak{l}\cap\mathfrak{g}$ *with respect to a certain G-action.*

Proof. Implication *ii)* is essentially obtained by exponentiation, from uniqueness results on Poisson-Lie groups and from the Poisson G-space (covariance) condition (3.22).

Let us focus on the first implication. Let us call H_m the stabilizer (2.13) of $m\in M$. Then the diffeomorphism (see Theorem 2.3)

$$\begin{aligned} \beta_m : G/H_m &\to M \\ gH_m &\to \alpha_g(m) \end{aligned} \tag{3.60}$$

identifies, on the one hand $T_mM \simeq T_{eH_m}(G/H_m) \simeq \mathfrak{g}/\mathfrak{h}_m$ and on the other hand, $T^*_mM \simeq T^*_{eH_m}(G/H_m) \simeq \mathfrak{h}_m^\perp$, where $\mathfrak{h}_m^\perp = \{x\in\mathfrak{g}^*\,|\,\langle x, X\rangle = 0, \quad \forall X\in\mathfrak{h}_m\}$. Now, the derivative of the map (3.9) at the origin eH_m of G/H_m, is a map

$$(\sharp_\pi)_{*,eH_m} : T^*_{eH_m}(G/H_m) \simeq \mathfrak{h}_m^\perp \to T_{eH_m}(G/H_m) \simeq \mathfrak{g}/\mathfrak{h}_m. \tag{3.61}$$

Now, for every PHS (M,π,α,G), consider the infinitesimal action

$$\begin{aligned} \rho : \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\to X^\alpha \end{aligned} \tag{3.62}$$

which assigns to each $X\in\mathfrak{g}$ its associated action vector field X^α (2.56). For each $m\in M$ the map $\rho_m : \mathfrak{g}\to T_m(M)$, provided by evaluation of $\rho(X) = X^\alpha$ at $m\in M$, is a Lie algebra homomorphism with kernel $\ker \rho_m = \mathfrak{h}_m = \mathrm{Lie}\,(H_m)$. Transitivity of α implies for every $m \in M$ that $T_m M = \mathrm{span}\{X_m^\alpha \,|\, X \in \mathfrak{g}\} \simeq \mathfrak{g}/\mathfrak{h}_m$ and $T_m^* M = (\mathfrak{g}/\mathfrak{h}_m)^* \simeq \mathfrak{h}_m^\perp$. Denoting by $\tilde{p}_m : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}_m$ the derivative of (2.54), we can identify the graph of the linear map defined by the derivative of (3.9) at $m \in M$, i.e.

$$\begin{aligned}(\sharp_\pi)_{*,m} : T_m^* M &\to T_m M \\ x &\to (x \otimes \mathrm{id})\pi_m,\end{aligned} \tag{3.63}$$

with the linear subspace

$$\mathfrak{l}_m = \{(X, x) \in \mathfrak{g} \oplus \mathfrak{h}_m^\perp \,|\, (\sharp_\pi)_{*,m}(x) = \tilde{p}_m(X)\} \subset \mathfrak{g} \oplus \mathfrak{g}^* = D(\mathfrak{g}). \tag{3.64}$$

The only remaining statement to be proven is that $\mathfrak{l} \equiv \mathfrak{l}_{eH_m}$ is a Lagrangian subalgebra of $D(\mathfrak{g})$, with respect to (3.48). Since working with an explicit basis will allow us to eventually give a simple constructive description of some Poisson homogeneous spaces we follow this path. Let us choose a basis $\{H_1, \dots, H_n\}$ of $\mathfrak{h} = \mathfrak{h}_{eH_m}$ and complete it to a basis $\{H_1, \dots, H_n, T_{n+1}, \dots, T_N\}$ of $\mathfrak{g}$. Call $\{h^1, \dots, h^n, t^{n+1}, \dots, t^N\}$ with pairing

$$\begin{aligned}\langle H_i, h^j\rangle &= \delta_i^j, & \langle T_\mu, t^\nu\rangle &= \delta_\mu^\nu \\ \langle H_i, t^\mu\rangle &= 0, & \langle T_\mu, h^i\rangle &= 0\end{aligned} \tag{3.65}$$

for all $i, j \in \{1, \dots, n\}$ and $\mu, \nu \in \{n+1, \dots, N\}$. By taking into account expressions (3.50) and (3.51) for the commutation relations of $D(\mathfrak{g})$, we have that

$$\begin{aligned}
&[H_i, H_j] = c_{ij}^k H_k + c_{ij}^\mu T_\mu, && [h^i, h^j] = f_k^{ij} H^k + f_\mu^{ij} T^\mu,\\
&[H_i, T_\mu] = c_{i\mu}^j H_j + c_{i\mu}^\nu T_\nu, && [h^i, t^\mu] = f_j^{i\mu} h^j + f_\nu^{i\mu} T^\nu,\\
&[T_\mu, T_\nu] = c_{\mu\nu}^\lambda T_\lambda + c_{\mu\nu}^i H_i, && [t^\mu, t^\nu] = f_\lambda^{\mu\nu} t^\lambda + f_i^{\mu\nu} h^i,\\
&[h^i, H_j] = c_{jk}^i h^k + c_{j\mu}^i t^\mu - f_j^{ik} H_k - f_j^{i\mu} T_\mu, && [h^i, T_\mu] = c_{\mu j}^i h^j + c_{\mu\nu}^i t^\nu - f_\mu^{ij} H_j - f_\mu^{i\nu} T_\nu,\\
&[t^\mu, H_i] = c_{ij}^\mu h^j + c_{i\nu}^\mu t^\nu - f_i^{\mu j} H_j - f_i^{\mu\nu} T_\nu, && [t^\mu, T_\nu] = c_{\nu i}^\mu h^i + c_{\nu\lambda}^\mu t^\lambda - f_\nu^{\mu i} H_i - f_\nu^{\mu\lambda} T_\lambda.
\end{aligned} \tag{3.66}$$

Clearly, as $\mathfrak{h} < \mathfrak{g}$, then $c_{ij}^\mu = 0$ for all i, j, μ. There is a neighborhood $U \subset G/H$ of the origin $eH \in G/H$, where $\{T_{n+1}^\alpha(u), \dots, T_N^\alpha(u)\}$ is a basis of $T_u U$ for all $u \in U$. Therefore, the Poisson bivector on U can be written as $\pi = \pi^{\mu\nu} T_\mu^\alpha \otimes T_\nu^\alpha$, where $\pi^{\mu\nu} \in \mathcal{C}^\infty(U)$. Particularizing (3.64) to the origin of the homogeneous space $eH \in M$ we have that

$$\mathfrak{l} \equiv \mathfrak{l}_{eH} = \mathfrak{h} \oplus \mathrm{span}\{t^\mu + \pi_o^{\mu\nu} T_\nu \,|\, \mu \in \{n+1, \dots, N\}\} \tag{3.67}$$

where we have written $\pi_o^{\mu\nu} \equiv \pi_{eH}^{\mu\nu}$ for the Poisson bivector components evaluated at the origin $o = eH$ of M. Then, a direct computation using (3.65) shows that $\mathfrak{l}$ is Lagrangian if and only if $\pi_o^{\mu\nu} = -\pi_o^{\nu\mu}$. Conversely, any Lagrangian subspace $\mathfrak{l} \subset D(\mathfrak{g})$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{h}$ can be put into the form (3.67) by a change of basis of the subspaces $\mathfrak{t} = \mathrm{span}\{T_{n+1}, \dots, T_N\}$ and $\mathfrak{h}^\perp = \mathrm{span}\{t^{n+1}, \dots, t^N\}$.

There only remains to be checked that $\mathfrak{l}$ is in fact a Lie subalgebra of $D(\mathfrak{g})$. First of all, let us introduce the following notation

$$\begin{aligned}M_\lambda^{\mu\nu} &\equiv (T_\lambda^\alpha\, \pi^{\mu\nu})(eH),\\ M_i^{\mu\nu} &\equiv (H_i^\alpha\, \pi^{\mu\nu})(eH).\end{aligned} \tag{3.68}$$

The Jacobi condition for the Poisson bracket on $\mathcal{U} \subset M$ imposes the set of constraints given by

$$\sum_{(\mu,\nu,\lambda)\in\Sigma_3} \left(\pi_o^{\rho\mu} M_\rho^{\nu\lambda} + \pi_o^{\sigma\mu}\pi^{\rho\nu} c_{\rho\sigma}^{\lambda}\right), \tag{3.69}$$

where the sum means cyclic permutations of those indexes. Moreover, the condition of Poisson G-space (covariance condition) for the Poisson bivector (3.22) implies that

$$\begin{aligned} M_\lambda^{\mu\nu} &= f_\lambda^{\mu\nu} + \pi_o^{\rho\nu} c_{\lambda\rho}^{\mu} + \pi_o^{\rho\mu} c_{\lambda\rho}^{\nu}, \\ M_i^{\mu\nu} &= 0. \end{aligned} \tag{3.70}$$

Inserting conditions (3.69) and (3.70) on the Lie bracket for general elements of

$$\mathfrak{l} = \mathrm{span}\{t^\mu + \pi_o^{\mu\nu} T_\nu, H_i | \mu \in \{n+1, \dots, N\}, i \in \{1, \dots, n\}\}, \tag{3.71}$$

which can be easily computed using (3.66), gives

$$\begin{aligned} [t^\mu + \pi_o^{\mu\lambda} T_\lambda, t^\nu + \pi_o^{\nu\rho} T_\rho] &= (f_\sigma^{\mu\nu} + \pi_o^{\nu\rho} c_{\rho\sigma}^{\mu} - \pi_o^{\mu\rho} c_{\rho\sigma}^{\nu})(t^\sigma + \pi_o^{\sigma\rho} T_\rho), \\ &\quad + (\pi_o^{\mu\lambda}\pi_o^{\nu\rho} c_{\lambda\rho}^{i} + \pi_o^{\mu\rho} f_\rho^{\nu i} - \pi_o^{\nu\rho} f_\rho^{\mu i}) H_i \\ [t^\mu + \pi_o^{\mu\lambda} T_\lambda, H_i] &= c_{i\nu}^{\mu}(t^\nu + \pi_o^{\nu\lambda} T_\lambda) - (f_i^{\mu j} - \pi_o^{\mu\lambda} c_{\lambda i}^{j}) H_j, \end{aligned} \tag{3.72}$$

proving that $\mathfrak{l}$ is indeed a Lie subalgebra of $D(\mathfrak{g})$. Conversely, if we suppose that $\mathfrak{l} < D(\mathfrak{g})$, then both conditions (3.69) and (3.70) hold.

Let us see now how Lagrangian Lie subalgebras change when we take a different point. Let us call $\mathfrak{l}_m$ the Lagrangian Lie subalgebra associated to $m \in M$. Let $m' \in M$ be a different point. Of course, by transitivity of α, it exists $g \in G$ such that $m' = \alpha_g(m)$, and the stabilizer $H_{m'}$ of m' is related to the stabilizer H_m of m by $H_{m'} = C_g(H_m)$ (see the discussion below Definition 2.15). If we denote by $\mathcal{L}(D(\mathfrak{g}))$ the algebraic variety of Lagrangian Lie subalgebras of $D(\mathfrak{g})$ [204] and define the adjoint action of G on $D(\mathfrak{g})$ by

$$\mathrm{Ad}^{D(\mathfrak{g})} : G \to \mathrm{GL}(D(\mathfrak{g})) \tag{3.73}$$

where we have that $\mathrm{Ad}_g^{D(\mathfrak{g})}$, for all $g \in G$, is the map given by

$$\begin{aligned} \mathrm{Ad}_g^{D(\mathfrak{g})} &: D(\mathfrak{g}) \to D(\mathfrak{g}) \\ &(X, x) \to \mathrm{Ad}_g(X) + \mathrm{Ad}_g^*(x) + (\mathrm{Ad}_g^*(x) \otimes \mathrm{id})\eta(g) \end{aligned} \tag{3.74}$$

for all $X \in \mathfrak{g}, x \in \mathfrak{g}^*$. Recall from (3.28) that $\eta(g) = (R_g)_*\Pi(g)$. This action passes to an action on the algebraic variety $\mathcal{L}(D(\mathfrak{g}))$ due to the fact that it preserves (3.48), so it sends Lagrangian Lie subalgebras of $D(\mathfrak{g})$ to Lagrangian Lie subalgebras of $D(\mathfrak{g})$. Therefore, isomorphism classes of PHS on $M = G/H$ are in one-to-one correspondence with the orbits of the adjoint action of G on $D(\mathfrak{g})$ (3.73) on $\mathcal{L}(D(\mathfrak{g}))$, and we can identify these orbits as models for PHS.

□

3.5.1 Coisotropic Poisson homogeneous spaces

Hereafter, we consider the special case of coisotropic Lie subalgebras of $D(\mathfrak{g})$ (recall from Definition 3.17 that they are those Lagrangian Lie subalgebras $\mathfrak{l}$ of $D(\mathfrak{g})$ for which $(\mathfrak{l}\cap\mathfrak{g})^\perp = \mathfrak{l}\cap\mathfrak{g}^*$). Hence, we have the following

Lemma 3.2. *[60] Let $\mathfrak{h} < \mathfrak{g}$ be a Lie subalgebra and $\mathfrak{l} < D(\mathfrak{g})$ a Lagrangian Lie subalgebra of $D(\mathfrak{g})$ such that $\mathfrak{l}\cap\mathfrak{g} = \mathfrak{h}$ as in (3.67). Then $\mathfrak{l}$ is coisotropic if and only if $\pi^{\mu\nu} = 0$ for all $\mu, \nu \in \{n+1, \dots, N\}$. In this case, one has that $\mathfrak{l} = \mathfrak{h}\oplus\mathfrak{h}^\perp = \operatorname{span}\{H_i, t^\mu\}$ and the Lie bracket on $\mathfrak{l}$ is given by*

$$[t^\mu, t^\nu] = f^{\mu\nu}_\lambda t^\lambda, \qquad [t^\mu, H_i] = c^\mu_{i\nu} t^\nu + f^{j\mu}_i H_j, \qquad [H_i, H_j] = c^k_{ij} H_k, \tag{3.75}$$

and the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy $c^\mu_{ij} = 0$, $f^{\mu\nu}_i = 0$ for all $i, j \in \{1, \dots, n\}$ and $\mu, \nu \in \{n+1, \dots, N\}$.

Consequently, for coisotropic Lie subalgebras we have that $\mathfrak{h} < \mathfrak{g}$ and $\mathfrak{h}^\perp < \mathfrak{g}^*$. Specially useful will be the precise cocommutator of the subalgebra $\mathfrak{h} = \mathfrak{l}\cap\mathfrak{g}$ that, by Lemma 3.2, is given by

$$\delta(H_i) = f^{jk}_i H_j \wedge H_k + f^{j\nu}_i H_j \wedge T_\nu. \tag{3.76}$$

From this expression it is clear that for every $\mathfrak{h} < \mathfrak{g}$, then $(\mathfrak{h}, \delta_\mathfrak{h})$, where we have written $\delta_\mathfrak{h} = \delta|_\mathfrak{h}$, is a sub-Lie bialgebra of $(\mathfrak{g}, \delta)$ (and therefore (H, Π_H) a Poisson-Lie subgroup of (G, Π_G)) if and only if $f^{j\nu}_i = 0$ for all $i, j \in \{1, \dots, n\}$ and $\nu \in \{n+1, \dots, N\}$. In this case, then (3.75) takes the simpler form

$$[t^\mu, t^\nu] = f^{\mu\nu}_\lambda t^\lambda, \qquad [t^\mu, H_i] = c^\mu_{i\nu} t^\nu, \qquad [H_i, H_j] = c^k_{ij} H_k, \tag{3.77}$$

so the Lagrangian Lie subalgebra $\mathfrak{l}$ of $D(\mathfrak{g})$ is a semidirect product $\mathfrak{l} = \mathfrak{h} \ltimes \mathfrak{h}^\perp$. In the remaining of this work, the explicit construction of PHS will play a central role, and coisotropic Lie subalgebras of $D(\mathfrak{g})$ will appear quite frequently. Moreover, the characterization by means of Lie subalgebras of $D(\mathfrak{g})$ is quite cumbersome, and in fact unnecessary for our purposes. Thus let us give the following definition of a coisotropic Lie bialgebra.

Definition 3.18. Let $(\mathfrak{g}, \delta)$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, with the Lie bialgebra structure given by $\delta_\mathfrak{h} = \delta|_\mathfrak{h}$. Then we say that $(\mathfrak{g}, \delta)$ *is coisotropic with respect to* $\mathfrak{h}$ if there exists a basis $\{H_1, \dots, H_n\}$ of $\mathfrak{h}$ such that

$$\delta(H_i) = f^{jk}_i H_j \wedge H_k + f^{j\mu}_i H_j \wedge T_\mu, \tag{3.78}$$

where $\{T_{n+1}, \dots, T_N\}$ is any basis of the complement of $\mathfrak{h}$ in $\mathfrak{g}$, $f^{jk}_i, f^{j\mu}_i$ are arbitrary constants and $i, j, k \in \{1, \dots, n\}$ and $\mu \in \{n+1, \dots, N\}$. When these conditions hold, we will write

$$\delta(\mathfrak{h}) \subset \mathfrak{h}\wedge\mathfrak{g}. \tag{3.79}$$

Consequently:

- We say that (M, π, α, G) is a *coisotropic Poisson homogeneous space* if M is diffeomorphic to G/H by the diffeomorphism (2.55) of Theorem 2.3 and the Lie bialgebra $(\mathfrak{g} = \operatorname{Lie} G, \delta)$ is coisotropic with respect to $\mathfrak{h} = \operatorname{Lie} H$.

- We say that (M, π, α, G) is a *Poisson homogeneous space of Poisson subgroup type* if M is diffeomorphic to G/H by the diffeomorphism (2.55) of Theorem 2.3 and $(\mathfrak{h} = \mathrm{Lie}\, H, \delta|_{\mathfrak{h}})$ is a sub-Lie bialgebra of $(\mathfrak{g} = \mathrm{Lie}\, G, \delta)$ (or equivalently, (H, Π_H) is a Poisson-Lie subgroup of (H, Π_H)), that is

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}. \tag{3.80}$$

3.5.2 Poisson homogeneous spaces from semidirect products

The following construction describes a PHS structure for a particular class of Lie groups. In particular, this will be relevant in Chapter 6 when we study PHS for the Poincaré and Euclidean groups in (2+1) dimensions, which are semidirect-product groups with the particularity that the dimension of the isotropy subgroup equals the dimension of the subalgebra of translations.

Let us consider a Lie group G of the form $G = H \ltimes_{\mathrm{Ad}^*} \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h} = \mathrm{Lie}(H)$, endowed with the trivial Lie algebra structure, so $\mathfrak{h}^* \simeq \mathbb{R}^n$. Then we have that $\mathfrak{g} = \mathfrak{h} \ltimes_{\mathrm{ad}^*} \mathfrak{h}^*$. If we denote a group element by (h, t) we can write the group multiplication as $(h_1, t_1) \cdot (h_2, t_2) = (h_1 h_2, t_1 + \mathrm{Ad}^*_{h_1^{-1}}(t_2))$. Let $\mathfrak{h} = \langle J_a \rangle$ be a basis for $\mathfrak{h}$ and $\mathfrak{h}^* = \langle P_a \rangle$ the dual basis, so $\langle J_a, P_b \rangle = \delta_{ab}$. The commutation relations read $[J_a, J_b] = c^c_{ab} J_c$ and $[P_a, P_b] = 0$, while the cross relations are obtained from the group law, namely $[J_a, P_b] = \mathrm{ad}^*_{J_a} P_b$. More explicitly

$$\langle [J_a, P_b], J_c \rangle = \langle \mathrm{ad}^*_{J_a} P_b, J_c \rangle = \langle -(\mathrm{ad}_{J_a})^*, J_c \rangle = -\langle P_b, \mathrm{ad}_{J_a} J_c \rangle = -c^b_{ac} = -c^b_{ac} \langle P_c, J_c \rangle, \tag{3.81}$$

so $\mathrm{ad}^*_{J_a} P_b = -c^b_{ac} P_c$. Therefore, the Lie algebra commutators will be

$$[J_a, J_b] = c^c_{ab} J_c, \qquad [J_a, P_b] = \mathrm{ad}^*_{J_a} P_b = -c^b_{ac} P_c, \qquad [P_a, P_b] = 0. \tag{3.82}$$

Theorem 3.7. *With the same notation as above the homogeneous space $M = G/H$ is a Poisson homogeneous space of Lie algebraic type isomorphic to $\mathfrak{h}$.*

Proof. It is straightforward from (3.51) to see that $G = D(H)$ is a Drinfel'd double with $Y_a = J_a$ and $y_a = P_a$, and so the classical skew-symmetric r-matrix $r = \sum_a P_a \wedge J_a$ defines a Poisson-Lie structure on G by means of the Sklyanin bracket, which in this case takes the simple form

$$\{f_1, f_2\} = \sum_a \left(X^L_{P_a} f_1 X^L_{J_a} f_2 - X^R_{P_a} f_1 X^R_{J_a} f_2 \right). \tag{3.83}$$

Let us introduce local coordinates in a neighborhood U of $g \in G$ by the map $\alpha : U \to \mathbb{R}^n$, $(g\, \Pi^n_{a=1} \exp x^a P_a\, \Pi^n_{b=1} \exp \theta^b J_b) \to (x^a, \theta^a)$. In this way x^a define a set of local coordinates in M and it suffices to compute the Sklyanin bracket for them

$$\{x^b, x^c\} = \sum_a \left(X^L_{P_a} x^b X^L_{J_a} x^c - X^R_{P_a} x^b X^R_{J_a} x^c \right). \tag{3.84}$$

Using right-invariance of x^a and the definition of local coordinates we have that

$$
\begin{aligned}
X^L_{J_a} x^c =& \frac{d}{dt}\Big|_{t=0} x^c(e^{x^k P_k} e^{\theta^k J_k} e^{tJ_a}) = \frac{d}{dt}\Big|_{t=0} x^c(e^{x^k P_k} e^{\theta^k J_k}) = 0,\\
X^R_{P_a} x^b =& \frac{d}{dt}\Big|_{t=0} x^b(e^{tP_a} e^{x^k P_k} e^{\theta^k J_k}) = \frac{d}{dt}\Big|_{t=0} x^b(e^{tP_a} e^{x^k P_k})\\
=& \frac{d}{dt}\Big|_{t=0} (t + x^k)\delta_{ab} = \delta_{ab},
\end{aligned}
\tag{3.85}
$$

and

$$
\begin{aligned}
X^R_{J_a} x^c =& \frac{d}{dt}\Big|_{t=0} x^c(e^{tJ_a} e^{x^k P_k} e^{\theta^k J_k}) = \frac{d}{dt}\Big|_{t=0} x^c(e^{tJ_a} e^{x^k P_k})\\
=& \frac{d}{dt}\Big|_{t=0} x^c(e^{tJ_a} e^{x^k P_k} e^{-tJ_a}) = \frac{d}{dt}\Big|_{t=0} x^c(\mathrm{Ad}^*_{e^{tJ_a}}(x^k P_k)) = x^c(x^k \mathrm{ad}^*_{J_a} P_k)\\
=& x^c(-c^k_{al} P_l x^k) = -c^k_{al} x^c(x^k P_l) = -c^k_{al}\delta_{cl} x^k = -c^k_{ac} x^k.
\end{aligned}
\tag{3.86}
$$

Therefore, the fundamental brackets for the Poisson structure π of the homogeneous space $M = G/H$ read

$$
\{x^b, x^c\} = c^k_{ac} x^k \delta_{ab} = c^k_{bc} x^k,
\tag{3.87}
$$

which is of Lie algebraic type (linear). □

Corollary 3.3. *Any linear Poisson structure on a manifold M with dimension d can be constructed as a Poisson homogeneous space for some Lie group G of dimensions $2d$.*

3.6 (Co)algebras, bialgebras and Hopf algebras

Until this moment, the description has been mainly geometrical. Let us now switch to a more algebraic language that will be useful in the rest of this Thesis, specially when introducing the quantization of Poisson structures.

Definition 3.19. An *algebra* (A, μ, i) over a commutative ring k is a k-module A equipped with two k-module maps: the multiplication (or product) $\mu : A \otimes_k A \to A$, and the unit $i : A \to A$, making the following diagrams commutative

$$
\begin{tikzcd}
A \otimes k \arrow[r, "\mathrm{id}\otimes i"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & k \otimes A \arrow[l, "i\otimes\mathrm{id}"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[dr, "\mu"] & \\
A \otimes A \otimes A \arrow[ur, "\mu\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\mu"'] & & A \\
& A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
\tag{3.88}
$$

These properties are equivalent to $i(\alpha) = \alpha 1$ and $\mu(a_1 \otimes a_2) = a_1 \cdot a_2$, for all $\alpha 1 \in k$ and $a_1, a_2 \in A$. In terms of the *flip operator*

$$\begin{aligned} \sigma : A \otimes A &\to A \otimes A \\ (a_1, a_2) &\to (a_2, a_1), \end{aligned} \tag{3.89}$$

an algebra is *commutative* if the following diagram is commutative

$$\begin{tikzcd} A \otimes A \arrow[rr, "\sigma"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{3.90}$$

In other words $\mu(a_1, a_2) = \mu \circ \sigma(a_1, a_2) = \mu(a_2, a_1)$ for all $a_1, a_2 \in A$.

An algebra homomorphism is a k-module map $\phi : A \to B$ which is compatible with the products and units in A and B, in the sense that $\phi(\mu_A(a_1 \otimes a_2)) = \mu_B((\phi \otimes \phi)(a_1 \otimes a_2))$ for all $a_1, a_2 \in A$, and $\phi(i_A(\alpha)) = i_B(\alpha)$ for all $\alpha \in k$. These conditions can be equivalently stated as the commutativity of the following diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "\phi \otimes \phi"'] & A \arrow[d, "\phi"] \\ B \otimes B \arrow[r, "\mu_B"] & B \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "i_A"] \arrow[d, "\simeq"'] & A \arrow[d, "\phi"] \\ k \arrow[r, "i_B"'] & B \end{tikzcd} \tag{3.91}$$

Definition 3.20. Let $\mathfrak{g}$ be a Lie algebra over the field k. As a vector space, we can consider its tensor algebra $T(\mathfrak{g}) = \oplus_{l \geq 0} \mathfrak{g}^{\otimes l}$(where $\mathfrak{g}^{\otimes 0} = k$). Let I be the two-sided ideal of $T(\mathfrak{g})$ generated by $X \otimes Y - Y \otimes X - [X, Y]$. Then the *universal enveloping algebra* of $\mathfrak{g}$ is $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$.

Definition 3.21. A *coalgebra* (A, Δ, ϵ) over a commutative ring k is a k-module A equipped with two k-module maps: the comultiplication (or coproduct) $\Delta : A \to A \otimes A$, and the counit $\epsilon : A \to A$, making the following diagrams commutative

$$\begin{tikzcd} A \otimes k & A \otimes A \arrow[l, "\mathrm{id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}"] & k \otimes A \\ & A \arrow[ul, "\simeq"] \arrow[u, "\Delta"'] \arrow[ur, "\simeq"'] & \end{tikzcd} \qquad \begin{tikzcd} & A \otimes A \arrow[dl, "\Delta \otimes \mathrm{id}"'] & \\ A \otimes A \otimes A & & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ & A \otimes A \arrow[ul, "\mathrm{id} \otimes \Delta"] & \end{tikzcd} \tag{3.92}$$

We say that a coalgebra is *cocommutative* if $\Delta(a) = \sigma \circ \Delta(a)$ for all $a \in A$, or equivalently if the following diagram is commutative

$$
\begin{tikzcd}
A \otimes A & & A \otimes A \arrow[ll, "\sigma"'] \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\tag{3.93}
$$

A coalgebra homomorphism is a k-module map $\phi : A \to B$ which is compatible with the coproducts and the counits in A and B in the sense that $(\phi \otimes \phi)(\Delta(a)) = \Delta(\phi(a))$ and $\epsilon_B(\phi(a)) = \epsilon_A(a)$ for all $a \in A$. These conditions can be equivalently stated as the commutativity of the following diagram

$$
\begin{tikzcd}
B \otimes B & B \arrow[l, "\Delta"'] \\
A \otimes A \arrow[u, "\phi\otimes\phi"] & A \arrow[l, "\Delta"] \arrow[u, "\phi"']
\end{tikzcd}
\tag{3.94}
$$

In the language of category theory, we say that a coalgebra is the dual of an algebra, where the duality should be understood as 'reversing arrows'.

Definition 3.22. A *bialgebra* $(A, \mu, \Delta, i, \epsilon)$ over a commutative ring k is a k-module A such that

i) A is both an algebra and a coalgebra over k,

ii) The comultiplication $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to A$ are algebra homomorphisms,

iii) The multiplication $\mu : A \otimes A \to A$ and the unit $i : A \to A$ are coalgebra homomorphisms.

The above definition means that a bialgebra is both an algebra and a coalgebra in such a way that both structures are compatible. Bialgebras are important objects because for them we know how to construct tensor product representations of algebras, a key fact in the applications to physics, where algebras describe physical observables.

Definition 3.23. A *Hopf algebra* $(A, \mu, \Delta, i, \epsilon, S)$ over a commutative ring k is a bialgebra $(A, \mu, \Delta, i, \epsilon)$ together with a bijective k-module map, the antipode $S : A \to A$, making the following diagram commutative

$$
\begin{tikzcd}
A \otimes A \arrow[r, "S\otimes\mathrm{id}"] & A \otimes A \arrow[d, "\mu"] & A \otimes A \arrow[l, "\mathrm{id}\otimes S"'] \\
A \arrow[u, "\Delta"] \arrow[r, "i\circ\epsilon"'] & A & A \arrow[l, "i\circ\epsilon"] \arrow[u, "\Delta"']
\end{tikzcd}
\tag{3.95}
$$

Let A and B be Hopf algebras. A *Hopf algebra homomorphism* is a k-module map $\phi : A \to B$ such that it is an algebra and coalgebra homomorphism.

Given a Hopf algebra A and a two-sided ideal I of A as an algebra, we say that I is a *Hopf ideal* if

$$\Delta(I) \subseteq I \otimes A + A \otimes I, \qquad \epsilon(I) = 0, \qquad S(I) \subseteq I. \tag{3.96}$$

The k-module A/I inherits a Hopf algebra structure from the one in A.

Note that, by definition, Hopf algebras are associative and coassociative, while they could be noncommutative, non-cocommutative or both. To illustrate these concepts, let us now consider three simple instances (see [35] for more examples) of Hopf algebra structures for finite or algebraic groups.

Example 3.2. Let G be a finite group and denote by e its identity element. We define the group algebra $k[G]$ of G over the commutative ring k as a free k-module with basis G, where the product of $k[G]$ is obtained by extending linearly the one in G. We obtain a Hopf algebra structure on $k[G]$ just by considering the following formulas for the coproduct, unit, counit and antipode

$$\Delta(g) = g \otimes g, \qquad i(1) = e, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1}, \tag{3.97}$$

for all $g \in G$, and extending them linearly. This Hopf algebra is cocommutative, but it is commutative only if G is commutative.

$\diamondsuit$

Motivated by this example, for any Hopf algebra A we say that an element $a \in A$ is group-like if $\Delta(a) = a \otimes a$.

Example 3.3. As in the previous Example, let G be a finite group. Then the set $\mathcal{F}(G)$ of regular functions with values in the ring k has a Hopf algebra structure where the multiplication and unit are defined pointwise, while the counit and antipode are given by $\epsilon(f) = f(e)$ and $S(f)(g) = f(g^{-1})$ for all $g \in G$, respectively. To define the coproduct consider the isomorphism $\mathcal{F}(G) \otimes \mathcal{F}(G) \to \mathcal{F}(G \times G)$ that assigns to every element $f_1 \otimes f_2 \in \mathcal{F}(G) \otimes \mathcal{F}(G)$ the function $f_{12} \in \mathcal{F}(G \times G)$ defined by $f_{12}(g_1, g_2) = f_1(g_1) f_2(g_2)$, for every $g_1, g_2 \in G$. This isomorphism allows us to have a well defined coproduct $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$ given by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$. This Hopf algebra structure is always commutative, but it is cocommutative if and only if G is commutative.

$\diamondsuit$

Example 3.4. Let G be an affine algebraic group over a field k and $\mathcal{F}(G)$ its algebra of regular functions. As in the previous Example, we have that $\mathcal{F}(G) \otimes \mathcal{F}(G) \simeq \mathcal{F}(G \times G)$ and the very same construction defines a Hopf algebra on $\mathcal{F}(G)$. As a concrete example, consider $G = GL(n, k)$ and denote by x_{ij} the matrix elements of $X \in G$, where $i, j \in \{1, \ldots, n\}$. Then $\mathcal{F}(G)$ is the commutative algebra with generators x_{ij} and D^{-1} with the relation $D^{-1}\det X = 1$. The coalgebra structure is defined by

$$\begin{aligned}
\Delta(x_{ij}) &= \sum_{k=1}^{n} x_{ik}\otimes x_{kj},\\
\Delta(D^{-1}) &= D^{-1}\otimes D^{-1},\\
\epsilon(x_{ij}) &= \delta_{ij},\\
\epsilon(D^{-1}) &= 1,
\end{aligned} \tag{3.98}$$

while the antipode reads

$$S(x_{ij}) = (X^{-1})_{ij}, \qquad S(D^{-1}) = D \tag{3.99}$$

where $(X^{-1})_{ij}$ are the matrix elements of the inverse matrix to X.

$\diamondsuit$

The following two examples will be extremely important in the rest of this work. The first one introduces a cocommutative Hopf algebra structure on the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, while the second one defines a commutative Hopf algebra structure on the algebra of functions on a topological group.

Example 3.5. With the notation of Definition 3.20, I is a Hopf ideal and $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra. By the Poincaré-Birkhoff-Witt theorem, we can see the monomials $x_{i_1}x_{i_2}\cdots x_{i_k}$ as generators of $\mathcal{U}(\mathfrak{g})$ if $\{x_{i_k}\}$ is a basis of $\mathfrak{g}$. It is sufficient to define the coproduct, counit and antipode on Lie algebra elements. They are given by

$$\Delta(X) = X\otimes 1 + 1\otimes X, \qquad i(1) = 1, \qquad \epsilon(X) = 0, \qquad S(X) = -X, \qquad \forall X\in\mathfrak{g}, \tag{3.100}$$

respectively. These conditions, together with the Hopf algebra axioms, are sufficient to compute the image by these maps of every element of $\mathcal{U}(\mathfrak{g})$, including

$$\Delta(1) = 1\otimes 1, \qquad \epsilon(1) = 1, \qquad S(1) = 1. \tag{3.101}$$

This is an example of a cocommutative Hopf algebra, which is commutative only if $\mathfrak{g}$ is commutative.

$\diamondsuit$

Example 3.6. Let G be a compact topological group (in particular a Lie group). Consider the algebra $A = \mathcal{C}^\infty(G)$ of continuous functions on G, where the multiplication μ is just the usual pointwise multiplication of functions, i.e. $\mu(f_1, f_2)(g) = f_1(g)f_2(g)$ for all $f_1, f_2 \in \mathcal{C}^\infty(G)$ and $g\in G$. Now define

$$\Delta(f)(g_1\otimes g_2) = f(g_1g_2), \qquad i(1) = u, \qquad \epsilon(f) = f(e), \qquad S(f)(g) = f(g^{-1}), \tag{3.102}$$

for all $f\in\mathcal{C}^\infty(G)$ and $g, g_1, g_2\in G$. $u : G\to\mathbb{R}$ is the constant function $u(g) = 1$ for all $g\in G$. In fact, this definition does not provide a well-defined Hopf algebra, since this coproduct takes values in $\mathcal{C}^\infty(G\times G)$ which is strictly larger that $\mathcal{C}^\infty(G)\otimes\mathcal{C}^\infty(G)$. In order to solve this problem (note that another option is to realize that the topology on $\mathcal{C}^\infty(G)$ allows us to regard $\mathcal{C}^\infty(G\times G)$ as a completion of $\mathcal{C}^\infty(G)\otimes\mathcal{C}^\infty(G)$), we will consider continuous representations $\rho : G \to GL(n,\mathbb{R})$. We call $\rho_{ij} = x_{ij}\circ\rho$ to the matrix elements of ρ. As ρ runs through all finite-dimensional representations of G, ρ_{ij} generate a subalgebra $\mathrm{Rep}(G)$ of $\mathcal{C}^\infty(G)$. Then it can be shown that $\mathrm{Rep}(G)$ is dense on $\mathcal{C}^\infty(G)$ and $\mathrm{Rep}(G\times G)$ is isomorphic to $\mathrm{Rep}(G)\otimes\mathrm{Rep}(G)$, so the maps defined previously endow $\mathrm{Rep}(G)$ with a Hopf algebra structure. Also, the coproduct

$$\Delta(\rho_{ij}) = \sum_{k=1}^{n}\rho_{ik}\otimes\rho_{kj}, \tag{3.103}$$

induce a Hopf algebra structure on $\mathrm{Rep}(G)$. This is the precise Hopf algebra structure that we will use in the main part of this work, when we consider matrix Lie groups.

$\diamondsuit$

When $\mathfrak{g} = \mathrm{Lie}(G)$ the previous examples 3.5 and 3.6 are related in a precise way: they are duals as Hopf algebras. Let us make this concept precise by considering a Hopf algebra $(A,\mu,\Delta,i,\epsilon,S)$, the dual vector space A^* to A and its canonical pairing $\langle\cdot,\cdot\rangle$. Then $(A^*,\mu^*,\Delta^*,i^*,\epsilon^*,S^*)$ is a Hopf algebra with the maps defined by

$$\begin{aligned}
&\langle\mu^*(f_1\otimes f_2),x\rangle = \langle f_1\otimes f_2,\Delta(x)\rangle,\\
&\langle\Delta^*(f),x_1\otimes x_2\rangle = \langle f,\mu(x_1,x_2)\rangle,\\
&\langle i^*(\alpha),x\rangle = \alpha\epsilon(x),\\
&\epsilon^*(f) = \langle f,1\rangle,\\
&\langle S^*(f),x\rangle = \langle f,S(x)\rangle.
\end{aligned} \tag{3.104}$$

We say that $(A,\mu,\Delta,i,\epsilon,S)$ *and* $(A^*,\mu^*,\Delta^*,i^*,\epsilon^*,S^*)$ *are duals as Hopf algebras.*

The explicit duality between Hopf algebras of Examples 3.5 and 3.6 is given [195] by considering the unique homomorphism

$$\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{C}^\infty(G)), \tag{3.105}$$

which is an extension of homomorphism which assigns to any element $T_a\in\mathfrak{g}$ its associated right-invariant vector field (2.23),

$$\begin{aligned}
\rho : \mathfrak{g} &\to \mathrm{End}(\mathcal{C}^\infty(G))\\
T_a &\to X_a^R.
\end{aligned} \tag{3.106}$$

In this way, we can see elements of higher order in $\mathcal{U}(\mathfrak{g})$ as differential operators on $\mathcal{C}^\infty(G)$. Then define the bilinear form

$$\begin{aligned}
\langle\cdot,\cdot\rangle : \mathcal{C}^\infty(G)\times\mathcal{U}(\mathfrak{g}) &\to k\\
(f,a) &\to (\rho(a)f)(e)
\end{aligned} \tag{3.107}$$

for all $f\in\mathcal{C}^\infty(G), a\in\mathcal{U}(\mathfrak{g})$ (e denotes as usual the identity in G). It is easy to see that the map $\mathcal{C}^\infty(G)\to(\mathcal{U}(\mathfrak{g}))^*$ defined by $f\to\langle f,\cdot\rangle$ is inyective, so $\mathcal{C}^\infty(G)$ can be embedded into $(\mathcal{U}(\mathfrak{g}))^*$ and so we have the duality between $\mathcal{C}^\infty(G)$ and $\mathcal{U}(\mathfrak{g})$. Now, using the duality (3.104) we find that the Hopf algebra structures defined in Examples 3.5 and 3.6 are indeed dual as Hopf algebras.

3.6.1 Poisson-Hopf algebras

In the following we introduce the notion of a Poisson-Hopf algebra, which will be a central one throughout this Thesis. With this aim, we need some previous definitions.

Definition 3.24. A *Poisson algebra* is a commutative and associative algebra (A, μ, i) together with a bilinear map

$$\{\cdot,\cdot\} : A \times A \to A \tag{3.108}$$

such that

PA1. $\{a, b\} = -\{b, a\}$,

PA2. $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$,

PA3. $\{ab, c\} = a\{b, c\} + \{a, c\}b$.

Note that PA1 and PA2 are just L1 and L2 for Lie algebras in Definition 2.6, so they just state that A is a Lie algebra with respect to $\{\cdot,\cdot\}$. PA3 is the condition that the Poisson bracket acts as a derivation.

Example 3.7. Let (M, π) be a Poisson manifold. Then the algebra of functions on M, $\mathcal{C}^\infty(M)$, is a Poisson algebra.

$\diamondsuit$

Let A, B be two algebras. Their tensor algebra $A \otimes B$ is an algebra with multiplication map given by

$$\begin{aligned} \mu_{A\otimes B} : (A \otimes B) \times (A \otimes B) &\to A \otimes B \\ (a_1 \otimes b_1, a_2 \otimes b_2) &\to \mu_A(a_1, a_2) \otimes \mu_B(b_1, b_2) \end{aligned} \tag{3.109}$$

We can also write $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (1 \otimes \sigma \otimes 1)$. Now, if $(A, \{\cdot,\cdot\}_A)$ and $(B, \{\cdot,\cdot\}_B)$ are Poisson algebras, their tensor product inherits a natural Poisson algebra structure given by

$$\{a_1 \otimes b_1, a_2 \otimes b_2\}_{A\otimes B} = \{a_1, a_2\}_A \otimes \mu_B(b_1, b_2) + \mu_A(a_1, a_2) \otimes \{b_1, b_2\}_B \tag{3.110}$$

for all $a_1, a_2 \in A, b_1, b_2 \in B$.

If we now consider bialgebras instead of just algebras the natural compatibility condition to require for is that the coproduct Δ be a Poisson map. Thus we have

Definition 3.25. A *Poisson bialgebra* $(A, \mu, i, \{\cdot,\cdot\}, \Delta)$ is a Poisson algebra $(A, \mu, i, \{\cdot,\cdot\})$ together with a coproduct Δ that is a Poisson map with respect to $\{\cdot,\cdot\}_A$ in A and $\{\cdot,\cdot\}_{A\otimes A}$ in $A \otimes A$.

A *Poisson-Hopf algebra* $(A, \mu, i, \{\cdot,\cdot\}, \Delta, S)$ is a Poisson bialgebra $(A, \mu, i, \{\cdot,\cdot\}, \Delta)$ which also has an antipode S.

3.6.2 (Co)actions of (co)algebras and Hopf algebra representations

Let us now introduce the notions of actions and coactions of algebras and coalgebras, respectively. We follow closely [35]. Let (A, μ, i) be an algebra (see Definition 3.19) over a commutative ring k.

Definition 3.26. A *left A-module* is a k-module V endowed with a k-module map (also called *left action)* $\lambda : A \times V \to V$ such that the following diagrams commute:

$$\begin{tikzcd} A\otimes A\otimes V \arrow[r, "\mu\otimes \mathrm{id_V}"] \arrow[d, "\mathrm{id_A}\otimes\lambda"'] & A\otimes V \arrow[d, "\lambda"] \\ A\otimes V \arrow[r, "\lambda"'] & V \end{tikzcd} \qquad \begin{tikzcd} k\otimes V \arrow[r, "i\otimes \mathrm{id_V}"] \arrow[d, "\simeq"'] & A\otimes V \arrow[d, "\lambda"] \\ V \arrow[r, "\mathrm{id_V}"'] & V \end{tikzcd} \tag{3.111}$$

Now, let (A, Δ, ϵ) be a coalgebra (see Definition 3.21) over a commutative ring k.

Definition 3.27. A *right A-comodule* is a k-module V endowed with a k-module map (also called *right coaction*) $\rho : V \to V \otimes A$ such that the following diagrams commute:

$$\begin{tikzcd} V\otimes A\otimes A & V\otimes A \arrow[l, "\mathrm{id_V}\otimes\Delta"'] \\ V\otimes A \arrow[u, "\rho\otimes \mathrm{id_A}"] & V \arrow[l, "\rho"] \arrow[u, "\rho"'] \end{tikzcd} \qquad \begin{tikzcd} V\otimes k & V\otimes A \arrow[l, "\mathrm{id_V}\otimes\epsilon"'] \\ V \arrow[u, "\simeq"] & V \arrow[l, "\mathrm{id_V}"] \arrow[u, "\rho"'] \end{tikzcd} \tag{3.112}$$

If the k-modules V have extra structure, it is therefore natural to require that such structure is preserved by left actions and right coactions. In particular, let us assume that $(A, \mu_A, \Delta_A, i_A, \epsilon_A)$ is a bialgebra (or a Hopf algebra if it is endowed with an antipode). Then we say that an algebra (V, μ_V, i_V) is a *left A-module algebra* if it is a left A-module and

$$\lambda(a \otimes \mu_V(v_1 \otimes v_2)) = \sum_i \mu_V(\lambda(a_i \otimes v_1) \otimes \lambda(a^i \otimes v_2)), \qquad \lambda(a \otimes 1) = \epsilon_A(a)1, \tag{3.113}$$

where $\Delta(a) = \sum_i a_i \otimes a^i$, for all $a \in A, v_1, v_2 \in V$ (Sweedler's notation). Similarly, we say that a coalgebra $(V, \Delta_V, \epsilon_V)$ is a *left A-module coalgebra* if it is a left A-module and

$$\Delta_V(\lambda(a \otimes v)) = \sum_{i,j} \lambda(a_i \otimes v_j) \otimes \lambda(a^i \otimes v^j), \qquad \epsilon_V(\lambda(a \otimes v)) = \epsilon_A(a)\epsilon_V(v), \tag{3.114}$$

where $\Delta_V(v) = \sum_j v_j \otimes v^j$, for all $a \in A, v \in V$.

For the case that the bialgebra A and the k-module V are endowed with Poisson structures (see Definition 3.25), we have:

Definition 3.28. Let $(A, \mu, i, \{\cdot,\cdot\}, \Delta, S)$ be a Poisson-Hopf algebra and $(V, \mu, i, \{\cdot,\cdot\})$ a Poisson algebra, then V is a *Poisson A-comodule algebra* if the right coaction $\rho : V \to A\otimes V$ is a Poisson map, where the Poisson structure in $A \otimes V$ is given by (3.110).

Example 3.8. Let $M = G/H$ be a G-space with action $\alpha : G \to \mathrm{Diff}\,(M)$. Let A be the Hopf algebra $\mathcal{C}^\infty(G)$ and V the algebra $\mathcal{C}^\infty(M)$. Recall that the action α satisfies that $\alpha_{g_1 g_2}(m) = \alpha_{g_1}(\alpha_{g_2}(m))$ because it is a homomorphism. This property is exactly what is implied by the axioms of left action and right coaction. In the first case we have that

$$\lambda(\mathrm{id} \otimes \lambda)(a \otimes b \otimes v) = \lambda(a \otimes (\lambda(b \otimes v))) \tag{3.115}$$

is equal to

$$\lambda(\mu \otimes \mathrm{id})(a \otimes b \otimes v) = \lambda(\mu(a \otimes b) \otimes v), \tag{3.116}$$

which is exactly the previous property for the action α in an algebraic language. For right coactions we have, if $\rho(v) = v \otimes a$, that

$$(\rho \otimes \mathrm{id})\rho(v) = (\rho \otimes \mathrm{id})(v \otimes a) = v \otimes a \otimes a, \tag{3.117}$$

should equal

$$(\mathrm{id} \otimes \Delta)\rho(v) = (\mathrm{id} \otimes \Delta)(v \otimes a) = v \otimes \Delta(a), \tag{3.118}$$

which is the dual of the previous property for actions. If moreover, the G-space $M = G/H$ is indeed a Poisson G-space, then the algebraic version of the covariance (compatibility of the action with the Poisson structures), is just the condition

$$\{\lambda(a_1 \otimes v_1), \lambda(a_2 \otimes v_2)\}_V = \lambda(\{a_1 \otimes v_1, a_2 \otimes v_2\}_{A \otimes V}) \tag{3.119}$$

for left actions, or its dual

$$\rho(\{v_1, v_2\}_V) = \{\rho(v_1), \rho(v_2)\}_{V \otimes A} \tag{3.120}$$

for right coactions. ♢

Definition 3.29. A *representation of a Hopf algebra* $(A, \mu, \Delta, i, \epsilon, S)$ over a commutative ring k is a left A-module. Similarly, a *co-representation of a Hopf algebra* is a right A-comodule.

Note that the definition of Hopf algebra representation only uses its algebra axioms, while the co-representation only uses the coalgebra ones.

3.7 Quantization

During this Thesis, Hopf algebras will play a central role. Moreover, we will be specially interested in 'deformations' of Hopf algebras, which roughly speaking are modifications depending on a parameter (the *deformation parameter* or *quantum parameter*) of Hopf algebras such that they keep the Hopf algebra structure and the original Hopf algebra is obtained as an appropriate analytic limit of the deformation parameter h, as in Theorem 3.4.

Definition 3.30. A *deformation of a Hopf algebra* $(A, \mu, \Delta, i, \epsilon, S)$ over a field k is a topological Hopf algebra $(A_h, \mu_h, \Delta_h, i_h, \epsilon_h, S_h)$ over the ring $k[[h]]$ of formal power series in an indeterminate h over k, such that

i) A_h is isomorphic to $A[[h]]$ as a $k[[h]]$-module,

ii) $\mu_h \equiv \mu \pmod{h}$,

iii) $\Delta_h \equiv \Delta \pmod{h}$,

where $A[[h]]$ denotes the algebra of formal power series in h with coefficients on A, i.e. elements of the form

$$\sum_{n=0}^{\infty} a_n h^n, \tag{3.121}$$

with $a_n \in A$.

Definition 3.31. A *quantum universal enveloping algebra (QUEA)* $\mathcal{U}_h(\mathfrak{g})$ is a deformation (in the sense of Definition 3.30) of the Hopf algebra structure on the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ defined in Example 3.5. We will also refer to a QUEA as a *quantum deformation* of $\mathfrak{g}$, or simply as a *quantum algebra*.

For any quantum algebra $\mathcal{U}_h(\mathfrak{g})$ with coproduct

$$\Delta_h : \mathcal{U}_h(\mathfrak{g}) \to \mathcal{U}_h(\mathfrak{g}) \otimes \mathcal{U}_h(\mathfrak{g}), \tag{3.122}$$

the Lie algebra $\mathfrak{g}$ inherits a Lie bialgebra structure with coproduct

$$\delta = \Delta_h - \sigma \circ \Delta_h \quad (\text{mod} h). \tag{3.123}$$

The first order deformation given by δ provides the relevant information about the full quantum deformation and this fact will be used throughout this Thesis. In this sense, we can say that the quantum algebra $\mathcal{U}_h(\mathfrak{g})$ is a deformation of the Lie algebra $\mathfrak{g}$ in the direction of the Lie bialgebra $(\mathfrak{g}, \delta)$ with deformation parameter h.

Definition 3.32. A *quantization of a Poisson algebra* $(A, \mu, i, \{\cdot,\cdot\})$ over a field k is a noncommutative but associative topological algebra $(A_h, \mu_h, i_h, [\cdot,\cdot])$ over the ring $k[[h]]$ of formal power series in an indeterminate h over k, such that

i) A_h is isomorphic to $A[[h]]$ as a $k[[h]]$-module,

ii) $\mu_h \equiv \mu \pmod{h}$,

iii) $\{a_1, a_2\} \equiv \frac{[\hat{a}_1, \hat{a}_2]}{h} \pmod{h}$,

where $a_i = \hat{a}_i \pmod{h}$, $\hat{a}_1, \hat{a}_2 \in A_h$, $a_1, a_2 \in A$, and

$$[\cdot,\cdot] = \mu_h - \mu_h \circ \sigma, \tag{3.124}$$

denotes the usual commutator on A_h. Also, $A[[h]]$ denotes the algebra of formal power series in h with coefficients on A, i.e. elements of the form

$$\sum_{l=0}^{\infty} a_l h^l, \tag{3.125}$$

with $a_l \in A$.

The concept of quantization of Poisson-Hopf algebras is stated as follows.

Definition 3.33. Let A be a commutative Poisson-Hopf algebra over a field k of characteristic zero, and let $\{\cdot,\cdot\}$ be its Poisson bracket. A *quantization of a Poisson-Hopf algebra* A is a Hopf algebra deformation A_h (see Definition 3.30) of A such that

$$\{a_1, a_2\} \equiv \frac{[\hat{a}_1, \hat{a}_2]}{h} \pmod{h}, \tag{3.126}$$

if $a_1 = \hat{a}_1 \pmod{h}$ and $a_2 = \hat{a}_2 \pmod{h}$, for all $\hat{a}_1, \hat{a}_2 \in A_h$ and $a_1, a_2 \in A$. A *quantization of an algebraic Poisson-Lie group* (G, Π) is a quantization $\mathcal{C}_h^\infty(G)$ of the algebra of regular functions $\mathcal{C}(G)$, regarded as a Poisson algebra, and (G, Π) is called the classical limit of $C_h^\infty(G)$.

Note that in this way a quantization of a Poisson-Hopf algebra is just a deformation of its Hopf algebra structure together with a quantization of its Poisson algebra structure in such a way that they are compatible.

Definition 3.34. A *quantum group* is a quantization of an algebraic Poisson-Lie group.

For Poisson comodule algebras the natural quantization is defined as follows.

Definition 3.35. Let V be a Poisson A-comodule algebra with right coaction $\rho : V \to A \otimes V$. A *quantization of a Poisson A-comodule algebra* is a quantization A_h of A as Poisson-Hopf algebra and V_h of V as a Poisson algebra, together with a map $\rho_h : V_h \to A_h \otimes V_h$ respecting the algebra structure (quantization of the Poisson bracket) on V_h and $A_h \otimes V_h$.

3.8 The basic Lorentzian example in (1+1) dimensions

In order to illustrate some of the concepts introduced, let us consider a simple but non-trivial example, which will be worked out in full detail. In fact, this will be the approach that we will follow in Chapters 4 and 5 where we generalize these results to the (2+1) and (3+1) dimensional cases.

Let us take G_0^{1+1}, i.e. the Poincaré group in (1+1) dimensions given by (2.121) in §2.3.4, so $\mathfrak{g}_0^{1+1} = \mathrm{Lie}(G_0^{1+1})$ (hereafter we omit the dimensional superscript). For convenience, we recall that the Lie algebra $\mathfrak{g}_0$ reads

$$[K, P_0] = P_1, \qquad [K, P_1] = P_0, \qquad [P_0, P_1] = 0, \tag{3.127}$$

and has the following quadratic Casimir

$$\mathcal{C} = P_0^2 - P_1^2. \tag{3.128}$$

A general element $\mathfrak{g}_0$ can be written as

$$Q_0 = x^\mu P_\mu + \xi K = \begin{pmatrix} 0 & 0 & 0 \\ x^0 & 0 & \xi \\ x^1 & \xi & 0 \end{pmatrix}. \tag{3.129}$$

and thus an element of G_0^{1+1}, sufficiently close to the identity, can be written

$$G_0^{1+1} = \begin{pmatrix} 1 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi \\ x^1 & \sinh\xi & \cosh\xi \end{pmatrix}. \tag{3.130}$$

First of all, let us introduce the so-called Lie-Poisson structure on the dual vector space $\mathfrak{g}_0^*$ (see the detailed construction of this Poisson structure on Example 3.1). Then we rigorously define what we will call in the remaining of this work the *Poisson version of* $\mathfrak{g}_0$. This Poisson structure, denoted by $\mathcal{P}(\mathfrak{g}_0)$, is defined by the fundamental brackets

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = P_0, \qquad \{P_0, P_1\} = 0. \tag{3.131}$$

where P_0, P_1, K are to be seen as a set of global coordinate functions on $\mathfrak{g}_0^*$. As explained in Example 3.1, these Poisson brackets are formally the Lie algebra commutation relations (3.127) where commutators have been replaced by Poisson brackets.

We know (see Example 3.5) that $\mathcal{U}(\mathfrak{g}_0)$ has a primitive Hopf algebra structure, i.e. for $\mathfrak{g}_0$ we have

$$\Delta_0(P_0) = P_0\otimes 1+1\otimes P_0, \qquad \Delta_0(P_1) = P_1\otimes 1+1\otimes P_1, \qquad \Delta_0(K) = K\otimes 1+1\otimes K, \tag{3.132}$$

and we extend this linearly to $\mathcal{U}(\mathfrak{g}_0)$. For instance

$$\Delta_0(P_0P_1) = \Delta_0(P_0)\Delta_0(P_1) = P_0P_1 \otimes 1 + P_0 \otimes P_1 + P_1 \otimes P_0 + 1 \otimes P_0P_1. \tag{3.133}$$

A well-known quantum quantum deformation of the Hopf algebra $\mathcal{U}(\mathfrak{g}_0, \Delta_0)$, the so-called κ-deformation, is defined by the element $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$, which is a solution of the mCYBE (3.36), given by

$$r = \frac{1}{\kappa} K \wedge P_1, \tag{3.134}$$

where $\kappa \in \mathbb{R}$ is the deformation (or quantum) parameter, that in this case should be physically thought as related to the Planck energy. From (3.134) we directly obtain the following cocommutator

$$\delta(P_0) = 0, \qquad \delta(P_1) = -\frac{1}{\kappa} P_0 \wedge P_1, \qquad \delta(K) = -\frac{1}{\kappa} P_0 \wedge K_1. \tag{3.135}$$

The associated quantum Poincaré algebra, denoted by $\mathcal{U}_\kappa(\mathfrak{g}_0)$, is given by the following deformed commutation relations

$$[K, P_0] = P_1, \qquad [K, P_1] = \frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) - \frac{1}{2\kappa} P_1^2, \qquad [P_0, P_1] = 0, \tag{3.136}$$

with coproduct

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_1) &= P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1, \\ \Delta(K) &= K \otimes 1 + e^{-P_0/\kappa} \otimes K. \end{aligned} \tag{3.137}$$

Note that the skew-symmetrized part of the first order of this coproduct is given by the previous cocommutator (3.135), i.e. $\delta(X) = (\Delta^{(1)} - \sigma \circ \Delta^{(1)})(X)$, for all $X \in \mathfrak{g}_0$.

We omit the counit and antipode because they can be computed from the Hopf algebra axioms. The deformed quadratic Casimir for $\mathcal{U}_\kappa(\mathfrak{g}_0)$ is given by

$$\mathcal{C}_\kappa = 4\kappa^2 \sinh^2(P_0/(2\kappa)) - e^{P_0/\kappa} P_1^2. \tag{3.138}$$

We remark that the limit $\kappa \to \infty$ of (3.136), (3.137) and (3.138) returns the undeformed expressions (3.127), (3.132) and (3.128), respectively.

3.8.1 The κ-Minkowski Poisson homogeneous space

First of all, let us consider the PHS associated to the Lie bialgebra (3.134). Consider the Lie group element $g \in G_0$ with coordinates (a^0, a^1, b) given by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ a^0 & \cosh b & \sinh b \\ a^1 & \sinh b & \cosh b \end{pmatrix}, \tag{3.139}$$

and the point $m = (1, y^0, y^1)^T \in M_0 = G_0/L$, where L is the $(1+1)$ Lorentz subgroup defined by (2.126). The action $\alpha : G_0 \times M_0 \to M_0$ given by left multiplication is transitive, so the G_0-space for this action is indeed a homogeneous space. This right action is explicitly given by

$$\alpha(g, m) = (1, a^0 + y^0 \cosh b + y^1 \sinh b, a^1 + y^0 \sinh b + y^1 \cosh b)^T \equiv (1, x^0, x^1)^T. \tag{3.140}$$

In order to have a PHS, the first step is to introduce a Poisson structure Π on G_0, such that (G_0, Π) is a Poisson-Lie group. We know that the Sklyanin bracket (3.44) for (3.134) satisfies this condition. In this case it takes the following simple form

$$\{f_1, f_2\} = \frac{1}{\kappa}\Big((X^L_K f_1 X^L_{P_1} f_2 - X^L_{P_1} f_1 X^L_K f_2) - (X^R_K f_1 X^R_{P_1} f_2 - X^R_{P_1} f_1 X^R_K f_2) \Big) \tag{3.141}$$

where the X^L and X^R left- and right-invariant vector fields for G_0 (see Table 2.4), namely

$$\begin{aligned} X^L_{P_1} &= \sinh\xi \partial_{x^0} + \cosh\xi \partial_{x^1}, & X^L_K &= \partial_\xi, \\ X^R_{P_1} &= \partial_{x^1}, & X^R_K &= x^1 \partial_{x^0} + x^0 \partial_{x^1} + \partial_\xi. \end{aligned} \tag{3.142}$$

This Poisson-Lie structure on G_0, which we hereafter call the *Poisson κ-Poincaré group* (G_0, Π), is given by the fundamental brackets

$$\{x^0, x^1\} = -\frac{1}{\kappa} x^1, \qquad \{x^0, \xi\} = -\frac{1}{\kappa}\sinh\xi, \qquad \{x^1, \xi\} = \frac{1}{\kappa}(1 - \cosh\xi) \tag{3.143}$$

where x^0, x^1, ξ are local coordinates (exponential coordinates of the second kind) on G_0 defined as explained in §2.3. In this way we have that $\tilde{x}^\alpha = x^\alpha \circ p$ (see (2.86)), and the *Poisson κ-Minkowski spacetime* (M_0, π) structure will be defined by the first of the previous brackets, i.e

$$\{x^0, x^1\} = -\frac{1}{\kappa} x^1, \tag{3.144}$$

where for simplicity we have written x^0, x^1 instead of $\tilde{x}^0, \tilde{x}^1$, but we recall that we are now dealing with functions $\mathcal{C}^\infty(M_0)$ instead of $\mathcal{C}^\infty(G_0)$. We will do this in the rest of this work since no confusion should arise (note that by the definition of these coordinates all the expressions are formally similar).

In order to check that this is a PHS we just need to verify the covariance condition (3.20) (or equivalently (3.21)). We have that the left hand side of (3.21) reads

$$\begin{aligned} \{x^0, x^1\}\,(\alpha(g,m)) &= -\frac{1}{\kappa} x^1 \left(1, a^0 + y^0 \cosh b + y^1 \sinh b, a^1 + y^0 \sinh b + y^1 \cosh b\right) \\ &= -\frac{1}{\kappa}\left(a^1 + y^0 \sinh b + y^1 \cosh b\right) \end{aligned} \tag{3.145}$$

while for the right hand side of (3.21) we compute

$$\begin{aligned} \{x^0 \circ (\alpha(g)), x^1 \circ (\alpha(g))\}_M(m) &= -\frac{1}{\kappa} y^1 \\ \{x^0 \circ \alpha_m, x^1 \circ \alpha_m\}_G(g) &= -\frac{1}{\kappa}\left(a^1 - y^1 + y^0 \sinh b + y^1 \cosh b\right) \end{aligned} \tag{3.146}$$

where we have used that $x^\mu(\alpha(g,m)) = (x^\mu \circ \alpha_m)(g) = (x^\mu \circ (\alpha(g)))\,(m)$. Thus, when summing up the two terms of the right hand side, we see that the condition for the action to be a Poisson map is satisfied, therefore (3.144) is a PHS. The PHS defined in this way is called the *Poisson κ-Minkowski spacetime*, and we have just proved that this is a covariant spacetime for the Poisson κ-Poincaré group in the sense of (3.20).

Note that in the very same way we could prove explicitly that the group multiplication for G_0 is a Poisson map for the Poisson structure Π (3.143). We will carry out this computation below in the dual algebraic language of Poisson coalgebras.

3.8.2 The κ-Poincaré Poisson-Hopf algebra

The previous PHS can be described in the algebraic language of comodule algebras, more specifically Poisson comodule algebras (see Definition 3.28), where $V = \mathcal{C}^\infty(M)$ and $A = \mathcal{C}^\infty(G_0)$. Recall that for matrix groups we have the coalgebra structure defined by $\Delta(g_{ij}) = \sum_k g_{ik} \otimes g_{kj}$, where g_{ij} are the matrix elements of $g \in G_0$ (see Example 3.6). In this case the coalgebra structure for $\mathcal{C}^\infty(G_0)$ can be easily computed from

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi \\ x^1 & \sinh\xi & \cosh\xi \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi \\ x^1 & \sinh\xi & \cosh\xi \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 \\ x^0 \otimes 1 + \cosh\xi \otimes x^0 + \sinh\xi \otimes x^1 & \frac{e^\xi \otimes e^\xi + e^{-\xi} \otimes e^{-\xi}}{2} & \frac{e^\xi \otimes e^\xi - e^{-\xi} \otimes e^{-\xi}}{2} \\ x^1 \otimes 1 + \sinh\xi \otimes x^0 + \cosh\xi \otimes x^1 & \frac{e^\xi \otimes e^\xi - e^{-\xi} \otimes e^{-\xi}}{2} & \frac{e^\xi \otimes e^\xi + e^{-\xi} \otimes e^{-\xi}}{2} \end{pmatrix} \end{aligned} \tag{3.147}$$

obtaining the following coproduct for the local coordinate functions:

$$
\begin{aligned}
\Delta(x^0) &= x^0 \otimes 1 + \cosh\xi \otimes x^0 + \sinh\xi \otimes x^1, \\
\Delta(x^1) &= x^1 \otimes 1 + \sinh\xi \otimes x^0 + \cosh\xi \otimes x^1, \\
\Delta(\xi) &= \xi \otimes 1 + 1 \otimes \xi,
\end{aligned} \tag{3.148}
$$

where we have taken into account that $\Delta(e^{\pm\xi}) = e^{\pm\Delta(\xi)} = e^{\pm\xi} \otimes e^{\pm\xi}$ for primitive elements, so that

$$
\begin{aligned}
\Delta(\sinh\xi) &= \sinh(\Delta\xi) = \sinh(\xi \otimes 1 + 1 \otimes \xi) = \frac{e^{\xi} \otimes e^{\xi} - e^{-\xi} \otimes e^{-\xi}}{2}, \\
\Delta(\cosh\xi) &= \cosh(\Delta\xi) = \cosh(\xi \otimes 1 + 1 \otimes \xi) = \frac{e^{\xi} \otimes e^{\xi} + e^{-\xi} \otimes e^{-\xi}}{2}.
\end{aligned} \tag{3.149}
$$

Both $V = \mathcal{C}^\infty(M)$ and $A = \mathcal{C}^\infty(G_0)$ are endowed with the Poisson structure of the Poisson κ-Poincaré group, explicitly given by (3.143). In order to prove that $V = \mathcal{C}^\infty(M)$ is indeed a Poisson (left) comodule algebra, where the left coaction $\psi = \Delta|_V : V \to A \otimes V$ just given by the restriction of the coproduct $\Delta : A \to A \otimes A$, i.e.

$$
\psi \begin{pmatrix} 1 \\ x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi \\ x^1 & \sinh\xi & \cosh\xi \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x^0 \\ x^1 \end{pmatrix}, \tag{3.150}
$$

we need to show that the coaction ψ is indeed a Poisson map, where the Poisson structure on $A \otimes V = \mathcal{C}^\infty(G) \otimes \mathcal{C}^\infty(M)$ is defined by (3.110). This is a simple computation:

$$
\begin{aligned}
&\{\psi(x^0), \psi(x^1)\}_{A\otimes V} = \{\Delta(x^0), \Delta(x^1)\}_{A\otimes V} = \\
&= \{x^0 \otimes 1 + \cosh\xi \otimes x^0 + \sinh\xi \otimes x^1, x^1 \otimes 1 + \sinh\xi \otimes x^0 + \cosh\xi \otimes x^1\}_{A\otimes V} = \\
&-\frac{1}{\kappa}\left(x^1 \otimes 1 + \sinh\xi \otimes x^0 + \cosh\xi \otimes x^1\right) = -\frac{1}{\kappa}\Delta(x^1) = \Delta\left(\{x^0, x^1\}_V\right) = \psi\left(\{x^0, x^1\}_V\right)
\end{aligned} \tag{3.151}
$$

In fact, from the previous coproduct (3.148), it is easy to check that $A = \mathcal{C}^\infty(G_0)$ is indeed a Poisson coalgebra, just by proving that the coproduct respects the full Poisson-Lie structure on G_0:

$$
\begin{aligned}
\{\Delta(x^0), \Delta(\xi)\} &= \{x^0 \otimes 1 + \cosh\xi \otimes x^0 + \sinh\xi \otimes x^1, \xi \otimes 1 + 1 \otimes \xi\} = \\
&= -\frac{1}{\kappa}\sinh(\xi \otimes \xi) = -\frac{1}{\kappa}\Delta(\sinh\xi) = \Delta\left(\{x^0, \xi\}\right) \\
\{\Delta(x^1), \Delta(\xi)\} &= \{x^1 \otimes 1 + \sinh\xi \otimes x^0 + \cosh\xi \otimes x^1, \xi \otimes 1 + 1 \otimes \xi\} = \\
&= \frac{1}{\kappa}(1 \otimes 1 - \cosh(\xi \otimes \xi)) = \frac{1}{\kappa}\Delta(1 - \cosh\xi) = \Delta\left(\{x^1, \xi\}\right)
\end{aligned} \tag{3.152}
$$

This coalgebra language will be the one employed throughout this Thesis.

3.8.3 Poisson-Lie duality principle

So far we have studied the κ-deformation from the viewpoint of the algebra of functions $\mathcal{C}^\infty(G_0)$, with a special emphasis on the PHS construction. Let us now focus on the dual part, the QUEA $\mathcal{U}_\kappa(\mathfrak{g}_0)$ defined by (3.136) and (3.137). We will now show how the coproduct, given by (3.137), of this QUEA $\mathcal{U}_\kappa(\mathfrak{g}_0)$, with commutation relations defined by (3.136), can be obtained by means of the 'quantum duality principle'. In order to do that we need to construct the dual Lie group G_0^*, so let us first describe this duality.

Consider the Poisson κ-Poincaré group (3.143) (G_0, π), i.e. the Poisson-Lie group given by the Sklyanin bracket for (3.134). From (3.134) the tangent Lie bialgebra to this Poisson-Lie group can be directly computed (remember from (3.31) that for coboundary Lie bialgebras we have $\delta(X) = \mathrm{ad}_X(r)$ for all $X \in \mathfrak{g}_0$. By the definition of a Lie bialgebra, we know that the cocommutator defines a Lie bialgebra structure on the dual Lie algebra $\mathfrak{g}_0^*$ to $\mathfrak{g}_0$, by ${}^t\delta : \mathfrak{g}_0^* \otimes \mathfrak{g}_0^* \to \mathfrak{g}_0^*$. Now, G_0^* is defined as the unique compact and simply-connected Lie group such that $\mathfrak{g}_0^* = \mathrm{Lie}(G_0^*)$.

Now, as previously stated, the cocommutator $\delta : \mathfrak{g}_0 \to \mathfrak{g}_0 \otimes \mathfrak{g}_0$ given by (3.135) has a dual map ${}^t\delta : \mathfrak{g}_0^* \otimes \mathfrak{g}_0^* \to \mathfrak{g}_0^*$ that canonically induces a Lie algebra structure on $\mathfrak{g}_0^*$. Denoting $[\cdot,\cdot]_* : \mathfrak{g}_0^* \times \mathfrak{g}_0^* \to \mathfrak{g}_0^*$ this Lie bracket and introducing an algebraic dual basis (X^0, X^1, L) on $\mathfrak{g}_0^*$ such that

$$\langle X^0, P_0\rangle = \langle X^1, P_1\rangle = \langle L, K\rangle = 1, \tag{3.153}$$

we have that

$$[X^0, X^1]_* = -\frac{1}{\kappa}X^1, \qquad [X^0, L]_* = -\frac{1}{\kappa}L, \qquad [X^1, L]_* = 0, \tag{3.154}$$

which is the so-called 3D 'book' Lie algebra [205]. In fact, this dual Lie algebra $\mathfrak{g}_0^*$ has a Lie bialgebra structure with cocommutator $\delta^* : \mathfrak{g}_0^* \to \mathfrak{g}_0^* \otimes \mathfrak{g}_0^*$ given by

$$\delta^*(X^0) = L \wedge X^1, \qquad \delta^*(X^1) = L \wedge X^0, \qquad \delta^*(L) = 0, \tag{3.155}$$

which is just the dual counterpart of the Lie algebra relations (3.127).

In order to describe locally the only compact and simply-connected Lie group G_0^* such that $\mathfrak{g}_0^* = \mathrm{Lie}(G_0^*)$, we need to find a faithful representation $\rho : \mathfrak{g}_0^* \to \mathrm{End}(\mathbb{R}^4)$ of $\mathfrak{g}_0^*$ and then exponentiate it to construct the embedding in $\mathrm{GL}(4, \mathbb{R}^4)$. We take this representation to be

$$\rho(X^0) = \frac{1}{\kappa}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \rho(X^1) = \frac{1}{\kappa}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \rho(L) = \frac{1}{\kappa}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.156}$$

and then we introduce exponential coordinates of the second kind on G_0^*, which we denote by $\{p_0, p_1, \chi\}$, by the inverse map of

$$G_0^* = \exp\left(p_1 \rho(X^1)\right) \exp\left(\chi \rho(L)\right) \exp\left(p_0 \rho(X^0)\right). \tag{3.157}$$

A straightforward computation leads to the following explicit form for the group element

$$G_0^* = \begin{pmatrix} \cosh(\frac{p_0}{\kappa}) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2+\chi^2) & \frac{p_1}{\kappa} & \frac{\chi}{\kappa} & \sinh(\frac{p_0}{\kappa}) + \frac{1}{2\kappa^2} e^{p_0/\kappa}(p_1^2+\chi^2) \\ \frac{p_1}{\kappa} e^{p_0/\kappa} & 1 & 0 & \frac{p_1}{\kappa} e^{p_0/\kappa} \\ \frac{\chi}{\kappa} e^{p_0/\kappa} & 0 & 1 & \frac{\chi}{\kappa} e^{p_0/\kappa} \\ \sinh(\frac{p_0}{\kappa}) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2+\chi^2) & -\frac{p_1}{\kappa} & -\frac{\chi}{\kappa} & \cosh(\frac{p_0}{\kappa}) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2+\chi^2) \end{pmatrix}. \tag{3.158}$$

Using again that $\Delta(m_{ij}) = \sum_k m_{ik} \otimes m_{kj}$, we can compute the coproduct for G_0^* from the expression above doing exactly the same as in (3.148). Thus one obtains

$$\begin{aligned} \Delta(p_0) &= p_0 \otimes 1 + 1 \otimes p_0, \\ \Delta(p_1) &= p_1 \otimes 1 + e^{-p_0/\kappa} \otimes p_1, \\ \Delta(\chi) &= \chi \otimes 1 + e^{-p_0/\kappa} \otimes \chi, \end{aligned} \tag{3.159}$$

which agrees with (3.137) under the identification $p_0 \equiv P_0, p_1 \equiv P_1, \chi \equiv K$. Moreover, by following the procedure described in [205] we are able to find the unique Poisson-Lie structure on G_0^* whose tangent Lie biagebra is $(\mathfrak{g}_0^*, \delta^*)$, where δ^* is given by (3.155) and whose linearization is given by the Poisson version $\mathcal{P}(\mathfrak{g}_0)$ of the Poincaré Lie algebra (3.131). This Poisson-Lie structure, which is a non-coboundary one, is found to be

$$\{\chi, p_0\} = p_1, \qquad \{\chi, p_1\} = \frac{\kappa}{2}(1 - e^{-2p_0/\kappa}) - \frac{1}{2\kappa} p_1^2, \qquad \{p_0, p_1\} = 0, \tag{3.160}$$

and gives just the Poisson version $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0))$ of the quantum universal enveloping algebra (3.136).

To finish we can explicitly show that the coproduct (3.159) is a Poisson map for the previous Poisson bracket (3.160) and the induced Poisson bracket on the tensor product algebra defined by (3.110), although this fact is guaranteed since imposing these conditions is one of the steps of the method described in [205]. We have

$$\begin{aligned} \Delta(\{P_0, P_1\}) &= \Delta(0) = 0 \otimes 0 = \\ &= \{P_0 \otimes 1 + 1 \otimes P_0, P_1 \otimes 1 + e^{P_0/\kappa} \otimes P_1\} = \{\Delta(P_0), \Delta(P_1)\}, \\ \Delta(\{K_1, P_0\}) &= \Delta(P_1) = P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1 = \\ &= \{K_1 \otimes 1 + e^{-P_0/\kappa} \otimes K_1, P_0 \otimes 1 + 1 \otimes P_0\} = \{\Delta(K_1), \Delta(P_0)\}, \\ \Delta(\{K_1, P_1\}) &= \frac{\kappa}{2}\Delta(1 - e^{-2P_0/\kappa}) - \frac{1}{2\kappa}\Delta(P_1^2) = \\ &= \frac{\kappa}{2}(1 \otimes 1 - e^{-2P_0/\kappa} \otimes e^{-2P_0/\kappa}) \\ &\quad - \frac{1}{2\kappa}(P_1^2 \otimes 1 + 2P_1 e^{-P_0/\kappa} \otimes P_1 + e^{-2P_0/\kappa} \otimes P_1^2) = \\ &= \{K_1 \otimes 1 + e^{-P_0/\kappa} \otimes K_1, P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1\} = \{\Delta(K_1), \Delta(P_1)\}, \end{aligned} \tag{3.161}$$

and finally this computation shows that, under the very same identification $p_0 \equiv P_0, p_1 \equiv P_1, \chi \equiv K_1$ as above, (3.160) together with (3.159) define a Poisson-Hopf algebra structure which is formally similar of $\mathcal{U}_\kappa(\mathfrak{g}_0)$ defined by (3.136) and (3.137). Therefore we can say that the former is the 'Poisson version' of the latter.

In this way we have illustrated the Poisson version of the so-called 'quantum duality principle', which will be a key ingredient in Chapter 5, where we will be using it to describe curved momentum spaces arising from quantum deformations.

Chapter 4

Poisson homogeneous spaces for quantum Lorentzian groups

This fourth Chapter is divided into two interrelated parts, due to the fact that the quantum deformation studied in both of them is the same, namely the so-called κ-deformation. In the first part, we firstly present in §4.1 the well-known κ-deformation of the universal enveloping algebra of the Poincaré Lie algebra $\mathfrak{g}_0$, and afterwards in §4.2 we construct the noncommutative spacetime associated to it, the so-called κ-Minkowski space. These well-known results will serve as an introduction to the notation and procedure followed in the rest of the Chapter.

The rest of the first part of this Chapter will be devoted to the analogous construction in the case of a non-vanishing cosmological constant Λ. Following the same structure, we will present in §4.3 the quantum universal enveloping algebra for the (anti-) de Sitter groups G_Λ [114] and afterwards, in §4.4 we present the construction of the noncommutative (anti-) de Sitter spacetime M_Λ. We will specifically start with the Poisson κ-(A)dS spacetime and afterwards we will proceed to its quantization. Regarding the latter, we firstly look at the first order in the parameter $\eta = \sqrt{-\Lambda}$ and we proof that it defines a quantum sphere generated by the space coordinates. Afterwards, we proceed to the quantization at all orders of the cosmological constant, which we achieve by introducing ambient coordinates. The quantization reveals an elegant result, because we obtain that the quantum Casimir of the full algebra is just a deformed version of the pseudosphere (2.96) defining the classical (A)dS spacetime M_Λ. All these results concerning the construction of the noncommutative κ-(A)dS spacetime have been presented in [115].

The second part of this Chapter will continue our study of the κ-deformation, but now for the space of timelike geodesics (worldlines) of Minkowski spacetime. In §4.5 we describe in detail this space and, in §4.6 show that a similar construction as the one employed previously allows us to construct a noncommutative space of timelike geodesics from the κ-Poincaré deformation, which has been published in [61]. Apart from presenting this construction in full detail we will analyze the structural differences of the κ-deformation on both the spacetime and the space of worldlines, and we will argue that this new space is somewhat more adapted to this precise quantum deformation. We will also discuss some potential physical applications of this new construction, like the one defining quantum observers from quantum group symmetries.

4.1 The κ-Poincaré quantum algebra

We start by considering the $(3+1)$-dimensional Poincaré Lie algebra $\mathfrak{g}_0^{3+1}$ (from now on we omit the superscript and write simply $\mathfrak{g}_0$ to refer to the $(3+1)$-dimensional case, unless otherwise stated) defined by the commutation relations (2.72) for $\Lambda = 0$, namely

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & [K_a, K_b] = -\epsilon_{abc} J_c, \\ [P_0, P_a] = 0, & [P_a, P_b] = 0, & [P_0, J_a] = 0. \end{array} \tag{4.1}$$

As we explained is Chapter 2 latin indices $a, b, c, \ldots$ will denote spatial coordinates so it will run from 1 to the spatial dimension of our space n, so during this Chapter $n = 3$ unless otherwise stated, while greek indices will run from 0 to n.

The so-called κ-deformation is a coboundary Lie bialgebra $(\mathfrak{g}, \delta)$ defined by the following skew-symmetric solution $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$ of the mCYBE

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) \tag{4.2}$$

from which the cocommutator $\delta(X) = \mathrm{ad}_X r$ for all $X \in \mathfrak{g}_0$ (3.31) is directly obtained:

$$\begin{aligned} \delta(P_0) &= \delta(J_a) = 0, \\ \delta(P_a) &= \frac{1}{\kappa} P_a \wedge P_0, \\ \delta(K_1) &= \frac{1}{\kappa}(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2), \\ \delta(K_2) &= \frac{1}{\kappa}(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3), \\ \delta(K_3) &= \frac{1}{\kappa}(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1). \end{aligned} \tag{4.3}$$

Note that time translation and rotation generators are primitive ones. This fact will be quite relevant in the next part of this Chapter, since we will see that in the non-vanishing cosmological constant case, a minimal assumption (that P_0 remains primitive, which as we will see is necessary from a physical point of view) imply not only that the rotation generators J_a are no more primitive, but also some kind of symmetry breaking in the rotation sector.

This cocommutator completely determines the κ-Poisson-Hopf Poincaré algebra structure $\mathcal{U}_\kappa(\mathfrak{g}_0)$ (see Definition 3.25) given, in the so-called bicrossproduct basis [64], by the following commutation relations

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, K_b] = -\epsilon_{abc} J_c, & [P_0, J_a] = 0, \\ [P_0, P_a] = 0, & [P_a, P_b] = 0, & \\ \multicolumn{3}{l}{[K_a, P_b] = \delta_{ab}\left(\frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}\mathbf{P}^2\right) - \frac{1}{\kappa} P_a P_b\,,} \end{array} \tag{4.4}$$

together with the compatible deformed coproduct map

$$
\begin{aligned}
\Delta_\kappa(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta_\kappa(P_a) &= P_a \otimes 1 + e^{-P_0/\kappa} \otimes P_a, \\
\Delta_\kappa(J_a) &= J_a \otimes 1 + 1 \otimes J_a, \\
\Delta_\kappa(K_a) &= K_a \otimes 1 + e^{-P_0/\kappa} \otimes K_a + \frac{1}{\kappa}\,\epsilon_{abc} P_b \otimes J_c \,.
\end{aligned}
\tag{4.5}
$$

where $\Delta_\kappa : \mathcal{U}_\kappa(\mathfrak{g}_0) \to \mathcal{U}_\kappa(\mathfrak{g}_0) \otimes \mathcal{U}_\kappa(\mathfrak{g}_0)$. It is worth noticing that the limit $\kappa \to \infty$ transforms (4.4) into the undeformed Lie algebra $\mathfrak{g}_0$ (4.1) while the limit $\kappa \to \infty$ of the coproduct (4.5) just describes the canonical Hopf algebra on $\mathcal{U}(\mathfrak{g}_0)$ described in Example (3.5), where every generator is primitive, i.e. $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ for all $X \in \mathfrak{g}_0$.

Before proceeding further, it will be useful to recall the notion of the 'Poisson version' of the preceding construction, as exemplified in Section 3.8 of Chapter 2. In order to construct it, recall that we can construct the so called Lie-Poisson structure (see Example 3.1) on the dual vector space $\mathfrak{g}_0^*$ to the Lie algebra $\mathfrak{g}_0$. This is a Poisson structure whose fundamental brackets have the same form as the commutation relations for $\mathfrak{g}_0$ (4.1). We have denoted this algebra by $\mathcal{P}(\mathfrak{g}_0)$ to remark this fact, and the Poisson brackets are

$$
\begin{aligned}
&\{J_a, J_b\} = \epsilon_{abc} J_c, \qquad &&\{J_a, P_b\} = \epsilon_{abc} P_c, \qquad &&\{J_a, K_b\} = \epsilon_{abc} K_c, \\
&\{K_a, P_0\} = P_a, &&\{K_a, K_b\} = -\epsilon_{abc} J_c, &&\{P_0, J_a\} = 0, \\
&\{P_0, P_a\} = 0, &&\{P_a, P_b\} = 0, && \\
&\{K_a, P_b\} = \delta_{ab}\left(\frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}\mathbf{P}^2\right) - \frac{1}{\kappa} P_a P_b \,, && &&
\end{aligned}
\tag{4.6}
$$

which is just a Poisson algebra deformation of the Lie-Poisson structure on $\mathfrak{g}_0^*$, so a Poisson version of (4.4) that we denote by $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0))$.

In this way the previous coproduct (4.5) can be alternatively seen as a Poisson algebra homomorphism for the previous Poisson structure and the Poisson structure defined in the tensor product algebra, which is just the canonical one defined by (3.110). So we have the map $\Delta_\kappa : \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0)) \to \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0)) \otimes \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0))$. Although we denote this map by the same symbol as the one for quantum universal enveloping algebras, it should be clear in each situation to which of these two maps we are referring to. In this way we have constructed a Poisson-Hopf structure, which we will refer to as the κ-Poisson-Hopf structure, and is the Poisson version $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0))$ of the quantum universal enveloping algebra $\mathcal{U}_\kappa(\mathfrak{g}_0)$. In fact, as explained in Section 3.8 of Chapter 2, this would be the Poisson-Hopf algebra obtained by applying the quantum duality principle.

This quantum deformation of the Poincaré algebra induces a deformed Casimir function $\mathcal{C}_\kappa$ for the Poisson algebra (4.6), given by

$$
\mathcal{C}_\kappa = 4\kappa^2 \sinh^2(P_0/2\kappa) - e^{P_0/\kappa}\mathbf{P}^2 = 2\kappa^2\left[\cosh(P_0/\kappa) - 1\right] - e^{P_0/\kappa}\mathbf{P}^2, \tag{4.7}
$$

which is a deformation of the quadratic Casimir for the Poincaré algebra (2.73).

This deformed Casimir constitutes the keystone for the interpretation of κ-Poincaré algebra as the modified kinematical symmetry underlying a class of deformed dispersion relations that arise in several quantum gravity contexts [102, 121]. This feature will be further studied in Chapter 5, where we generalize the construction of non-trivial momentum spaces to the case of a non-vanishing cosmological constant (see [145, 146] where these results were presented). It is also worth recalling that a second invariant $\mathcal{W}_\kappa$ does exist for the Poisson structure (4.6), which is just a deformed analogue of the square of the norm of the Pauli-Lubanski four-vector, which in the undefeormed case is given by (2.74). This second invariant $\mathcal{W}_\kappa$ is given by

$$\mathcal{W}_\kappa = \left(\cosh(P_0/\kappa) - \frac{1}{4\kappa^2}\, e^{P_0/\kappa}\mathbf{P}^2\right) W_{\kappa,0}^2 - \mathbf{W}_\kappa^2\,, \tag{4.8}$$

where the deformed components $W_{\kappa,0}$ and $W_{\kappa,a}$ are

$$W_{\kappa,0} = e^{P_0/2\kappa}\,\mathbf{J}\cdot\mathbf{P}, \qquad W_{\kappa,a} = -\kappa J_a \sinh(P_0/\kappa) + e^{P_0/\kappa}\epsilon_{abc}\left(K_b + \frac{1}{2\kappa}\,\epsilon_{bkl}J_k P_l\right) P_c\,. \tag{4.9}$$

We would like to stress that all the expressions given in this Chapter are analytic in the deformation parameter κ, unless otherwise stated, and the non-deformed limit $\kappa \to \infty$ gives rise to the usual $(3+1)$ relativistic symmetries.

Note that when dealing with Hopf algebra kinematical symmetries, the coproduct can be interpreted as the composition law for observables. In particular, the coproduct (4.5) is such that the κ-deformation induces a nonlinear composition rule for momenta in interaction vertices. As we are going to show in Chapter 5, it is because of this deformed composition rule that curvature in the κ-Poincaré momentum space emerges. In more technical terms, the curvature of the momentum space arises as a consequence of the non cocommutativity of the coproduct map for the translation generators.

4.2 The κ-Minkowski noncommutative spacetime

Once the κ-Poisson-Hopf algebra deformation (recall that it is simply the Poisson version of $\mathcal{U}_\kappa(\mathfrak{g}_0)$) has been described in the previous Section, we proceed to construct its associated noncommutative spacetime, the so-called κ-Minkowski spacetime. Although this noncommutative spacetime has been known for a long time [62, 63, 64], and in fact it is one of the most studied noncommutative spacetimes, our construction as a Poisson homogeneous space (PHS) followed by its (trivial in this case) quantization, will proof useful both because of its simplicity and the fact that this procedure can be generalized to the construction of different spacetimes, as we will in fact do in §4.4.

As it was explained in Section 2.2 of Chapter 2, our strategy for the construction of a Poisson homogeneous space for the Poisson-Lie group (G, Π) starts with a suitable parametrization of the Lie group G, i.e. the introduction of local coordinates that descend to functions in the quotient space G/H. This local coordinates are obtained exponentiating the matrix representation of the Lie algebra given by (2.78) in an appropriate order. In our case we will use geodesics parallel coordinates (see the discussion after (2.97)) which are defined by the inverse map of (2.88). In this way we obtain a local parametrization of G_0, and the same can be done in order to introduce local coordinates on the Lorentz group L (2.89). In this way, we introduce local coordinates $\bar{x} : M_0 = G_0/L \to \mathbb{R}^4$ on the four-dimensional Minkowski spacetime.

The following step in our construction involves the introduction of a Poisson structure on M_0, which we shall call π, such that it be covariant in the sense of (3.22), under the action of the Lie group G_0. This is precisely what a Poisson homogeneous space for the Poisson-Lie group (G_0, Π) gives us, so let us apply all the machinery described in Chapter 3 to this particular case, where it will be useful to recall the low dimensional example at the end of that Chapter, since the procedure is analogous. First of all, in order to have a well-defined Poisson homogeneous space in such a way that its Poisson structure can be easily obtained, a sufficient condition is given by the coisotropy condition (see Definition (3.79)), which in this case reads

$$\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{g}_0, \tag{4.10}$$

where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ are the Lie algebras of the Lorentz and Poincaré groups, respectively. In our case, this condition is fulfilled as shown in (4.3). It is important to note that the Lorentz subalgebra $\mathfrak{l}$ is not a Lie subalgebra, so the Lorentz group will not be a Poisson subgroup, but a coisotropic one. This situation should be compared with the one considered at the end of this Chapter, where the Poisson homogeneous space constructed in the space of time-like geodesics of Minkowski space, is indeed a Poisson homogeneous space of Poisson subgroup type.

Once we have checked that the quotient space $M_0 = G_0/L$ can be given the structure of a coisotropic Poisson homogeneous space for the κ-Poincaré Poisson-Lie group, the following step is to explicitly write down the only coboundary Poisson-Lie structure on G_0 defined by the κ-Poincaré r-matrix (4.2). This Poisson-Lie structure (G_0, Π) is given by the Sklyanin bracket (3.44), which in this particular case reduces to

$$\{f_1, f_2\} = \frac{1}{\kappa}\sum_{a=1}^{3}\Big(\big(X^L_{K_a} f_1 X^L_{P_a} f_2 - X^L_{P_a} f_1 X^L_{K_a} f_2\big) - \big(X^R_{K_a} f_1 X^R_{P_a} f_2 - X^R_{P_a} f_1 X^R_{K_a} f_2\big) \Big) \tag{4.11}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(G_0)$. Introducing exponential coordinates of the second kind on G_0, defined as in (2.88), the fundamental brackets in this coordinates for this Poisson structure Π read:

$$\{x^0, x^a\} = -\frac{1}{\kappa} x^a, \qquad \{x^a, x^b\} = 0, \qquad \{\xi^a, \xi^b\} = \{\xi^a, \theta^b\} = \{\theta^a, \theta^b\} = 0,$$

$$\{x^0, \xi^1\} = -\frac{1}{\kappa}\frac{\sinh \xi^1}{\cosh \xi^2 \cosh \xi^3}, \qquad \{x^0, \xi^2\} = -\frac{1}{\kappa}\frac{\cosh \xi^1 \sinh \xi^2}{\cosh \xi^3},$$

$$\{x^0, \xi^3\} = -\frac{1}{\kappa}\cosh \xi^1 \cosh \xi^2 \sinh \xi^3$$

$$\{x^0, \theta^1\} = \frac{1}{\kappa}\left(\tan\theta^2 \sinh\xi^1 \left(\frac{\cos\theta^1 \tanh\xi^2}{\cosh\xi^3} + \sin\theta^1 \tanh\xi^3 \right) - \cosh\xi^1 \sinh\xi^2 \tanh\xi^3 \right),$$

$$\{x^0, \theta^2\} = \frac{1}{\kappa}\sinh\xi^1 \left(\cos\theta^1 \tanh\xi^3 - \frac{\sin\theta^1 \tanh\xi^2}{\cosh\xi^3} \right),$$

$$\{x^0, \theta^3\} = -\frac{1}{\kappa}\frac{\sinh\xi^1}{\cos\theta^2}\left(\frac{\cos\theta^1 \tanh\xi^2}{\cosh\xi^3} + \sin\theta^1 \tanh\xi^3 \right),$$

$$
\begin{aligned}
&\{x^1,\xi^1\} = \frac{1}{\kappa}\left(1-\frac{\cosh\xi^1}{\cosh\xi^2\cosh\xi^3}\right), \qquad \{x^1,\xi^2\} = -\frac{1}{\kappa}\frac{\sinh\xi^1\sinh\xi^2}{\cosh\xi^3},\\
&\{x^1,\xi^3\} = -\frac{1}{\kappa}\sinh\xi^1\cosh\xi^2\sinh\xi^3,\\
&\{x^1,\theta^1\} = \frac{1}{\kappa}\left(\tan\theta^2\cosh\xi^1\left(\frac{\cos\theta^1\tanh\xi^2}{\cosh\xi^3}+\sin\theta^1\tanh\xi^3\right)-\sinh\xi^1\sinh\xi^2\tanh\xi^3\right),\\
&\{x^1,\theta^2\} = \frac{1}{\kappa}\cosh\xi^1\left(\cos\theta^1\tanh\xi^3-\frac{\sin\theta^1\tanh\xi^2}{\cosh\xi^3}\right),\\
&\{x^1,\theta^3\} = -\frac{1}{\kappa}\frac{\cosh\xi^1}{\cos\theta^2}\left(\frac{\cos\theta^1\tanh\xi^2}{\cosh\xi^3}+\sin\theta^1\tanh\xi^3\right),\\
&\{x^2,\xi^1\} = -\frac{1}{\kappa}\sinh\xi^1\tanh\xi^2, \qquad \{x^2,\xi^2\} = \frac{1}{\kappa}\left(\cosh\xi^1-\frac{\cosh\xi^2}{\cosh\xi^3}\right),\\
&\{x^2,\xi^3\} = -\frac{1}{\kappa}\sinh\xi^2\sinh\xi^3,\\
&\{x^2,\theta^1\} = -\frac{1}{\kappa}\cosh\xi^2\left(\frac{\cos\theta^1\tan\theta^2\sinh\xi^1}{\cosh^2\xi^2}+\tanh\xi^3\right),\\
&\{x^2,\theta^2\} = \frac{1}{\kappa}\frac{\sin\theta^1\sinh\xi^1}{\cosh\xi^2},\\
&\{x^2,\theta^3\} = \frac{1}{\kappa}\frac{\cos\theta^1\sinh\xi^1}{\cos\theta^2\cosh\xi^2},\\
&\{x^3,\xi^1\} = -\frac{1}{\kappa}\frac{\sinh\xi^1\tanh\xi^3}{\cosh\xi^2}, \qquad \{x^3,\xi^2\} = -\frac{1}{\kappa}\cosh\xi^1\sinh\xi^2\tanh\xi^3,\\
&\{x^3,\xi^3\} = \frac{1}{\kappa}\left(\cosh\xi^1\cosh\xi^2-\cosh\xi^3\right),\\
&\{x^3,\theta^1\} = \frac{1}{\kappa}\left(\tan\theta^2\sinh\xi^1\left(\cos\theta^1\tanh\xi^2\tanh\xi^3-\frac{\sin\theta^1}{\cosh\xi^3}\right)+\frac{\cosh\xi^1\sinh\xi^2}{\cosh\xi^3}\right),\\
&\{x^3,\theta^2\} = -\frac{1}{\kappa}\sinh\xi^1\left(\sin\theta^1\tanh\xi^2\tanh\xi^3+\frac{\cos\theta^1}{\cosh\xi^3}\right),\\
&\{x^3,\theta^3\} = \frac{1}{\kappa}\frac{\sinh\xi^1}{\cos\theta^2}\left(\frac{\sin\theta^1}{\cosh\xi^3}-\cos\theta^1\tanh\xi^2\tanh\xi^3\right).
\end{aligned}
\tag{4.12}
$$

In terms of these local coordinates on G_0, it is straightforward to write down the Poisson structure π on $M_0 = G_0/L$ such that (M_0, π) becomes a Poisson homogeneous space for the above Poisson-Lie structure on G_0. As explained in detail in (2.86), $\bar{x}$ are local coordinates on M_0 (indeed they are the so-called parallel geodesic coordinates) and the fundamental Poisson brackets for them are simply

$$
\{x^0, x^a\} = -\frac{1}{\kappa}x^a, \qquad \{x^a, x^b\} = 0. \tag{4.13}
$$

In other words, the Poisson structure on M_0 is just the restriction of the Poisson structure on G_0. If we call

$$
\begin{aligned}
p : G_0 &\to M_0\\
g &\to gL
\end{aligned}
\tag{4.14}
$$

then we have simply that

$$
\{\tilde{f}_1, \tilde{f}_2\}_{M_0}(m) = \{f_1 \circ p, f_2 \circ p\}_{M_0}(m) = \{f_1, f_2\}_{G_0}(g) \tag{4.15}
$$

where $f_1, f_2 \in \mathcal{C}^\infty(G)^L$ and $m = gL \in M_0$. In terms of the Poisson bivector, this just implies that the Poisson bivector π on M_0 is the pushforward of the Poisson bivector Π on G_0, i.e. $\pi = p_*\Pi$. Checking explicitly that (M_0, π) is a Poisson homogeneous space for (G_0, Π), i.e. checking (3.21) for these two Poisson structures is quite cumbersome, but it can indeed be done (see the last Section of Chapter 2 in which this checking is done for the $(1+1)$-dimensional case).

4.2.1 Quantization of the κ-Minkowski Poisson homogeneous space

Let us now say something about the definition of the proper (quantum) κ-Minkowski spacetime. By this we mean a quantization of the Poisson algebra (see Definition 3.32) given by the fundamental brackets (4.13). In fact, performing this quantization is trivial in this case, because for linear Poisson structures (also called Poisson structures of Lie algebraic type) it suffices to replace the Poisson brackets by commutators, so we have

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0. \tag{4.16}$$

These commutation relations define the so-called *κ-Minkowski spacetime*. To be precise we would need to multiply the previous expressions by the quantization parameter h, so obtaining $[\hat{x}_0, \hat{x}^a] = -\frac{h}{\kappa}\hat{x}^a$. However, then we could set $h/\kappa = \kappa'$ and just work with the new parameter κ'. For the sake of clarity, and provided that this will not affect neither the physical interpretation of our results (recall that the relevant parameter for us is the one appearing in the Hopf algebra deformation, but it is irrelevant for us whether this parameter is κ or κ') or the mathematical procedure, we choose to set the quantization parameter h to 1 during this Thesis.

Moreover, recall from (4.12) that the Sklyanin bracket on G_0 defines a Poisson commuting algebra on the Lorentz subgroup. This fact, together with explicit form of the rest of the Poisson structure, which just involves Lorentz coordinates for the mixed (space-time coordinates and Lorentz coordinates) Poisson brackets, shows that in fact the full Poisson-Lie group can be quantized (formally replacing Poisson brackets by commutators and taking into account the considerations above), giving rise to the quantum κ-Poincaré group. In the next Section, when we consider the case of a non-vanishing cosmological constant, we will see that this is no longer the case.

4.2.2 Twisted κ-Minkowski Poisson homogeneous spaces

There is another deformation of G_0 related to the κ-Poincaré one. This is the so-called twisted κ-Poincaré quantum deformation [62, 63, 64, 65], which is structurally similar to κ-Poincaré, in the sense that it is also a coboundary one and it defines a Poisson homogeneous space on Minkowski spacetime. The element r^t defining this deformation is obtained from (4.2) just by adding the twist term $J_3 \wedge P_0$ (note that $[P_0, J_a] = 0$ (4.1)). So, we have that this deformation is given by the following skew-symmetric $r \in \mathfrak{g}_0 \wedge \mathfrak{g}_0$ solution of the mCYBE:

$$r^t = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + \vartheta J_3 \wedge P_0, \tag{4.17}$$

where the parameter ϑ is the one associated to the twist term. The previous procedure for constructing the associated PHS can be straightforwardly applied to this deformation, since the cocommutator

$$\delta_t(X) = \mathrm{ad}_X(r^t) = \mathrm{ad}_X(r) + \vartheta\, \mathrm{ad}_X(J_3 \wedge P_0), \tag{4.18}$$

takes the following explicit form (see [114])

$$\begin{aligned}
&\delta_t(P_0) = 0,\\
&\delta_t(P_1) = \frac{1}{\kappa} P_1 \wedge P_0 - \vartheta(P_2 \wedge P_0),\\
&\delta_t(P_2) = \frac{1}{\kappa} P_2 \wedge P_0 + \vartheta(P_1 \wedge P_0),\\
&\delta_t(P_3) = \frac{1}{\kappa} P_3 \wedge P_0,\\
&\delta_t(K_1) = \frac{1}{\kappa}(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2) - \vartheta(K_2 \wedge P_0 - J_3 \wedge P_1),\\
&\delta_t(K_2) = \frac{1}{\kappa}(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3) + \vartheta(K_1 \wedge P_0 + J_3 \wedge P_2),\\
&\delta_t(K_3) = \frac{1}{\kappa}(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1) + \vartheta J_3 \wedge P_3,\\
&\delta_t(J_1) = -\vartheta J_2 \wedge P_0,\\
&\delta_t(J_2) = \vartheta J_1 \wedge P_0,\\
&\delta_t(J_3) = 0.
\end{aligned} \tag{4.19}$$

This cocomutator clearly satisfies the coisotropy condition (3.79) for the Lorentz algebra, i.e. $\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{g}_0$. It is relevant to note here that P_0 is primitive, allowing to keep the interpretation of the deformation parameter κ as a (Planck) mass.

The Poisson homogeneous space associated to this deformation is defined by the Poisson-Lie group (G_0, Π_t) whose Poisson structure is the one given by (4.12) modified by adding the following terms to some of the fundamental brackets. It should be noted that here only list those brackets that should be modified, the remaining ones will be the same as those in (4.12):

$$\begin{aligned}
&\{x_0, x^1\}_t = \{x_0, x^1\} - \vartheta x^2,\\
&\{x_0, x^2\}_t = \{x_0, x^2\} + \vartheta x^1,\\
&\{x^0, \xi^1\}_t = \{x^0, \xi^1\} - \vartheta \cosh\xi^1 \tanh\xi^2,\\
&\{x^0, \xi^2\}_t = \{x^0, \xi^2\} + \vartheta \sinh\xi^1,\\
&\{x^0, \theta^1\}_t = \{x^0, \theta^1\} - \vartheta \frac{\cos\theta^1 \cosh\xi^1 \tan\theta^2}{\cosh\xi^2},\\
&\{x^0, \theta^2\}_t = \{x^0, \theta^2\} + \vartheta \frac{\cosh\xi^1 \sin\theta^1}{\cosh\xi^2},\\
&\{x^0, \theta^3\}_t = \{x^0, \theta^3\} + \vartheta \cosh\xi^1 \left(\cosh\xi^2 \cosh\xi^3 + \frac{\cos\theta^1}{\cos\theta^2 \cosh\xi^2}\right),
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\{x^1,\theta^3\}_t &= \{x^1,\theta^3\} - \vartheta \sinh\xi^1 \cosh\xi^2 \cosh\xi^3,\\
\{x^2,\theta^3\}_t &= \{x^2,\theta^3\} - \vartheta \sinh\xi^2 \cosh\xi^3,\\
\{x^3,\theta^3\}_t &= \{x^3,\theta^3\} - \vartheta \sinh\xi^3,
\end{aligned}$$

while the twisted κ Poisson homogeneous space on (M_0), denoted by (M_0,π_t), will be

$$\begin{aligned}
\{x^0,x^1\}_t &= -\frac{1}{\kappa}x^1 - \vartheta x^2,\\
\{x^0,x^2\}_t &= -\frac{1}{\kappa}x^2 + \vartheta x^1,\\
\{x^0,x^3\}_t &= -\frac{1}{\kappa}x^3,\\
\{x^a,x^b\}_t &= 0.
\end{aligned} \tag{4.21}$$

This deformation has some striking differences with respect to the non-twisted one, in particular, it is interesting to note that one spatial direction is privileged with respect to the others, in this case x^3 (although it can be rotated to any other spatial direction). The fact that one spatial direction becomes privileged when adding the twist term should be compared with the situation in which we introduce a non-vanishing cosmological constant Λ because, as we will show in the following, in this case also one spatial direction becomes privileged. Similarly to the non-twisted case, the quantization of the Poisson algebra (4.21) can be directly performed, given that this Poisson structure is Lie algebraic, obtaining the *quantum twisted κ-Minkowski spacetime*, whose defining commutation relations are

$$\begin{aligned}
[\hat{x}^0,\hat{x}^1]_t &= -\frac{1}{\kappa}\hat{x}^1 - \vartheta \hat{x}^2,\\
[\hat{x}^0,\hat{x}^2]_t &= -\frac{1}{\kappa}\hat{x}^2 + \vartheta \hat{x}^1,\\
[\hat{x}^0,\hat{x}^3]_t &= -\frac{1}{\kappa}\hat{x}^3,\\
[\hat{x}^a,\hat{x}^b]_t &= 0.
\end{aligned} \tag{4.22}$$

It is also a remarkable feature that the twist does not affect the commutation rules between space coordinates, which again define a commutative subalgebra.

4.3 The κ-deformation of the (3+1) (A)dS algebra

As discussed in the Introduction (see Chapter 1), both the κ-Poincaré quantum universal enveloping algebra $\mathcal{U}_\kappa(\mathfrak{g}_0)$ and the κ-Minkowski noncommutative spacetime (4.16) were already well-known and thoroughly studied in the bibliography. However, the generalization of these structures to the case of a non-vanishing cosmological constant was an intriguing open problem. Only very recently, the Poisson version $\mathcal{PV}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ of the quantum universal enveloping algebra $\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)$ (see [114]) and the κ-(A)dS spacetime (see [115]) have been explicitly constructed. Following the same construction as above, in this Section we present the full Poisson version of $\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)$.

The first step consists in proving that there is such a deformation that could be called a κ-(A)dS one, and that it is in fact unique. In order to do that, we start at the tangent level, so let us recall the commutation relations (2.72) for the Lie algebra $\mathfrak{g}_\Lambda$, which read

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & [K_a, K_b] = -\epsilon_{abc} J_c, \\ [P_0, P_a] = -\Lambda K_a, & [P_a, P_b] = \Lambda \epsilon_{abc} J_c, & [P_0, J_a] = 0. \end{array} \tag{4.23}$$

The following automorphism of $\mathfrak{g}_\Lambda$ will be relevant in what follows

$$\tilde{P}_0 = P_0, \qquad \tilde{P}_a = \sqrt{-\Lambda} K_a, \qquad \tilde{K}_a = -\frac{1}{\sqrt{-\Lambda}} P_a, \qquad J_a = J_a. \tag{4.24}$$

Now let us give a clear definition of what we understand by a κ-deformation.

Definition 4.1. Let $\mathfrak{g}_\Lambda$ be the Lie algebra defined by (4.23). Then a Lie bialgebra $(\mathfrak{g}_\Lambda, \delta)$ will be called a *κ-deformation (or κ-like deformation)* if it is a coboundary one, i.e. if the cocommutator can be written as $\delta(X) = \mathrm{ad}_X r_\Lambda$ for all $X \in \mathfrak{g}_\Lambda$, for some $r \in \mathfrak{g}_\Lambda \otimes \mathfrak{g}_\Lambda$ and provided it satisfies the following two conditions:

i) P_0 is primitive, i.e. $\delta(P_0) = 0$,

ii) $\lim_{\Lambda\to 0} r_\Lambda = r_0$, with $r_0 = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3)$.

If *ii*) is replaced by

ii)' $\lim_{\Lambda\to 0} r_\Lambda = r_0^t$, with $r_0^t = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + \vartheta J_3 \wedge P_0$,

then we say that it is *κ-twisted-like.*

Condition *i*) of the previous definition is necessary in order to guarantee that the parameter κ has the dimension of mass, a key physical requirement. This is due to the fact that Condition *i*) is necessary for the coproduct of the κ-Poincaré quantum algebra to be a primitive generator, namely $\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$, which is essential in order to allow exponentials $e^{P_0/\kappa}$ to emerge as the building blocks of the quantum κ-deformation and of the dispersion relation arising from the deformed Casimir, thus implying that κ has dimensions of a (Planck) mass. Condition *ii*) just states that by means of a Lie bialgebra contraction procedure [80, 206] we recover the κ-Poincaré Lie bialgebra (4.3). So these are the minimal assumptions one could take in order to construct a sensible generalization of the κ Poincaré deformation when $\Lambda \neq 0$. Now we are ready to state the following

Theorem 4.1. *With the above notation, there is a unique (up to Lie algebra automorphisms) κ-like Lie bialgebra structure on $\mathfrak{g}_\Lambda$. It is defined by the skew-symmetric solution of the mCYBE $r_\Lambda \in \mathfrak{g}_\Lambda \otimes \mathfrak{g}_\Lambda$ given by*

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2), \tag{4.25}$$

where $\eta = \sqrt{-\Lambda}$. In the same way, there is a unique (up to Lie algebra automorphisms) κ-twisted-like Lie bialgebra structure on $\mathfrak{g}_\Lambda$, defined by

$$r_\Lambda^t = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2) + \vartheta J_3 \wedge P_0. \tag{4.26}$$

In contradistinction to the κ-Poincaré Lie bialgebra, these two Lie bialgebra structures are quastriangular, and their defining r-matrices can be quantized to obtain a quantum R-matrix.

Proof. The last part is straightforward. Quasitriangularity is directly obtained from 3.4 due to the fact that $\mathfrak{g}_\Lambda$ is semisimple and thus metric (see the discussion after (3.40)), with the Killing-Cartan form defining the non-degenerate symmetric associative bilinear form. Quantization follows directly from Theorem 5.1 of [200].

For the main part of the Theorem, we start by a long computer-assisted calculus (which starts from a completely generic skew-symmetric $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$ depending on 45 parameters onto which the mCYBE is imposed). This shows that the only family of multiparametric (A)dS r-matrices compatible with these two conditions is given by:

$$\begin{aligned} r_\Lambda = \frac{1}{\kappa}&(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + P_0 \wedge (\beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3) \\ &+ \alpha_3 J_1 \wedge J_2 - \alpha_2 J_1 \wedge J_3 + \alpha_1 J_2 \wedge J_3 \,, \end{aligned} \tag{4.27}$$

together with the following quadratic relations among the parameters:

$$\begin{gathered} \beta_1\alpha_3 - \beta_3\alpha_1 = 0, \qquad \beta_1\alpha_2 - \beta_2\alpha_1 = 0, \qquad \beta_2\alpha_3 - \beta_3\alpha_2 = 0, \\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \left(\frac{\eta}{\kappa}\right)^2 . \end{gathered} \tag{4.28}$$

Notice that the term $P_0 \wedge (\beta_1 J_1 + \beta_2 J_2 + \beta_2 J_3)$ in (4.27) is given by the superposition of three twists (recall from (4.23) that $[P_0, J_a] = 0$) and therefore these three terms would lead to the (A)dS generalization of the twisted κ-Poincaré, completely defined by (4.17). The equations (4.28) have a neat geometrical interpretation: non-twisted solutions (with parameters α_i) are given by the vector of a point in the sphere with radius η/κ, while twisted solutions (with parameters β_i) are defined by another vector orthogonal to the former. Note also that equations (4.28) are valid for $\Lambda = 0$ ($\eta = 0$); in this Poincaré case $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and the twists parameters are free.

In order to solve the equations (4.28), let us firstly consider the non-twisted case with $\beta_1 = \beta_2 = \beta_3 = 0$. Then the only non-vanishing equation in (4.28) defines a sphere of radius $R = \eta/\kappa$, so we can write

$$\alpha_3 = R\cos\theta, \qquad \alpha_2 = -R\sin\theta\sin\varphi, \qquad \alpha_1 = R\sin\theta\cos\varphi, \tag{4.29}$$

where $\theta \in [0, \pi], \varphi \in [0, 2\pi)$. Now, the solution (4.27) reads

$$\begin{aligned} r_\Lambda = \frac{1}{\kappa}&(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) \\ &+ \frac{\eta}{\kappa}(\cos\theta J_1 \wedge J_2 + \sin\theta\sin\varphi J_1 \wedge J_3 + \sin\theta\cos\varphi J_2 \wedge J_3) \,. \end{aligned} \tag{4.30}$$

The last term within the r-matrix (4.30) is represented by a point on the 2D sphere parametrized by (4.29), and it is straightforward to prove that the Lie algebra generator

$$\tilde{J}_3 = \sin\theta\cos\varphi J_1 - \sin\theta\sin\varphi J_2 + \cos\theta J_3\,, \tag{4.31}$$

becomes primitive under the deformation defined by the r-matrix (4.30), i.e $\delta(\tilde{J}_3) = 0$. Now, since there exists an automorphism of $\mathfrak{g}_\Lambda$ (4.23) that corresponds to the rotation providing the new $\tilde{J}_3$ generator (4.31), we can apply it to the r-matrix (4.30), and we find the following transformed r-matrix (tildes will be omitted for the sake of simplicity)

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2)\,. \tag{4.32}$$

This shows that we can simply take $\theta = 0$ in (4.30) with no loss of generality, and we arrive at the only possible solution for the r-matrix which has previously been considered as the one generating the (non-twisted) κ-(A)dS deformation [80, 114, 145, 146]. Moreover, this computation provides a neat geometrical intuition of the fact discussed in [160] that a rotation generator becomes privileged when $\Lambda \neq 0$. Also, this proves that, modulo Lie algebra automorphisms, the (A)dS r-matrix (4.32) is the only (non-twisted) skew-symmetric solution of the mCYBE which generalizes the κ-Poincaré deformation.

For the twisted case we have that $(\beta_1, \beta_2, \beta_3) \neq (0, 0, 0)$. With no loss of generality we can assume that $\beta_3 \neq 0$. By taking into account (4.28) and (4.29) we find that

$$\beta_1 = \beta_3 \tan\theta\cos\varphi, \qquad \beta_2 = -\beta_3\tan\theta\sin\varphi, \tag{4.33}$$

$(\theta \neq \pi/2)$ which inserted in (4.30) gives

$$\begin{aligned} r_\Lambda = \frac{1}{\kappa}&(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + \beta_3 P_0 \wedge (\tan\theta\cos\varphi J_1 - \tan\theta\sin\varphi J_2 + J_3) \\ &+ \frac{\eta}{\kappa}(\cos\theta J_1 \wedge J_2 + \sin\theta\sin\varphi J_1 \wedge J_3 + \sin\theta\cos\varphi J_2 \wedge J_3). \end{aligned} \tag{4.34}$$

Now, if we consider the rotated basis such that $\theta = 0$ and rename the twist parameter as $\beta_3 = -\vartheta$ we arrive at

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2) + \vartheta J_3 \wedge P_0\,. \tag{4.35}$$

□

The κ-twisted-like r-matrix is just the quasitriangular r-matrix presented in [113] as the one arising from a Drinfel'd double structure of the (A)dS Lie algebra (see also [114]). The Poincaré $\Lambda \to 0$ limit of this r-matrix, r_0^t, along with its Galilean counterpart were studied in [207].

Once the existence and uniqueness (up to Lie algebra automorphisms) of the κ-like deformation have been proved, we proceed to its explicit construction. First of all, from

(4.25) the cocommutator $\delta(X) = \mathrm{ad}_X r$ for all $X \in \mathfrak{g}_0$ is directly obtained and it reads

$$
\begin{aligned}
&\delta(P_0) = \delta(J_3) = 0, \qquad \delta(J_1) = \frac{\eta}{\kappa} J_1 \wedge J_3, \qquad \delta(J_2) = \frac{\eta}{\kappa} J_2 \wedge J_3,\\
&\delta(P_1) = \frac{1}{\kappa}(P_1 \wedge P_0 - \eta P_3 \wedge J_1 - \eta^2 K_2 \wedge J_3 + \eta^2 K_3 \wedge J_2),\\
&\delta(P_2) = \frac{1}{\kappa}(P_2 \wedge P_0 - \eta P_3 \wedge J_2 + \eta^2 K_1 \wedge J_3 - \eta^2 K_3 \wedge J_1),\\
&\delta(P_3) = \frac{1}{\kappa}(P_3 \wedge P_0 + \eta P_1 \wedge J_1 + \eta P_2 \wedge J_2 - \eta^2 K_1 \wedge J_2 + \eta^2 K_2 \wedge J_1),\\
&\delta(K_1) = \frac{1}{\kappa}(K_1 \wedge P_0 + P_2 \wedge J_3 - P_3 \wedge J_2 - \eta K_3 \wedge J_1),\\
&\delta(K_2) = \frac{1}{\kappa}(K_2 \wedge P_0 - P_1 \wedge J_3 + P_3 \wedge J_1 - \eta K_3 \wedge J_2),\\
&\delta(K_3) = \frac{1}{\kappa}(K_3 \wedge P_0 + P_1 \wedge J_2 - P_2 \wedge J_1 + \eta K_1 \wedge J_1 + \eta K_2 \wedge J_2).
\end{aligned}
\tag{4.36}
$$

Here it becomes clear that the $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ Lie subalgebra generated by the rotation generators $\{J_1, J_2, J_3\}$ defines a sub-Lie bialgebra structure, which becomes non-trivial when the cosmological constant is different from zero, a fact that will be relevant in the sequel.

Now we have all the ingredients needed to construct the unique κ-Poisson-Hopf (A)dS algebra (see Definition (3.25)), which is the Poisson version $\mathcal{PV}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ of $\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)$, defined by (4.25). The Poisson version of the 'quantum duality principle' (see [19, 208, 59, 205] and references below) implies that the Poisson structure we are looking for is just a Poisson structure on the dual group G^*_Λ, the unique connected and simply-connected Lie group such that $\mathrm{Lie}(G^*_\Lambda) = \mathfrak{g}^*_\Lambda$, where $\mathfrak{g}^*_\Lambda$ is the dual Lie algebra to (4.36). This dual algebra can be found in [114, 146]. Introducing an algebraic basis $\{X^\alpha, L^a, R^a\}$ in $\mathfrak{g}^*_\Lambda$, such that

$$
\langle X^\alpha, P_\beta\rangle = \delta^\alpha_\beta, \qquad \langle L^a, K_b\rangle = \delta^a_b, \qquad \langle R^a, J_b\rangle = \delta^a_b, \tag{4.37}
$$

this dual Lie algebra is defined by the following commutation relations

$$
\begin{array}{lll}
[R^1, R^2] = 0, & [R^1, R^3] = \frac{\sqrt{-\Lambda}}{\kappa} R^1, & [R^2, R^3] = \frac{\sqrt{-\Lambda}}{\kappa} R^2,\\
[R^1, X^1] = -\frac{\sqrt{-\Lambda}}{\kappa} X^3, & [R^1, X^2] = \frac{1}{\kappa} L^3, & [R^1, X^3] = -\frac{1}{\kappa}(L^2 - \sqrt{-\Lambda} X^1),\\
[R^2, X^1] = -\frac{1}{\kappa} L^3, & [R^2, X^2] = -\frac{\sqrt{-\Lambda}}{\kappa} X^3, & [R^2, X^3] = \frac{1}{\kappa}(L^1 + \sqrt{-\Lambda} X^2),\\
[R^3, X^1] = \frac{1}{\kappa} L^2, & [R^3, X^2] = -\frac{1}{\kappa} L^1, & [R^3, X^3] = 0,\\
[R^1, L^1] = -\frac{\sqrt{-\Lambda}}{\kappa} L^3, & [R^1, L^2] = \frac{\Lambda}{\kappa} X^3, & [R^1, L^3] = \frac{1}{\kappa}(\sqrt{-\Lambda} L^1 + -\Lambda X^2),\\
[R^2, L^1] = \frac{-\Lambda}{\kappa} X^3, & [R^2, L^2] = -\frac{\sqrt{-\Lambda}}{\kappa} L^3, & [R^2, L^3] = \frac{1}{\kappa}(\sqrt{-\Lambda} L^2 - -\Lambda X^1),\\
[R^3, L^1] = \frac{\Lambda}{\kappa} X^2, & [R^3, L^2] = -\frac{\Lambda}{\kappa} X^1, & [R^3, L^3] = 0,\\
[L^a, X^0] = \frac{1}{\kappa} L^a, & [L^a, L^b] = 0, & [L^a, X^b] = 0,\\
[X^a, X^0] = \frac{1}{\kappa} X^a, & [X^a, X^b] = 0, & [X^0, R^a] = 0\,,
\end{array}
\tag{4.38}
$$

Notice that, by quantum duality, the automorphism (4.24) of $\mathfrak{g}_\Lambda$ leads to the following automorphism (whenever $\Lambda \neq 0$) of $\mathfrak{g}_\Lambda$

$$\tilde{X}^0 = X^0, \qquad \tilde{X}^a = \frac{1}{\sqrt{-\Lambda}} L^a, \qquad \tilde{L}^a = -\sqrt{-\Lambda}\, X^a, \qquad \tilde{R}^a = R^a, \tag{4.39}$$

which keeps the commutation relations (5.57) invariant and shows the interchanging between the X^a and L^a generators.

As it should be, the limit $\kappa \to \infty$ produces an abelian Lie algebra and so its Lie group is also abelian, thus giving rise to the undeformed, or primitive, coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in \mathfrak{g}_\Lambda$.

Therefore, at the Poisson level, this deformation induces a non-abelian group G^*_Λ whose multiplication is dual to the coproduct for the Poisson-Hopf algebra $\mathcal{PV}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$. Now we need to find a faithful representation in order to construct the group element by exponentiation, and from here compute the group multiplication. The adjoint representation will work, but note that as we are interested in real representations, the cases of $\Lambda < 0$ and $\Lambda > 0$ need to be treated separately. For clarity, we omit the explicit expressions here, but they can be found in Chapter 5 where the quantum duality principle will be used to construct the non-trivial momentum spaces associated to these deformations. Now, if $\rho : \mathfrak{g}^*_\Lambda \to \mathrm{End}(\mathbb{R}^{10})$ stands for the appropriate adjoint representation, we can always locally write a group element as

$$\begin{aligned} g^* = (\theta, p, \chi) = & e^{\theta_3 \rho(R^3)} e^{\theta_2 \rho(R^2)} e^{\theta_1 \rho(R^1)} e^{p_1 \rho(X^1)} e^{p_2 \rho(X^2)} e^{p_3 \rho(X^3)} \times \\ & \times e^{\chi_1 \rho(L^1)} e^{\chi_2 \rho(L^2)} e^{\chi_3 \rho(L^3)} e^{p_0 \rho(X^0)}. \end{aligned} \tag{4.40}$$

where $\{p_0, p_1, p_2, p_3, \chi_1, \chi_2, \chi_3, \theta_1, \theta_2, \theta_3\}$ are exponential coordinates of the second kind on the dual group G^*_Λ. A long but straightforward computer-assisted computation (in [205] the details of the procedure are explained in detail) leads to the group law for G^*_Λ, which once dualized allows us to write down the coproduct map for the Poisson-Hopf $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ algebra. Explicitly, after the identification $p_\alpha = P_\alpha, \chi_a = K_a, \theta_a = J_a$, we have that the coproduct

$$\Delta_\kappa : \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)) \to \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)) \otimes \mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)) \tag{4.41}$$

takes the following form:

$$\begin{aligned} \Delta_\kappa(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \Delta_\kappa(J_1) &= J_1 \otimes e^{\sqrt{-\Lambda} J_3/\kappa} + 1 \otimes J_1, \\ \Delta_\kappa(J_2) &= J_2 \otimes e^{\sqrt{-\Lambda} J_3/\kappa} + 1 \otimes J_2, \end{aligned} \tag{4.42}$$

$$
\begin{aligned}
\Delta_\kappa(P_0) &= P_0\otimes 1+1\otimes P_0,\\
\Delta_\kappa(P_1) &= P_1\otimes\cosh(\sqrt{-\Lambda}J_3/\kappa)+e^{-P_0/\kappa}\otimes P_1-\sqrt{-\Lambda}K_2\otimes\sinh(\sqrt{-\Lambda}J_3/\kappa)\\
&\quad -\frac{\sqrt{-\Lambda}}{\kappa}P_3\otimes J_1-\frac{\Lambda}{\kappa}K_3\otimes J_2-\frac{\Lambda}{\kappa^2}\left(\sqrt{-\Lambda}K_1-P_2\right)\otimes J_1J_2e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad +\frac{\Lambda}{2\kappa^2}\left(\sqrt{-\Lambda}K_2+P_1\right)\otimes\left(J_1^2-J_2^2\right)e^{-\sqrt{-\Lambda}J_3/\kappa},\\
\Delta_\kappa(P_2) &= P_2\otimes\cosh(\sqrt{-\Lambda}J_3/\kappa)+e^{-P_0/\kappa}\otimes P_2+\sqrt{-\Lambda}K_1\otimes\sinh(\sqrt{-\Lambda}J_3/\kappa) \qquad (4.43)\\
&\quad -\frac{\sqrt{-\Lambda}}{\kappa}P_3\otimes J_2+z\Lambda K_3\otimes J_1+\frac{\Lambda}{\kappa^2}\left(\sqrt{-\Lambda}K_2+P_1\right)\otimes J_1J_2e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad +\frac{\Lambda}{2\kappa^2}\left(\sqrt{-\Lambda}K_1-P_2\right)\otimes\left(J_1^2-J_2^2\right)e^{-\sqrt{-\Lambda}J_3/\kappa},\\
\Delta_z(P_3) &= P_3\otimes 1+e^{-P_0/\kappa}\otimes P_3+\frac{1}{\kappa}\left(-\Lambda K_2+\sqrt{-\Lambda}P_1\right)\otimes J_1e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad -z\left(-\Lambda K_1-\sqrt{-\Lambda}P_2\right)\otimes J_2e^{-\sqrt{-\Lambda}J_3/\kappa},
\end{aligned}
$$

$$
\begin{aligned}
\Delta_\kappa(K_1) &= K_1\otimes\cosh(\sqrt{-\Lambda}J_3/\kappa)+e^{-P_0/\kappa}\otimes K_1+P_2\otimes\frac{\sinh(\sqrt{-\Lambda}J_3/\kappa)}{\sqrt{-\Lambda}}\\
&\quad -\frac{1}{\kappa}P_3\otimes J_2-\frac{\sqrt{-\Lambda}}{\kappa}K_3\otimes J_1-\frac{1}{\kappa^2}\left(-\Lambda K_2+\sqrt{-\Lambda}P_1\right)\otimes J_1J_2e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad -\frac{1}{2\kappa^2}\left(-\Lambda K_1-\sqrt{-\Lambda}P_2\right)\otimes\left(J_1^2-J_2^2\right)e^{-\sqrt{-\Lambda}J_3/\kappa},\\
\Delta_\kappa(K_2) &= K_2\otimes\cosh(\sqrt{-\Lambda}J_3/\kappa)+e^{-P_0/\kappa}\otimes K_2-P_1\otimes\frac{\sinh(\sqrt{-\Lambda}J_3/\kappa)}{\sqrt{-\Lambda}} \qquad (4.44)\\
&\quad +\frac{1}{\kappa}P_3\otimes J_1-\frac{\sqrt{-\Lambda}}{\kappa}K_3\otimes J_2-\frac{1}{\kappa^2}\left(-\Lambda K_1-\sqrt{-\Lambda}P_2\right)\otimes J_1J_2e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad +\frac{1}{2\kappa^2}\left(-\Lambda K_2+\sqrt{-\Lambda}P_1\right)\otimes\left(J_1^2-J_2^2\right)e^{-\sqrt{-\Lambda}J_3/\kappa},\\
\Delta_\kappa(K_3) &= K_3\otimes 1+e^{-P_0/\kappa}\otimes K_3+z(\sqrt{-\Lambda}K_1-P_2)\otimes J_1e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&\quad +\frac{1}{\kappa}(\sqrt{-\Lambda}K_2+P_1)\otimes J_2e^{-\sqrt{-\Lambda}J_3/\kappa}.
\end{aligned}
$$

Notice that this coproduct is written in a 'bicrossproduct-type' basis that generalizes the one corresponding to the (2+1) $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ algebra [111, 144].

As it can be easily checked, the κ-Poincaré coproduct (4.5) is obtained from the above expressions in the limit $\Lambda\to 0$. A direct comparison between both sets of expressions makes it evident that the degree of complexity of the κ-deformation is greatly increased when the cosmological constant Λ is turned on. In fact, the $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ algebra can be thought of as a two-parametric deformation, which is ruled by a 'quantum' deformation parameter κ (the Planck scale) and a 'classical' deformation parameter Λ (the cosmological constant) which has a well-defined geometrical meaning. As we will show in the Chapter 5, the roles of the two deformation parameters are interchanged when the dual Poisson-Lie group is considered, in the spirit of the 'semidualization' approach to (2+1) quantum gravity [159, 209].

There are several differences between the coproducts (4.42)-(4.44) and (4.5) that have to be emphasized. First, $\Delta_\kappa(K_a)$ and $\Delta_\kappa(P_a)$ are structurally similar when $\Lambda\neq 0$, in contrast with (4.5). Second, translations in (4.43) do not close a Hopf subalgebra, since when $\Lambda \neq 0$ the coproducts $\Delta_\kappa(P_a)$ contain boosts and rotations as well. Finally, in the non-vanishing cosmological constant case the rotation sector (4.42) is deformed, whilst in (4.5) all of the coproducts for J_a are primitive ones. These three features are induced by the interplay between the cosmological constant Λ and the quantum deformation κ, and the first two will be essential for the construction of the curved momentum space when $\Lambda \neq 0$, which will be topic of the first part of Chapter 5.

The unique Poisson brackets compatible with the previous coproduct Δ_κ as a Poisson-Hopf algebra $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ are given, by Drinfeld's theorem [210], as the unique Poisson-Lie structure on G_Λ^* whose tangent Lie bialgebra is given by the dual of $\mathfrak{g}_\Lambda$ (2.72), as explained in the (1+1)-dimensional example at the end of Chapter 2. This dual Lie bialgebra is not coboundary, so we do not have a canonical way to compute it, in particular there is no Sklyanin bracket (3.44) defined on G_Λ^*. However, by following the computational approach introduced in [205] and assuming that such Poisson bivector is at most quadratic in the functions appearing in the coproduct, such Poisson structure was explicitly computed in [114]. For the sake of brevity, we omit the details of such computation, but we do write down explicitly the fundamental brackets:

$$
\begin{array}{lll}
\{J_1,J_2\} = \dfrac{e^{2z\sqrt{-\Lambda}J_3}-1}{2z\sqrt{-\Lambda}} - \dfrac{z\sqrt{-\Lambda}}{2}\left(J_1^2+J_2^2\right), & \{J_1,J_3\} = -J_2, & \{J_2,J_3\} = J_1, \\
\{J_1,P_1\} = z\sqrt{-\Lambda}J_1P_2, & \{J_1,P_2\} = P_3 - z\sqrt{-\Lambda}J_1P_1, & \{J_1,P_3\} = -P_2, \\
\{J_2,P_1\} = -P_3 + z\sqrt{-\Lambda}J_2P_2, & \{J_2,P_2\} = -z\sqrt{-\Lambda}J_2P_1, & \{J_2,P_3\} = P_1, \\
\{J_3,P_1\} = P_2, & \{J_3,P_2\} = -P_1, & \{J_3,P_3\} = 0, \\
\{J_1,K_1\} = z\sqrt{-\Lambda}J_1K_2, & \{J_1,K_2\} = K_3 - z\sqrt{-\Lambda}J_1K_1, & \{J_1,K_3\} = -K_2, \\
\{J_2,K_1\} = -K_3 + z\sqrt{-\Lambda}J_2K_2, & \{J_2,K_2\} = -z\sqrt{-\Lambda}J_2K_1, & \{J_2,K_3\} = K_1, \\
\{J_3,K_1\} = K_2, & \{J_3,K_2\} = -K_1, & \{J_3,K_3\} = 0, \\
\{K_a,P_0\} = P_a, & \{P_0,P_a\} = -\Lambda K_a, & \{P_0,J_a\} = 0,
\end{array}
$$

$$
\begin{aligned}
\{K_1,P_1\} &= \frac{1}{2z}\left(\cosh(2z\sqrt{-\Lambda}J_3) - e^{-2zP_0}\right) + \frac{z^3\Lambda^2}{4}e^{-2z\sqrt{-\Lambda}J_3}\left(J_1^2+J_2^2\right)^2 + \frac{z}{2}\left(P_2^2+P_3^2-P_1^2\right) \\
&\quad + \frac{-z\Lambda}{2}\left[K_2^2+K_3^2-K_1^2+J_1^2\left(1-e^{-2z\sqrt{-\Lambda}J_3}\right)+J_2^2\left(1+e^{-2z\sqrt{-\Lambda}J_3}\right)\right], \\
\{K_2,P_2\} &= \frac{1}{2z}\left(\cosh(2z\sqrt{-\Lambda}J_3) - e^{-2zP_0}\right) + \frac{z^3\Lambda^2}{4}e^{-2z\sqrt{-\Lambda}J_3}\left(J_1^2+J_2^2\right)^2 + \frac{z}{2}\left(P_1^2+P_3^2-P_2^2\right) \\
&\quad + \frac{-z\Lambda}{2}\left[K_1^2+K_3^2-K_2^2+J_1^2\left(1+e^{-2z\sqrt{-\Lambda}J_3}\right)+J_2^2\left(1-e^{-2z\sqrt{-\Lambda}J_3}\right)\right], \\
\{K_3,P_3\} &= \frac{1-e^{-2zP_0}}{2z} + \frac{z}{2}\left[(P_1+\sqrt{-\Lambda}K_2)^2+(P_2-\sqrt{-\Lambda}K_1)^2-P_3^2+\Lambda K_3^2\right] \\
&\quad - z\Lambda e^{-2z\sqrt{-\Lambda}J_3}\left(J_1^2+J_2^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\{P_1,K_2\} &= z\left(P_1P_2-\Lambda K_1K_2-\sqrt{-\Lambda}P_3K_3-\Lambda J_1J_2e^{-2z\sqrt{-\Lambda}J_3}\right),\\
\{P_2,K_1\} &= z\left(P_1P_2-\Lambda K_1K_2+\sqrt{-\Lambda}P_3K_3-\Lambda J_1J_2e^{-2z\sqrt{-\Lambda}J_3}\right),\\
\{P_1,K_3\} &= \frac{1}{2}\sqrt{-\Lambda}J_1\left(1-e^{-2z\sqrt{-\Lambda}J_3}\left[1+z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\left(P_1P_3-\Lambda K_1K_3+\sqrt{-\Lambda}K_2P_3\right),\\
\{P_3,K_1\} &= \frac{1}{2}\sqrt{-\Lambda}J_1\left(1-e^{-2z\sqrt{-\Lambda}J_3}\left[1+z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\left(P_1P_3-\Lambda K_1K_3-\sqrt{-\Lambda}P_2K_3\right),\\
\{P_2,K_3\} &= \frac{1}{2}\sqrt{-\Lambda}J_2\left(1-e^{-2z\sqrt{-\Lambda}J_3}\left[1+z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\left(P_2P_3-\Lambda K_2K_3-\sqrt{-\Lambda}K_1P_3\right),\\
\{P_3,K_2\} &= \frac{1}{2}\sqrt{-\Lambda}J_2\left(1-e^{-2z\sqrt{-\Lambda}J_3}\left[1+z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\left(P_2P_3-\Lambda K_2K_3+\sqrt{-\Lambda}P_1K_3\right),
\end{aligned}
$$

$$
\begin{aligned}
&\{K_1,K_2\}=-\frac{\sinh(2z\sqrt{-\Lambda}J_3)}{2z\sqrt{-\Lambda}}-\frac{z\sqrt{-\Lambda}}{2}\left(J_1^2+J_2^2+2K_3^2\right)-\frac{z^3(-\Lambda)^{3/2}}{4}e^{-2z\sqrt{-\Lambda}J_3}\left(J_1^2+J_2^2\right)^2,\\
&\{K_1,K_3\}=\frac{1}{2}J_2\left(1+e^{-2z\sqrt{-\Lambda}J_3}\left[1+-z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\sqrt{-\Lambda}K_2K_3,\\
&\{K_2,K_3\}=-\frac{1}{2}J_1\left(1+e^{-2z\sqrt{-\Lambda}J_3}\left[1+-z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)-z\sqrt{-\Lambda}K_1K_3,
\end{aligned}
$$

$$
\begin{aligned}
&\{P_1,P_2\}=\Lambda\frac{\sinh(2z\sqrt{-\Lambda}J_3)}{2z\sqrt{-\Lambda}}-\frac{z\sqrt{-\Lambda}}{2}\left(2P_3^2-\Lambda(J_1^2+J_2^2)\right)-\frac{z^3\Lambda^{5/2}}{4}e^{-2z\sqrt{-\Lambda}J_3}\left(J_1^2+J_2^2\right)^2,\\
&\{P_1,P_3\}=-\frac{1}{2}\Lambda J_2\left(1+e^{-2z\sqrt{-\Lambda}J_3}\left[1+-z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)+z\sqrt{-\Lambda}P_2P_3,\\
&\{P_2,P_3\}=\frac{1}{2}\Lambda J_1\left(1+e^{-2z\sqrt{-\Lambda}J_3}\left[1+-z^2\Lambda\left(J_1^2+J_2^2\right)\right]\right)-z\sqrt{-\Lambda}P_1P_3.
\end{aligned}
\tag{4.45}
$$

Regarding this Poisson structure it is worth stressing that, in contradistinction to (4.6), the full Lorentz sector has now deformed Poisson brackets. These expressions make the parallelism between translations and boosts evident also at the Poisson bracket level, and show the interplay between deformed rotations and the rest of the quantum algebra.

The deformed quadratic Casimir for the $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ Poisson-Hopf algebra reads [114]

$$
\begin{aligned}
\mathcal{C}_\kappa = {}& 2\kappa^2\left[\cosh(P_0/\kappa)\cosh(\sqrt{-\Lambda}J_3/\kappa)-1\right]-\Lambda\cosh(P_0/\kappa)(J_1^2+J_2^2)e^{-\sqrt{-\Lambda}J_3/\kappa}\\
&-e^{P_0/\kappa}\left(\mathbf{P}^2-\Lambda\mathbf{K}^2\right)\left[\cosh(\sqrt{-\Lambda}J_3/\kappa)-\frac{\Lambda}{2\kappa^2}(J_1^2+J_2^2)e^{-\sqrt{-\Lambda}J_3/\kappa}\right]\\
&-2\Lambda e^{P_0/\kappa}\left[\frac{\sinh(\sqrt{-\Lambda}J_3/\kappa)}{\sqrt{-\Lambda}}T_3+\frac{1}{\kappa}\left(J_1T_1+J_2T_2+\frac{\sqrt{-\Lambda}}{2\kappa}(J_1^2+J_2^2)T_3\right)e^{-\sqrt{-\Lambda}J_3/\kappa}\right],
\end{aligned}
\tag{4.46}
$$

where $T_a=\epsilon_{abc}K_bP_c$. Again, the $\Lambda\to 0$ limit of (4.45) is just (4.6), and comparing (4.46) to its 'flat' limit (4.7) gives a clear idea of the kind of deformation we are dealing with. This deformed invariant (4.46) (compare it with the undeformed one (2.80) to see clearly the complexity increment) will be very relevant in the first part of Chapter 5, since it is connected to the deformed dispersion relation that can be deduced from the curved momentum space with cosmological constant that we will construct.

Indeed, what we have obtained is a commutative Poisson-Hopf algebra $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$, whose quantization has to be performed in order to obtain the proper quantum universal enveloping algebra $\mathcal{U}_\kappa(\mathfrak{g}_\Lambda)$. Once this had been completed, a (presumably nonlinear and complicated) change of basis should exist between the kinematical basis and the Cartan-Weyl or Cartan-Chevalley basis used in [62, 74, 75, 76]. Nevertheless, most of the physically relevant features of this quantum deformation can already be extracted from the kinematical Poisson-Hopf structure here obtained.

4.4 The κ-(A)dS noncommutative spacetime

Similarly to the flat case, once the Poisson-Hopf $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda))$ algebra has been studied, we focus on the construction of the associated noncommutative spacetime. In this case, as we will show in the following, the PHS so obtained is much more complicated than its vanishing constant version, the κ-Minkowski spacetime previously presented. This is the main reason why this noncommutative spacetime has only recently been constructed, in contradistinction to the κ-Minkowski spacetime, which as explained above has been extensively studied since its introduction.

The first step in the construction, as usual, is the introduction of local coordinates on G_Λ following the strategy described in §2.3, such that the diagram (2.86) becomes commutative. Coordinates defined by the inverse map of (2.88) satisfy all these conditions, and are the ones we will use. These coordinates satisfy that $\bar{x} \in \mathcal{C}^\infty(G)^H$, so they define a set of coordinates on the coset space, i.e. $\bar{x} : M_\Lambda = G_\Lambda/L \to \mathbb{R}^4$. They are precisely the *geodesic parallel coordinates on* M_Λ introduced in §2.2.

Before constructing the Poisson homogeneous space associated to the κ-deformation of G_Λ, we need to check that this PHS fulfills the coisotropy condition (3.79). This can be checked directly from (4.36) and so we have that

$$\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{g}_\Lambda, \tag{4.47}$$

where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{g}_\Lambda = \mathrm{Lie}(G_\Lambda)$. In this case, similarly to the κ-Minkowski spacetime, this PHS will not be a Poisson subgroup PHS, but a coisotropic one, since the Lorentz group L is not a Poisson subgroup of G_Λ.

The procedure to equip the manifolds G_Λ and M_Λ with relevant Poisson structures is the same that we used in the Minkowski case. If we call (G_Λ, Π) to the κ-Poisson-Lie structure on G_Λ and (M_Λ, π) to the Poisson structure in M_Λ that makes

$$\begin{aligned} \alpha : G_\Lambda \times M_\Lambda &\to M_\Lambda \\ (g, g'L) &\to gg'L \end{aligned} \tag{4.48}$$

a Poisson homogeneous space, then Π is given by the Sklyanin bracket (3.44) defined by the r-matrix (4.25). The Sklyanin bracket in this particular case reads

$$\begin{aligned}\{f_1, f_2\} = \frac{1}{\kappa}\sum_{a=1}^{3}\big(\big(X^L_{K_a}f_1X^L_{P_a}f_2 - X^L_{P_a}f_1X^L_{K_a}f_2\big) - \big(X^R_{K_a}f_1X^R_{P_a}f_2 - X^R_{P_a}f_1X^R_{K_a}f_2\big)\big)\\ + \eta\,\big(\big(X^L_{J_1}f_1X^L_{J_2}f_2 - X^L_{J_2}f_1X^L_{J_1}f_2\big) - \big(X^R_{J_1}f_1X^R_{J_2}f_2 - X^R_{J_2}f_1X^R_{J_1}f_2\big)\big)\,.\end{aligned} \tag{4.49}$$

Similarly, π can be obtained from (4.49) just by particularizing it to the L-invariant functions. We have that

$$\{\tilde{f}_1, \tilde{f}_2\}_{M_\Lambda}(m) = \{f_1\circ p, f_2\circ p\}_{M_\Lambda}(m) = \{f_1, f_2\}_{G_\Lambda}(g) \tag{4.50}$$

where $f_1, f_2 \in \mathcal{C}^\infty(G)^L$, $m = gL \in M_\Lambda$ and

$$\begin{aligned} p: G_\Lambda &\to M_\Lambda\\ g &\to gL\end{aligned} \tag{4.51}$$

is the canonical projection. In terms of the Poisson bivector, this just implies that the Poisson bivector π on M_Λ is the pushforward of the Poisson bivector Π on G_0, i.e. $\pi = p_*\Pi$. In order to write down the explicit fundamental brackets, left- and right-invariant vector fields for G_Λ have to be obtained (through a really cumbersome computer-assisted computation). They are extremely complicated, so we do not present them in this work. Moreover, the fundamental brackets on G_Λ are also really cumbersome, so we shall omit their complete expressions since they do not provide any meaningful information for our purposes. However, we do write down the fundamental brackets associated to (4.50), i.e. the Poisson structure π on M_Λ, which read

$$\begin{aligned}\{x^0, x^1\} &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^1)}{\eta\cosh^2(\eta x^2)\cosh^2(\eta x^3)},\\ \{x^0, x^2\} &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^2)}{\eta\cosh^2(\eta x^3)},\\ \{x^0, x^3\} &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^3)}{\eta},\end{aligned} \tag{4.52}$$

$$\begin{aligned}\{x^1, x^2\} &= -\frac{1}{\kappa}\,\frac{\cosh(\eta x^1)\tanh^2(\eta x^3)}{\eta},\\ \{x^1, x^3\} &= \frac{1}{\kappa}\,\frac{\cosh(\eta x^1)\tanh(\eta x^2)\tanh(\eta x^3)}{\eta},\\ \{x^2, x^3\} &= -\frac{1}{\kappa}\,\frac{\sinh(\eta x^1)\tanh(\eta x^3)}{\eta},\end{aligned} \tag{4.53}$$

These expressions can be thought of as a (complicated) cosmological constant deformation of the (Poisson) κ-Minkowski spacetime (4.13) in terms of the parameter η. Moreover, a striking feature of the κ-(A)dS spacetime suddenly arises from them: brackets between space coordinates do not vanish, in contradistinction with the κ-Minkowski case (and also with the (2+1) κ-(A)dS spacetime presented in [111], which can be obtained from (4.52)–(4.53) by projecting $x^3 \to 0$). In order to stress the relationship with the κ-Minkowski expressions, we can take the power series expansion of (4.52) in terms of η, and we get

$$
\begin{aligned}
\{x^0, x^1\} &= -\frac{1}{\kappa}\,(x^1 + o[\eta^2]),\\
\{x^0, x^2\} &= -\frac{1}{\kappa}\,(x^2 + o[\eta^2]),\\
\{x^0, x^3\} &= -\frac{1}{\kappa}\,(x^3 + o[\eta^2]),
\end{aligned}
\tag{4.54}
$$

whose zeroth-order in η is just the κ-Minkowski spacetime, whilst the first order deformation in η of the space subalgebra (4.53) defines the following homogeneous quadratic algebra

$$
\begin{aligned}
\{x^1, x^2\} &= -\frac{1}{\kappa}\,(\eta\,(x^3)^2 + o[\eta^2]),\\
\{x^1, x^3\} &= \frac{1}{\kappa}\,(\eta\,x^2x^3 + o[\eta^2]),\\
\{x^2, x^3\} &= -\frac{1}{\kappa}\,(\eta\,x^1x^3 + o[\eta^2]).
\end{aligned}
\tag{4.55}
$$

This essential novelty of the κ-(A)dS spacetime deserves further discussion. Firstly, note that the quadratic Poisson algebra arising in (4.55) and given by

$$
\{x^1, x^2\} = -\frac{\eta}{\kappa}\,(x^3)^2, \qquad \{x^1, x^3\} = \frac{\eta}{\kappa}\,x^2x^3, \qquad \{x^2, x^3\} = -\frac{\eta}{\kappa}\,x^1x^3, \tag{4.56}
$$

can be identified [211, 212] as a subalgebra of the semiclassical limit of Woronowicz's quantum SU(2) group [213, 214] (see also [215, 216]). We also recall that the brackets

$$
\{x^1, x^2\} = f\,\frac{\partial F}{\partial x^3}, \qquad \{x^2, x^3\} = f\,\frac{\partial F}{\partial x^1}, \qquad \{x^3, x^1\} = f\,\frac{\partial F}{\partial x^2}, \tag{4.57}
$$

always define a three-dimensional Poisson algebra for any choice of the smooth functions f and F, and the Casimir function for (4.57) is just the function F [217]. Therefore, the algebra (4.56) can directly be obtained by taking

$$
F(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2, \qquad f(x^1, x^2, x^3) = -\frac{1}{2}\,\frac{\eta}{\kappa}\,x^3. \tag{4.58}
$$

This implies that two-dimensional spheres

$$
S = (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{4.59}
$$

define symplectic leaves for the Poisson structure (4.56). Moreover, it is straightforward to check that the Poisson brackets (4.53) arise in the Sklyanin bracket just from the $J_1 \wedge J_2$ term of the r-matrix (4.32). This explains why the Poisson algebra (4.56) is naturally linked to the semiclassical limit of the quantum SU(2) subgroup of the κ-(A)dS deformation, albeit realized on the 3-space coordinates. In this respect, we recall that the $\mathfrak{su}(2)$ subalgebra generated by $\{J_1, J_2, J_3\}$ becomes a quantum $\mathfrak{su}(2)$ subalgebra when the full quantum deformation is constructed [114], a fact that can already be envisaged from the cocommutator (4.36) where the $\mathfrak{su}(2)$ generators define a sub-Lie bialgebra.

4.4.1 Quantization of the κ-(A)dS Poisson homogeneous space

Furthermore, the algebra (4.56) can be quantized as

$$[\hat{x}^1, \hat{x}^2] = -\frac{\eta}{\kappa}(\hat{x}^3)^2, \qquad [\hat{x}^1, \hat{x}^3] = \frac{\eta}{\kappa}\hat{x}^3\hat{x}^2, \qquad [\hat{x}^2, \hat{x}^3] = -\frac{\eta}{\kappa}\hat{x}^1\hat{x}^3, \tag{4.60}$$

since associativity is ensured by the Jacobi identity, which can be checked by considering the ordered monomials $(\hat{x}^1)^l\,(\hat{x}^3)^m\,(\hat{x}^2)^n$. The Casimir operator for (4.60) can be proven to be

$$\hat{S}_{\eta/\kappa} = (\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 + \frac{\eta}{\kappa}\hat{x}^1\hat{x}^2, \tag{4.61}$$

which defines the 'quantum spheres generated by the noncommuting κ-(A)dS local coordinates. Thus space coordinates become noncommutative, while at first order in η the time-space sector is kept invariant with respect to the κ-Minkowski case. Moreover, the space-space brackets (4.56) are just a subalgebra of the quantum SU(2) group. So, at first order in the quantum parameter η, the *quantum κ-AdS spacetime* is given by the following commutation relations

$$\begin{aligned}
&[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\hat{x}^a, \\
&[\hat{x}^1, \hat{x}^2] = -\frac{\eta}{\kappa}(\hat{x}^3)^2, \qquad [\hat{x}^1, \hat{x}^3] = \frac{\eta}{\kappa}\hat{x}^3\hat{x}^2, \qquad [\hat{x}^2, \hat{x}^3] = -\frac{\eta}{\kappa}\hat{x}^1\hat{x}^3.
\end{aligned} \tag{4.62}$$

From this explicit form it is perfectly clear that the noncommutativity between space and time coordinates is different from the noncommutativity between space-space coordinates, because of the factor η in the last ones. It should be noticed that when considering propagation of particles on noncommutativity spacetime this cosmological constant contribution could be non-negligible.

Now, the quantum κ-(A)dS spacetime for any order in η should be obtained as the quantization of the full Poisson algebra (4.52)-(4.53), which is by no means a trivial task due to the noncommutativity of the space coordinates given by (4.53). However, by considering the five ambient coordinates $(s^4, s^0, \mathbf{s})$ defined by (2.98) and fulfilling the constraint (2.96), we get that their Sklyanin bracket leads to the following quadratic algebra

$$\begin{aligned}
&\{s^0, s^a\} = -\frac{1}{\kappa}s^a s^4, \qquad \{s^4, s^a\} = \frac{\eta^2}{\kappa}s^a s^0, \\
&\{s^1, s^2\} = -\frac{\eta}{\kappa}(s^3)^2, \qquad \{s^1, s^3\} = \frac{\eta}{\kappa}s^2 s^3, \qquad \{s^2, s^3\} = -\frac{\eta}{\kappa}s^1 s^3, \\
&\{s^0, s^4\} = -\frac{\eta^2}{\kappa}\left((s^1)^2 + (s^2)^2 + (s^3)^2\right),
\end{aligned} \tag{4.63}$$

which is, at most, quadratic in the cosmological constant parameter η. Since the subalgebra generated by the three ambient space coordinates $\mathbf{s}$ is formally the same as (4.56), its quantization would give the same result as (4.60), but now with $\hat{\mathbf{s}}$ instead of $\hat{\mathbf{x}}$. By taking into account this fact and by considering the ordered monomials $(\hat{s}^0)^k\,(\hat{s}^1)^l\,(\hat{s}^3)^m\,(\hat{s}^2)^n\,(\hat{s}^4)^j$, a long but straightforward computation shows that the following quadratic brackets give rise to an associative algebra (i.e. Jacobi identities are satisfied) which becomes the full quantization of the Poisson brackets (4.63)

$$
\begin{aligned}
&[\hat{s}^0, \hat{s}^a] = -\frac{1}{\kappa}\, \hat{s}^a \hat{s}^4, &&[\hat{s}^4, \hat{s}^a] = \frac{\eta^2}{\kappa}\, \hat{s}^0 \hat{s}^a, &&[\hat{s}^0, \hat{s}^4] = -\frac{\eta^2}{\kappa}\, \hat{\mathcal{S}}_{\eta/\kappa}, \\
&[\hat{s}^1, \hat{s}^2] = -\frac{\eta}{\kappa}(\hat{s}^3)^2, &&[\hat{s}^1, \hat{s}^3] = \frac{\eta}{\kappa}\, \hat{s}^3 \hat{s}^2, &&[\hat{s}^2, \hat{s}^3] = -\frac{\eta}{\kappa}\, \hat{s}^1 \hat{s}^3,
\end{aligned}
\tag{4.64}
$$

and defines the κ-(A)dS spacetime for all orders in η. Here $\hat{\mathcal{S}}_{\eta/\kappa}$ is given by

$$
\hat{\mathcal{S}}_{\eta/\kappa} = (\hat{s}^1)^2 + (\hat{s}^2)^2 + (\hat{s}^3)^2 + \frac{\eta}{\kappa}\, \hat{s}^1 \hat{s}^2 \,, \tag{4.65}
$$

and this operator is the analogue of the quantum sphere (4.61) in quantum ambient coordinates, since (4.65) is just the Casimir operator for the subalgebra spanned by $\hat{\mathbf{s}}$, namely $[\hat{\mathcal{S}}_{\eta/\kappa}, \hat{s}^a] = 0$. However, $\hat{\mathcal{S}}_{\eta/\kappa}$ does not commute with the remaining quantum ambient coordinates

$$
\begin{aligned}
&[\hat{\mathcal{S}}_{\eta/\kappa}, \hat{s}^0] = \frac{1}{\kappa}\big(\hat{s}^4\, \hat{\mathcal{S}}_{\eta/\kappa} + \hat{\mathcal{S}}_{\eta/\kappa}\, \hat{s}^4\big) - \frac{\eta^2}{\kappa^2}\, \hat{s}^0\, \hat{\mathcal{S}}_{\eta/\kappa}, \\
&[\hat{\mathcal{S}}_{\eta/\kappa}, \hat{s}^4] = -\frac{\eta^2}{\kappa}\big(\hat{s}^0\, \hat{\mathcal{S}}_{\eta/\kappa} + \hat{\mathcal{S}}_{\eta/\kappa}\, \hat{s}^0\big) + \frac{\eta^2}{\kappa^2}\, \hat{\mathcal{S}}_{\eta/\kappa}\, \hat{s}^4.
\end{aligned}
\tag{4.66}
$$

In fact, the Casimir operator for the full κ-(A)dS quantum space (4.64) is found to be

$$
\hat{\Sigma}_{\eta,\kappa} = (\hat{s}^4)^2 + \eta^2(\hat{s}^0)^2 - \frac{\eta^2}{\kappa}\, \hat{s}^0 \hat{s}^4 - \eta^2 \hat{\mathcal{S}}_{\eta/\kappa}, \tag{4.67}
$$

which is just the quantum analogue of the pseudosphere (2.96) that defines the (A)dS space.

Indeed, the quantization of the algebra (4.63) that we have obtained should coincide with the corresponding subalgebra of the full κ-(A)dS quantum group relations obtained by applying the usual FRT approach [33] onto the quantum matrix group arising from (2.91). Note that the ambient coordinates are entries of this matrix and the quantum R-matrix for the κ-(A)dS quantum algebra should be derived from the one associated to the Drinfel'd-Jimbo deformation [19, 31] of the corresponding complex simple Lie algebra.

We would like to stress that from a physical perspective the relevant parameter appearing in the κ-(A)dS 3-space (4.60) is just η/κ, which is actually very small. This fact could preclude the need of considering higher order terms in the algebras (4.54)-(4.55) for all physically relevant purposes. Therefore, the noncommutative algebra (4.62) should suffice in order to provide the essential information concerning the novelties introduced by the κ-(A)dS spacetime with respect to the κ-Minkowski one. In particular, the changes introduced by the cosmological constant in the representation theory of the latter [95, 96] are worth studying as a first step, and we recall that the irreducible representations for a complex C^*-version of the algebra (4.60) were presented in [215] (see also [216]).

Some words concerning the (2+1)-dimensional counterpart of the results here presented are in order, since it is well-known that the κ-(A)dS deformation leads to a vanishing commutation rule $[\hat{x}^1, \hat{x}^2] = 0$ for the space coordinates (see [111]). This can easily be explained by taking into account that in (2+1) dimensions the κ-(A)dS r-matrix reads

$$r_\Lambda = \frac{1}{\kappa}\left(K_1 \wedge P_1 + K_2 \wedge P_2\right), \tag{4.68}$$

and the term $J_1 \wedge J_2$ which generates the space-space noncommutativity (4.60) in (3+1) dimensions cannot exist.

Let us finish this first part of the Chapter by recalling that the result here presented solves an important problem, both from the theoretical and phenomenological points of view. In particular, the noncommutative κ-(A)dS spacetime, constructed by quantizing its semiclassical counterpart, is shown to have a quadratic subalgebra of local spatial coordinates whose first order brackets in terms of the cosmological constant parameter define a quantum sphere, while the commutators between time and space coordinates preserve the same structure of the κ-Minkowski spacetime. When expressed in ambient coordinates, the quantum κ-(A)dS spacetime is shown to be defined as a noncommutative pseudosphere. Moreover, we have proven that this is the only possible generalization to the case of non-vanishing cosmological constant of the well-known κ-Minkowski spacetime, under minimal physical assumptions.

4.4.2 Twisted κ-(A)dS Poisson homogeneous space

Similarly to the case of vanishing cosmological constant, we want to finish the discussion considering the twisted-κ-deformation of G_Λ defined by (4.26). Since it again provides a coisotropic Lie bialgebra for the Lorentz subalgebra $\mathfrak{l} \subset \mathfrak{g}_\Lambda$, as can be directly checked from the cocommutator

$$\begin{aligned}
&\delta_t(P_0) = 0,\\
&\delta_t(P_1) = z(P_0 \wedge P_1 - \eta P_3 \wedge J_1 + \eta^2 K_2 \wedge J_3 - \eta^2 K_3 \wedge J_2) - \vartheta(P_0 \wedge P_2 + \eta^2 K_1 \wedge J_3),\\
&\delta_t(P_2) = z(P_0 \wedge P_2 - \eta P_3 \wedge J_2 - \eta^2 K_1 \wedge J_3 + \eta^2 K_3 \wedge J_1) + \vartheta(P_0 \wedge P_1 - \eta^2 K_2 \wedge J_3),\\
&\delta_t(P_3) = z(P_0 \wedge P_3 + \eta P_1 \wedge J_1 + \eta P_\wedge J_2 + \eta^2 K_1 \wedge J_2 - \eta^2 K_2 \wedge J_1) - \vartheta \eta^2 K_3 \wedge J_3,\\
&\delta_t(K_1) = z(P_0 \wedge K_1 - P_2 \wedge J_3 + P_3 \wedge J_2 - \eta K_3 \wedge J_1) + \vartheta(-P_0 \wedge K_2 + P_1 \wedge J_3),\\
&\delta_t(K_2) = z(P_0 \wedge K_2 + P_1 \wedge J_3 - P_3 \wedge J_1 - \eta K_3 \wedge J_2) + \vartheta(P_0 \wedge K_1 + P_2 \wedge J_3),\\
&\delta_t(K_3) = z(P_0 \wedge K_3 - P_1 \wedge J_2 + P_2 \wedge J_1 + \eta K_1 \wedge J_1 + \eta K_2 \wedge J_2) + \vartheta P_3 \wedge J_3,\\
&\delta(J_3) = 0,\\
&\delta(J_1) = \eta J_1 \wedge J_3 - \vartheta P_0 \wedge J_2,\\
&\delta(J_2) = \eta J_2 \wedge J_3 + \vartheta P_0 \wedge J_1,
\end{aligned} \tag{4.69}$$

the same construction as before can be mimicked.

As it happened with the twisted κ-Minkowski spacetime (4.21), the computation of the Sklyanin bracket shows that the twist does not affect the Poisson brackets between space coordinates, which are again (4.53), and the twisted brackets involving x^0 and the space coordinates x^a are given by

$$
\begin{aligned}
\{x^0, x^1\}_t &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^1)}{\eta\cosh^2(\eta x^2)\cosh^2(\eta x^3)} - \vartheta\,\frac{\cosh(\eta x^1)\tanh(\eta x^2)}{\eta}, \\
\{x^0, x^2\}_t &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^2)}{\eta\cosh^2(\eta x^3)} + \vartheta\,\frac{\sinh(\eta x^1)}{\eta}, \\
\{x^0, x^3\}_t &= -\frac{1}{\kappa}\,\frac{\tanh(\eta x^3)}{\eta},
\end{aligned}
\tag{4.70}
$$

which again provide a nonlinear algebra deformation of the twisted κ-Minkowski whose zeroth-order in η leads to (4.21). We omit the complete Poisson-Lie structure (G_Λ, Π_t) compatible with this PHS due to its complexity.

4.5 Noncommutative space of worldlines

While the construction of maximally symmetric Lorentzian spacetimes with constant curvature as homogeneous spaces of its Lie group of isometries is well-known, the fact that their set of time-like (or space-like) geodesics has the structure of a smooth manifold is much less common in the literature. Recall that for a generic spacetime M, i.e. a smooth manifold endowed with a Lorentzian metric, the space of oriented geodesics is a quite complicated object. In fact, it is a topological space but not necessarily Hausdorff, and even when this is the case the topological manifold could not admit a smoothable atlas and henceforth could not be a smooth manifold. These problems have been previously considered in the literature (see, for instance, [218] and [219], where the space of null geodesics is described). For manifolds whose geodesics are closed many results are known (see [220]).

However, in the case of M being a simply connected pseudo-Riemannian space of constant curvature, or a rank one Riemannian symmetric space, its set of oriented time-like geodesics $L(M)$ is a smooth homogeneous manifold. From now on we will refer to time-like geodesics as worldlines, because they are precisely the worldlines for a free massive particle moving in the appropriate spacetime. In [221] all symplectic, complex and metric structures were described, and we recall that in [222] all homogeneous spaces of worldlines corresponding to kinematical groups were studied in detail, including the pseudo-Riemannian metrics defined on them, and in [144] the (2+1) Lorentzian spaces of worldlines were considered. In particular, for the Poincaré case it was found that an invariant foliation exists in the space of worldlines and that the resulting homogeneous space is of negative curvature (see [222] for details). This result provides a neat geometrical description of the hyperbolic nature of the space of velocities in special relativity, and all these classical geometric notions should admit some rigorous generalization to the quantum (noncommutative) setting.

In the rest of this Chapter, we investigate the Poisson homogeneous space associated to the κ-Poincaré group (G_0, Π) acting on the homogenous space of time-like geodesics of Minkowski spacetime, which we call $(\mathcal{W}, \pi)$. This homogeneous space $\mathcal{W}$ is the coset space of the Poincaré group divided by the stabilizer of the worldline that passes through the origin of Minkowski space with zero velocity. This stabilizer can be easily computed and turns out to be the full subgroup of rotations of the Poincaré group together with time translations, which we hereafter denote by H and parametrize by

$$H = \exp\phi^1\rho(J_1)\exp\phi^2\rho(J_2)\exp\phi^3\rho(J_3)\exp y^0\rho(P_0). \tag{4.71}$$

The Lie algebra $\mathfrak{h}$ of H is given by (2.76). While the parametrization in terms of coordinates for M_Λ has been worked out in detail in §2.2, the corresponding one for the space of worldlines is more subtle, due to the fact that the coordinates x^a and ξ^a defined by the inverse of (2.88) do not define a set of coordinates in the coset space $\mathcal{W} = G/H$, since they do not satisfy that the diagram (2.86) is commutative and so (x^a, ξ^a) are not H-invariant functions on G, which is equivalent to say that they are not functions on $\mathcal{W} = G/H$, i.e. $(x^a, \xi^a) \notin \mathcal{C}^\infty(G)^H = \mathcal{C}^\infty(\mathcal{W})$. Therefore, we introduce a different set of local coordinates

$$(\eta^a, y^\alpha, \phi^a) : U \subset G_0 \to \mathbb{R}^{10} \tag{4.72}$$

by the inverse map of the following exponentiation

$$\begin{aligned} G_{\mathcal{W}} = {} & \exp\eta^1\rho(K_1)\exp y^1\rho(P_1)\exp\eta^2\rho(K_2)\exp y^2\rho(P_2)\exp\eta^3\rho(K_3)\exp y^3\rho(P_3) \\ & \times \exp\phi^1\rho(J_1)\exp\phi^2\rho(J_2)\exp\phi^3\rho(J_3)\exp y^0\rho(P_0). \end{aligned} \tag{4.73}$$

In this way it is straightforward to check that we have well-defined coordinates

$$(y^\alpha, \eta^a) : U' \subset \mathcal{W} = G/H \to \mathbb{R}^6 \tag{4.74}$$

on $\mathcal{W} = G/H$, as they are invariant by right multiplication by an element of H.

Now the Poincaré group element has the form

$$G_{\mathcal{W}} = \begin{pmatrix} 1 & \bar{0} \\ \bar{f}^T & \mathbf{L} \end{pmatrix}, \tag{4.75}$$

where $\mathbf{L}$ is the same matrix as in (2.92), i.e. the Lorentz subgroup, and $\bar{f}$ are functions given by

$$\begin{aligned} f^0(y^\alpha, \eta^a) &= y^1\sinh\eta^1 + \cosh\eta^1\left(y^2\sinh\eta^2 + \cosh\eta^2(y^0\cosh\eta^3 + y^3\sinh\eta^3)\right), \\ f^1(y^\alpha, \eta^a) &= y^1\cosh\eta^1 + \sinh\eta^1\left(y^2\sinh\eta^2 + \cosh\eta^2(y^0\cosh\eta^3 + y^3\sinh\eta^3)\right), \\ f^2(y^\alpha, \eta^a) &= y^2\cosh\eta^2 + \sinh\eta^2(y^0\cosh\eta^3 + y^3\sinh\eta^3), \\ f^3(y^\alpha, \eta^a) &= y^0\sinh\eta^3 + y^3\cosh\eta^3. \end{aligned} \tag{4.76}$$

We stress that the previous construction allows us to obtain the explicit relationships among the local coordinates of the Poincaré group G_0 in both parametrizations, namely

$$\bar{x} = \bar{f}(y^\alpha, \eta^a), \qquad \xi^a = \eta^a, \qquad \theta^a = \phi^a. \tag{4.77}$$

Note that the position coordinates x^a on the Minkowski spacetime $M_0 = G_0/L$ cannot be naively identified with the 'position' coordinates y^a on the space of worldlines, since they only coincide when all rapidities vanish (i.e. for an observer at rest). Moreover, in the representation (4.75) the action of the Poincaré group on the space of worldlines is not linear.

The space of worldlines $\mathcal{W}$ has also a metric structure which is rather different from the flat Lorentzian metric on the Minkowskian spacetime M (2.100). In particular, the metric on $\mathcal{W}$ is degenerate, and an invariant foliation under the Poincaré group action arises in such a manner that a 'subsidiary' metric restricted to each leaf of the foliation has to be considered [222]. It can be shown that in terms of the coordinates y^a and η^a the degenerate 'main' metric $g^{(1)}$ on $\mathcal{W}$ has a line element

$$\mathrm{d}s^2_{(1)} = (\cosh\eta^2)^2(\cosh\eta^3)^2(\mathrm{d}\eta^1)^2 + (\cosh\eta^3)^2(\mathrm{d}\eta^2)^2 + (\mathrm{d}\eta^3)^2. \tag{4.78}$$

This is a Riemannian metric of negative constant curvature, whose value is just $-1/c^2$ (in this work we are using units in which $c = 1$), which only involves rapidities, and thus provides the relative rapidity between two free motions. This, in turn, shows that the three-velocity space is hyperbolic. The invariant foliation is determined by a uniform motion with $\boldsymbol{\eta} = \boldsymbol{\eta}_0 =$constant and the 'subsidiary' metric $g^{(2)}$ defined on each leaf reads

$$\mathrm{d}s^2_{(2)} = (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2 + (\mathrm{d}y^3)^2, \qquad \boldsymbol{\eta} = \boldsymbol{\eta}_0, \tag{4.79}$$

that is, each leaf is isometric to the three-dimensional Euclidean space. We also point out that in the three-velocity space the geodesic distance χ corresponding to the relative speed from an observer at rest and one with a uniform motion with rapidity $\boldsymbol{\eta}$ is given by

$$\cosh\chi = \cosh\eta^1\cosh\eta^2\cosh\eta^3. \tag{4.80}$$

Finally, notice that in the low rapidity regime (i.e. take $c \to \infty$), the expressions (4.78) and (4.80) reduce to the usual ones for velocities in classical mechanics

$$\mathrm{d}s^2_{(1)} = (\mathrm{d}\eta^1)^2 + (\mathrm{d}\eta^2)^2 + (\mathrm{d}\eta^3)^2, \qquad \chi^2 = (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2. \tag{4.81}$$

4.6 The κ-Poincaré homogeneous space of worldlines

In the sequel we show that the noncommutativity in the space of worldlines induced by the κ-deformation of Poincaré symmetries can be obtained by mimicking the previous construction of the Poisson homogeneous Minkowski spacetime, but taking into account the appropriate isotropy subgroup of worldlines. These results have been recently presented in [61].

As a first step, we have to check whether the κ-Poincaré Lie bialgebra structure (4.3) is coisotropic with respect to the Lie subalgebra of the isotropy subgroup of time-like worldlines $\mathfrak{h} = \operatorname{span}\{P_0, J_a\}$ (2.76). This is indeed the case, since we have that $\delta(P_0) = \delta(J_a) = 0$ and the coisotropy condition (3.79) is trivially satisfied. Furthermore, in this case the stronger Poisson-subgroup condition [60] $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$ is also trivially fulfilled. We stress that this does not occur for the κ-Minkowski isotropy subalgebra $\mathfrak{l}$, and implies that after quantization the isotropy subgroup $\mathfrak{h}$ is promoted to a Hopf subalgebra (again, this is not the case for the Lorentz sector of the κ-deformation).

This fact provides a first signature that the κ-deformation is more naturally realized on the space of worldlines than on Minkowski spacetime. As a consequence, the coisotropy condition guarantees that the homogeneous Poisson structure on the space of worldlines $\mathcal{W}$ can be obtained as a canonical projection from the coboundary Poisson structure on G induced by the r-matrix (4.2). This is directly connected with the precise ordering (4.73) chosen for the construction of $G_{\mathcal{W}}$ (4.75), which is the one that provides the appropriate description of the projected Poisson structure onto $\mathcal{W}$.

Indeed, left- and right-invariant vector fields (2.23) for $G_{\mathcal{W}}$ have to be computed (we omit their explicit expressions for the sake of brevity), and the Sklyanin bracket (4.11) for the κ-Poincaré r-matrix (4.2) has to be written in terms of such vector fields expressed in terms of the local Poincaré coordinates $\{y^a, \eta^a, \phi^a, y^0\}$. With all these ingredients at hand, the explicit form of the κ-Poincaré Poisson homogeneous space of worldlines is just given by the canonical projection of the Skyanin bracket to the coordinates y^a and η^a of the space $\mathcal{W}$, and reads

$$
\begin{aligned}
\{y^1, y^2\} &= \frac{1}{\kappa}\left(y^2 \sinh\eta^1 - \frac{y^1 \tanh\eta^2}{\cosh\eta^3}\right), \\
\{y^1, y^3\} &= \frac{1}{\kappa}\left(y^3 \sinh\eta^1 - y^1 \tanh\eta^3\right), \\
\{y^2, y^3\} &= \frac{1}{\kappa}\left(y^3 \cosh\eta^1 \sinh\eta^2 - y^2 \tanh\eta^3\right), \\
\{y^1, \eta^1\} &= \frac{1}{\kappa}\frac{\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right)}{\cosh\eta^2 \cosh\eta^3}, \\
\{y^2, \eta^2\} &= \frac{1}{\kappa}\frac{\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right)}{\cosh\eta^3}, \\
\{y^3, \eta^3\} &= \frac{1}{\kappa}\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right), \\
\{y^a, \eta^b\} &= 0, \quad a \neq b, \qquad \{\eta^a, \eta^b\} = 0\,.
\end{aligned}
\tag{4.82}
$$

These expressions for the Poisson version of the noncommutative space of worldlines show that this noncommutative space contains a commutative subalgebra of rapidities η^a, while the 'position' worldline coordinates y^a are noncommutative. The Poisson bracket between a given rapidity η^a and its corresponding 'position' y^a does not vanish and depends on the geodesic distance function (4.80).

The structure of these Poisson brackets for $\mathcal{W}$ becomes more symmetric and manifestly spatially isotropic if they are expanded as power series in the coordinates of $\mathcal{W}$ up to second-order, namely

$$
\begin{aligned}
\{y^a, y^b\} &= \frac{1}{\kappa}(\eta^a y^b - \eta^b y^a) + \mathcal{O}(\mathbf{y}, \boldsymbol{\eta})^3, \qquad \{\eta^a, \eta^b\} = 0, \\
\{y^a, \eta^b\} &= \delta_{ab}\,\frac{1}{2\kappa}\left((\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2\right) + \mathcal{O}(\mathbf{y}, \boldsymbol{\eta})^3.
\end{aligned}
\tag{4.83}
$$

We stress that the quadratic terms in (4.83) are the ones coming from the non-relativistic limit $c \to \infty$, since they are just the angular momenta $(\eta^a y^b - \eta^b y^a)$ and the relative speed χ^2 (4.81). Note also that the linearization of the brackets (4.83) in terms of the local coordinates y^a and η^b vanish, in contradistinction to the κ-Minkowski spacetime (4.13), which is a purely linear bracket having no higher-order terms in the coordinates.

If we now compute the κ-Poincaré Sklyanin bracket relations between y^0 (which does not belong to $\mathcal{W}$) and the coordinates of the space of worldlines we obtain

$$
\begin{aligned}
\{y^0, y^1\} &= -\frac{1}{\kappa}\left(y^1 - y^2\,\frac{\sinh\eta^1\tanh\eta^2}{\cosh\eta^3} - y^3\sinh\eta^1\tanh\eta^3\right),\\
\{y^0, y^2\} &= -\frac{1}{\kappa}\left(y^2 + y^1\,\frac{\sinh\eta^1\tanh\eta^2}{\cosh\eta^3} - y^3\cosh\eta^1\sinh\eta^2\tanh\eta^3\right),\\
\{y^0, y^3\} &= -\frac{1}{\kappa}\left(y^3 + y^1\sinh\eta^1\tanh\eta^3 + y^2\cosh\eta^1\sinh\eta^2\tanh\eta^3\right),\\
\{y^0, \eta^1\} &= -\frac{1}{\kappa}\,\frac{\sinh\eta^1}{\cosh\eta^2\cosh\eta^3},\\
\{y^0, \eta^2\} &= -\frac{1}{\kappa}\,\frac{\cosh\eta^1\sinh\eta^2}{\cosh\eta^3},\\
\{y^0, \eta^3\} &= -\frac{1}{\kappa}\cosh\eta^1\cosh\eta^2\sinh\eta^3,
\end{aligned}
\tag{4.84}
$$

which means that the smooth functions on $\mathcal{W}$ enlarged with y^0 (*i.e.* the coset G/R where R is the rotations subgroup) still define a Poisson subalgebra where $\mathcal{C}^\infty(\mathcal{W})$ is a non-abelian ideal. All these properties are fully consistent with the transformation (4.76) which provides the Minkowski spacetime coordinates x^α in terms of the ones for $\mathcal{W}$ and y^0. Indeed, if we use the transformation (4.76) to compute the Poisson structure given by (4.82) and (4.84) for the four Minkowski coordinates $x^\alpha \equiv f^\alpha$, we just obtain the defining relations of the Poisson version of the κ-Minkowski spacetime (4.13). This is a direct consequence of the fact that both the noncommutative κ-Minkowski spacetime and the noncommutative space of κ-Poincaré worldlines are just two realizations in two different geometric contexts of the very same noncommutative structure provided by the κ-deformation.

4.6.1 Quantum κ-Poincaré worldlines

At this point, the space of quantum worldlines would be defined by the quantization of the Poisson algebra of worldline coordinates (4.82), which can be obtained by substituting the Poisson brackets into commutators with exactly the same expressions (4.82), but now in terms of the noncommutative quantum worldline coordinates $\hat{y}^a$ and $\hat{\eta}^a$. Indeed, no ordering ambiguities appear when the Poisson bracket (4.82) is transformed into a commutator: the quantum rapidities $\hat{\eta}^a$ commute, and this implies that the crossed commutators $[\hat{y}^a, \hat{\eta}^b]$ have no ordering problems. Finally, at the r.h.s. of the commutators $[\hat{y}^a, \hat{y}^b]$, the $\hat{y}^a$ coordinates are always multiplied by a function depending on the $\hat{\eta}^b$ coordinates with $b \neq a$, which means that the latter commute with the former and no quantization ambiguities do exist.

Moreover, the fact that the Poisson brackets $\{y^a, \eta^b\}$ only involve the η^a coordinates provides a natural ansatz for a new set of classical variables whose quantization is straightforward. Let us consider the following diffeomorphism

$$
\begin{aligned}
q^1 &= \frac{\cosh\eta^2 \cosh\eta^3}{\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1}\, y^1, \qquad & p^1 &= \eta^1, \\
q^2 &= \frac{\cosh\eta^3}{\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1}\, y^2, \qquad & p^2 &= \eta^2, \\
q^3 &= \frac{1}{\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1}\, y^3, \qquad & p^3 &= \eta^3,
\end{aligned}
\tag{4.85}
$$

which is well-defined whenever $(\eta^1, \eta^2, \eta^3) \neq (0,0,0)$ since its Jacobian determinant

$$
|\mathrm{J}(\mathbf{y}, \boldsymbol{\eta})| = \frac{\cosh\eta^2 (\cosh\eta^3)^2}{(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1)^3}\,, \tag{4.86}
$$

is different from zero everywhere on such a domain (note again the presence of the geodesic distance χ (4.80) in all these expressions). In fact, in terms of the elements of $g^{(1)}$ (4.78) the expressions above take the simple form

$$
q^a = \frac{\sqrt{g^{(1)}_{aa}}}{\cosh\chi - 1}\, y^a, \qquad |\mathrm{J}(\mathbf{y}, \boldsymbol{\eta})| = \frac{\sqrt{\det g^{(1)}}}{(\cosh\chi - 1)^3}\,, \tag{4.87}
$$

where in this case sum over repeated indices should not be assumed. These expressions neatly shows the interconnection among the Poisson homogeneous structure underlying the quantum deformation and the (hyperbolic) geometry of the space of velocities of special relativity.

Surprisingly enough, in terms of these new coordinates on $\mathcal{W}$ the noncommutative (Poisson) algebra of the worldline coordinates turns out to be

$$
\{q^a, q^b\} = \{p^a, p^b\} = 0, \qquad \{q^a, p^b\} = \frac{1}{\kappa}\, \delta_{ab}. \tag{4.88}
$$

Obviously, the chosen notation (q^a, p^a) for these new coordinates is not arbitrary, since what we have found is that the homogeneous Poisson structure induced by the κ-Poincaré r-matrix on the space or worldlines is just a symplectic structure on $\mathcal{W}$ (without the origin). Note that the diffeomorphism (4.85) is defined everywhere but in a point is a direct consequence of the fact that a Poisson-Lie group is never symplectic [35], because it always vanishes at the identity and this property descends to the quotient through canonical projection. Hence, outside the origin eH of $\mathcal{W}$ (the projection of the identity element e of G_0 to the coset space) we have obtained a symplectic form onto $\mathcal{W}$ given by

$$
\omega = \kappa \sum_{a=1}^{3} \mathrm{d}q^a \wedge \mathrm{d}p^a\,, \tag{4.89}
$$

and in this way our new worldline coordinates $(\mathbf{q}, \mathbf{p})$ have a direct interpretation as canonical phase space coordinates. Notice that while the change $\frac{y^a}{\{y^a, \eta^a\}} \to q^a$ is clearly suggested by the precise form of the fundamental brackets (4.82), the fact that the new coordinates q^a Poisson commute, which is essential for them in order to be Darboux coordinates, is a quite surprising result. Also, it is worth stressing that the symplectic structure (4.89) is the result of the κ-deformation of the Poincaré symmetry, and this explains why the deformation parameter κ explicitly appears within the symplectic form. Note that taking the limit $\kappa \to \infty$ implies that the Poisson-Lie structure on the Poincaré group becomes the trivial one and the space of worldlines (4.82) and (4.88) (as well as the spacetime (4.13)) becomes commutative.

Obviously, the quantization of the symplectic algebra (4.88) is straightforward in terms of canonical worldline position operators $\hat{\mathbf{q}}$ and their conjugate momenta $\hat{\mathbf{p}}$, and taking into account that κ^{-1} plays exactly the same role as the Planck constant. Therefore, all quantum gravity effects amenable to be described through the κ-deformation should be fully understood as a standard deformation-quantization on the space of worldlines with deformation parameter κ^{-1}, just in the same way as ordinary quantum mechanics arises as a deformation-quantization (with parameter $\hbar$) on the classical mechanical phase space.

4.6.2 Remarks

In this second part of the Chapter we have proposed that quantum deformations of spaces of worldlines arising as quantizations of Poisson homogeneous spaces of kinematical groups should also be considered as noncommutative spaces amenable to describe quantum gravity effects. We have shown that this is a completely general construction that can be applied to any quantum deformation provided that the coisotropy condition (3.79) of its associated Lie bialgebra with respect to the isotropy subalgebra of worldlines is fulfilled. In this way, a non-trivial Poisson homogenous structure on the space of worldlines can be introduced, and from the latter the noncommutativity between worldline coordinates arises in a natural way.

Furthermore, when this construction is applied to the κ-deformation of Poincaré symmetries, the quantum space of worldlines so obtained turns out to be isomorphic to three copies of the Heisenberg-Weyl algebra, where the constant κ^{-1} plays the role of $\hbar$. This is a straightforward consequence of the symplectic structure of the space of worldlines induced by the κ-Poincaré Poisson-Lie structure. This result suggests that the κ-deformation is a natural one for the space of worldlines, an idea enforced by the fact that the isotropy subgroup of worldlines behaves as a Poisson-Lie subgroup under this deformation. Moreover, the fact that the κ-Poincaré r-matrix (4.2) is connected in a natural way to a symplectic structure on the space of worldlines seems quite natural if we realize that the relativistic Newton-Wigner position operators [223] are defined in terms of the Poincaré Lie algebra generators as

$$Q_a = \frac{1}{2P_0} K_a + K_a \frac{1}{2P_0}, \tag{4.90}$$

since in this way the canonical brackets $[Q_a, P_b] = \delta_{ab}$ are obtained in terms of the generators of the Poincaré Lie algebra (4.1). Therefore, the bivector

$$B = Q_1 \wedge P_1 + Q_2 \wedge P_2 + Q_3 \wedge P_3, \tag{4.91}$$

should define a symplectic form under the appropriate realization, and if we substitute (4.90) into this expression we get

$$B = \left(\frac{1}{2P_0}\, K_a + K_a\, \frac{1}{2P_0} \right) \wedge P_a, \tag{4.92}$$

which implies that both the symplectic bivector B and the κ-Poincaré r-matrix (4.2) are closely related (recall that P_0 is one of the generators of the the stabilizer of the origin of the space of worldlines).

Consequently, the noncommutative spaces of worldlines seem to provide a privileged arena in order to explore the physical role of the κ-deformation. In particular, once the canonical coordinates have been found, noncommutativity in the space of worldlines could be rephrased in more physical terms as the impossibility of determining simultaneously and with infinite precision the six $(\mathbf{q}, \mathbf{p})$ coordinates of a given worldline. In this respect, note that (4.85) implies that, before introducing the quantum deformation, the $\mathbf{p}$ coordinates are just the usual rapidities $\boldsymbol{\eta}$ and the 'positions' $\mathbf{q}$ for a worldline are defined as the product of Poincaré coordinates $\mathbf{y}$ associated to translations with certain functions depending on $\boldsymbol{\eta}$. Indeed, the precise physical meaning of the coordinates (4.85) has to be studied in detail.

The construction here presented is fully general and can thus be applied to any other quantum deformation (provided it is coisotropic with respect to the isotropy subgroup of worldlines) of any kinematical group. This opens the path to several future investigations, and the first of them consists in the construction of the noncommutative space of worldlines associated with the κ-deformation of the (A)dS groups studied in the first part of this Chapter. As has been proved, this κ-(A)dS deformation is significantly more complicated than its $\Lambda \to 0$ limit, at least when considering the noncommutative spacetime. It would be interesting to check if this is also the case when considering noncommutative spaces of worldlines, or on the other hand, it is possible to find 'symplectic' coordinates and thus the results would be somehow similar.

Finally, the model here introduced provides a purely kinematical framework for a schematic theory of quantum ('noncommutative') free observers, and strongly suggests that if the quantum deformation is assumed to encode quantum gravity effects, the latter are simply reflected as a canonical Heisenberg-Weyl noncommutativity on the (phase) space of worldlines, which is the simplest possible algebraic framework to deal with.

Chapter 5

Curved momentum spaces from quantum groups

In this Chapter, the concept that quantum symmetries describe theories with nontrivial momentum space properties is generalized to the case of a non-vanishing cosmological constant Λ. In particular, the momentum space associated to the κ-deformation of the de Sitter algebra is explicitly constructed as a dual Poisson-Lie group manifold parametrized by Λ. Such momentum space includes both the momenta associated to spacetime translations and the 'hyperbolic' momenta associated to boost transformations, and has the geometry of (half of) a higher-dimensional de Sitter manifold. Known results for the momentum space of the κ-Poincaré algebra are smoothly recovered in the limit $\Lambda \to 0$, where hyperbolic momenta decouple from translational momenta. It should be stressed that the approach here presented is general and can be applied to other quantum deformations of kinematical symmetries.

In §5.1 we illustrate the construction we will perform during this Chapter in the simpler case of a vanishing cosmological constant, thus constructing the curved momentum space associated to the κ-Poincaré deformation. Sections §5.2 and §5.3 consider the low dimensional (1+1) and (2+1) cases, which are simpler, while in §5.4 the physical (3+1)-dimensional case is analyzed. More in detail, the Poisson version of the κ-dS Hopf algebra in (1+1) and in (2+1) dimensions is defined in section 5.2, where it is shown that the main differences with respect to the corresponding κ-Poincaré structures fully arise in the (2+1) setting: whilst in the vanishing cosmological constant limit the translation generators $\{P_0, P_1, P_2\}$ close a Hopf subalgebra, this is no longer the case for the κ-dS algebra, since the cosmological constant mixes the translation and Lorentz sectors within both the coproduct map and the deformed Casimir function. Thus, for non-vanishing Λ it seems natural to consider an enlarged momentum space including also the dual coordinates to the Lorentz generators. This idea allows us to construct the curved (generalized) momentum manifold in the non-vanishing cosmological constant setting as the full dual Poisson-Lie group manifold, whose explicit construction can be achieved through the Poisson version of the 'quantum duality principle', which has been already used in §3.8 (see [19, 208, 59, 205] and references therein).

The κ-dS dual Poisson-Lie groups are explicitly constructed in section 5.3. In (1+1) dimensions the dual group coordinates are those associated to both the spacetime translations and boosts, and a certain linear action of the dual group on the origin of momentum space generates (half of) a (2+1)-dimensional dS manifold M_{dS_3}, spanned by the orbit of the group passing through the origin. In this case, the fact that boosts have the same role in the momentum space as translation generators can be understood since their coproducts have the same formal structure. In (2+1) dimensions one spatial rotation comes into play and the structure of the κ-dS Hopf algebra is apparently much more involved. Nevertheless, the construction of the full dual Poisson-Lie group G^*_Λ gives the clue for the full geometrical description of the associated momentum space. The dual Lie algebra and its associated Poisson-Lie group are explicitly constructed in section 5.3.2, and the corresponding linear action on the enlarged momentum space can be defined in such a way that the dual rotation generates the isotropy subgroup of the origin of the momentum space. As a consequence, we find that a (4+1)-dimensional space of momenta associated to translations and boosts arises as a dual group orbit passing through the origin, and such a space again has the geometry of (half of) a dS manifold M_{dS_5}. Moreover, in the vanishing cosmological constant limit, the Lorentz sector completely decouples both in the dispersion relation and in the coproduct, thus recovering the well-known κ-Poincaré momentum space.

A similar construction for the (3+1)-dimensional case is performed in §5.4, where the main difference come from the extra term on the κ-(A)dS r-matrix, which involves the rotation sector,and as it was already emphasized in the previous Chapter. However, the qualitative description of the momentum space is similar. The Chapter ends with some remarks regarding the construction of curved momentum space and its interest from the phenomenological point of view §5.5.

5.1 The κ-Poincaré momentum space

The starting point for the construction is the Poisson version $\mathcal{P}(\mathfrak{g}_0^{3+1})$ (hereafter we omit the dimensional superindex) of the (3+1) κ-Poincaré algebra $\mathfrak{g}_0$ in the so-called bicrossproduct basis [64]. This is the Poisson-Hopf algebra defined by the fundamental Poisson brackets (4.6) together with the coproduct $\Delta_\kappa : \mathcal{P}(\mathfrak{g}_0) \to \mathcal{P}(\mathfrak{g}_0) \otimes \mathcal{P}(\mathfrak{g}_0)$ given by (4.5).

This quantum deformation of the Poincaré algebra posses two deformed Casimir functions, $\mathcal{C}_\kappa$ given by (4.7) and $\mathcal{W}_\kappa$ defined by (4.8). The first one is specially relevant because it constitutes the keystone for the interpretation of κ-Poincaré algebra as the modified kinematical symmetry underlying a class of deformed dispersion relations that arise in several quantum gravity contexts [103, 121].

When dealing with Hopf algebra kinematical symmetries, the coproduct can be interpreted as the composition law for observables. In particular, the coproduct (4.5) is such that the κ-deformation induces a nonlinear composition rule for momenta in interaction vertices. As we are going to show, it is because of this deformed composition rule that curvature in the κ-Poincaré momentum space emerges. In more technical terms, the curvature of the momentum space arises as a consequence of the non cocommutativity of the coproduct map for the translation generators, and such a curved momentum space can be explicitly constructed as follows.

Firstly, the non cocommutativity of the κ-translations can be characterized by writing the cocommutator $\delta : \mathfrak{g}_0 \to \mathfrak{g}_0 \otimes \mathfrak{g}_0$, which is just the skew-symmetric part of the first order deformation in $1/\kappa$ of the coproduct. Namely we have

$$\begin{aligned} \delta(P_0) &= 0, \\ \delta(P_1) &= \frac{1}{\kappa}\left(P_1 \wedge P_0\right), \\ \delta(P_2) &= \frac{1}{\kappa}\left(P_2 \wedge P_0\right), \\ \delta(P_3) &= \frac{1}{\kappa}\left(P_3 \wedge P_0\right). \end{aligned} \tag{5.1}$$

This cocommutator map endows the Poincaré algebra $\mathfrak{g}_0$ with a Lie bialgebra structure. Moreover, the cocommutator $\delta : \mathfrak{g}_0 \to \mathfrak{g}_0 \otimes \mathfrak{g}_0$ completely characterizes the Hopf algebra deformation through the first order information it encodes (see [35, 34, 205] and references therein for details).

Secondly, the dual ${}^t\delta : \mathfrak{g}_0^* \otimes \mathfrak{g}_0^* \to \mathfrak{g}_0^*$ of the cocommutator map defines the Lie algebra $\mathfrak{g}_0^*$ of the dual Poisson-Lie group G_0^*. In the κ-Poincaré case, if we denote by $\{X^0, X^1, X^2, X^3\}$ the generators in $\mathfrak{g}_0^*$ such that

$$\langle X^\alpha, P_\beta \rangle = \delta^\alpha_\beta, \tag{5.2}$$

then their dual Lie brackets are given by

$$\left[X^0, X^a\right] = -\frac{1}{\kappa} X^a, \qquad \left[X^a, X^b\right] = 0. \tag{5.3}$$

Note that in the limit $\kappa \to \infty$ all the coproducts are primitive, $\Delta(X) = X \otimes 1 + 1 \otimes X$. As a consequence, δ vanishes and the dual Lie algebra (and group) is abelian. It is also worth recalling that the dual Lie algebra of the translations sector given by (5.33) is just the so-called κ-Minkowski spacetime [63, 64, 65, 139]. Obviously, the restriction to (1+1) and (2+1) dimensions of (5.33) leads to the lower dimensional κ-Minkowski spacetimes.

5.1.1 Dual Poisson-Lie group and curved momentum space

The dual Poisson-Lie group G_Λ^* can be explicitly constructed starting from the 5-dimensional faithful representation ρ of the dual Lie algebra $\mathfrak{g}_0^*$, given by:

$$\rho(Q) = p_0\,\rho(X^0) + p_1\,\rho(X^1) + p_2\,\rho(X^2) + p_3\,\rho(X^3) = \frac{1}{\kappa}\begin{pmatrix} 0 & p_1 & p_2 & p_3 & p_0 \\ p_1 & 0 & 0 & 0 & p_1 \\ p_2 & 0 & 0 & 0 & p_2 \\ p & 0 & 0 & 0 & p_3 \\ p_0 & -p_1 & -p_2 & -p_3 & 0 \end{pmatrix} \tag{5.4}$$

In terms of these local dual group coordinates $\{p_0, p_1, p_2, p_3\}$, then the dual Lie group element can be constructed through the exponentiation

$$G_0^*(p_0, p_1, p_2, p_3) = \exp\left(p_1 \rho(X^1)\right) \exp\left(p_2 \rho(X^2)\right) \exp\left(p_3 \rho(X^3)\right) \exp\left(p_0 \rho(X^0)\right), \tag{5.5}$$

which explicitly reads

$$G_0^*(p) = \begin{pmatrix} \cosh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\, \mathbf{p}^2 & \frac{p_1}{\kappa} & \frac{p_2}{\kappa} & \frac{p_3}{\kappa} & \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\, \mathbf{p}^2 \\ \frac{p_1}{\kappa} e^{p_0/\kappa} & 1 & 0 & 0 & \frac{p_1}{\kappa} e^{p_0/\kappa} \\ \frac{p_2}{\kappa} e^{p_0/\kappa} & 0 & 1 & 0 & \frac{p_2}{\kappa} e^{p_0/\kappa} \\ \frac{p_3}{\kappa} e^{p_0/\kappa} & 0 & 0 & 1 & \frac{p_3}{\kappa} e^{p_0/\kappa} \\ \sinh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{z\, p_0}\, \mathbf{p}^2 & -\frac{p_1}{\kappa} & -\frac{p_2}{\kappa} & -\frac{p_3}{\kappa} & \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\, \mathbf{p}^2 \end{pmatrix}. \tag{5.6}$$

The significance of the dual Poisson-Lie group relies on the fact that the coproduct (4.5) is just the group law for G_0^* (see [205] for details). In fact, if we multiply two matrices of the type (5.6) we get another group element

$$G_0^*(p'') = G_0^*(p) \cdot G_0^*(p')\,. \tag{5.7}$$

It can be straightforwardly checked that the group law $p'' = f(p, p')$ reads:

$$p_0'' = p_0 + p_0', \qquad p_i'' = p_i + e^{-zp_0'}\, p_i, \tag{5.8}$$

which is consistent with (5.33) in the sense that X^0 generates a dilation and the X^i generators correspond to (dual) translations.

Now, by making use of the Poisson version of the quantum duality principle (see [19, 208, 59] and the Section 3.8 for the (1+1)-dimensional example), the group multiplication law (5.8) can be immediately rewritten in algebraic terms as the comultiplication map Δ_κ through the identification of the two copies of the dual group coordinates as

$$p \equiv p \otimes 1, \qquad p' \equiv 1 \otimes p. \tag{5.9}$$

In this algebraic language, the multiplication law for the group G_0^* can be written as a co-product in the form:

$$\Delta_z(p_0) \equiv p_0'' = p_0 \otimes 1 + 1 \otimes p_0, \qquad \Delta_z(p_a) \equiv p_a'' = p_a \otimes 1 + e^{-zp_0} \otimes p_a. \tag{5.10}$$

This coproduct is just the one for the translation sector of the κ-Poincaré algebra once the following identification between the dual group coordinates and the generators of the κ-Poincaré algebra is performed:

$$p_0 \equiv P_0, \quad p_1 \equiv P_1, \quad p_2 \equiv P_2, \quad p_3 \equiv P_3. \tag{5.11}$$

Moreover, the unique Poisson-Lie structure on G_0^* that is compatible with the coproduct (5.10) and has the undeformed Poincaré Lie algebra as its linearization is given by the κ-Poincaré Poisson brackets for the translation sector (see Section 3.8).

Under this approach, the κ-Poincaré momentum space admits a straightforward geometric interpretation [121]. The entries of the fourth column in G_0^* can be rewritten as the following S_i functions:

$$\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2}\, e^{p_0/\kappa}\, z^2\, \mathbf{p}^2, \\
S_1 &= \frac{p_1}{\kappa} e^{p_0/\kappa}, \\
S_2 &= \frac{p_2}{\kappa} e^{p_0/\kappa}, \\
S_3 &= \frac{p_3}{\kappa} e^{p_0/\kappa}, \\
S_4 &= \cosh(p_0/\kappa) - \frac{1}{2}\, e^{p_0/\kappa}\, z^2\, \mathbf{p}^2 .
\end{aligned} \tag{5.12}$$

Surprisingly enough, these satisfy the defining relation of the (3+1)-dimensional dS space:

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 = 1. \tag{5.13}$$

This means that the κ-Poincaré momentum space parametrized by the ambient coordinates $(S_0, S_1, S_2, S_3, S_4)$ can be obtained as the orbit arising from a linear action of the Lie group matrix $G_0^*(p)$ onto a five-dimensional ambient Minkowski space and passing through the point $(0, 0, 0, 0, 1)$. Namely:

$$G^* \cdot (0, 0, 0, 0, 1)^T = (S_0, S_1, S_2, S_3, S_4)^T . \tag{5.14}$$

Moreover, the ambient coordinates fulfil the condition:

$$S_0 + S_4 = e^{p_0/\kappa} > 0, \tag{5.15}$$

which means that only half of the (3+1)-dimensional dS space is generated through the action (5.14). We will denote this manifold as M_{dS_4}. Note that in the limit $\kappa \to \infty$ the dual Lie group G_0^* generated by (5.33) is abelian.

5.2 The κ-dS Poisson-Hopf algebra

Let us start by reviewing the structural properties of the κ-deformation of the (1+1) and (2+1) dS algebra, which will be presented by considering the cosmological constant $\Lambda > 0$ as an explicit parameter whose $\Lambda \to 0$ limit provides automatically the expressions for the κ-Poincaré algebra. In this way, the specific features of the construction leading to the κ-Poincaré momentum space will become transparent, and the proposed path to its non-vanishing cosmological constant generalization will arise in a natural way.

In the subsection on the (1+1)-dimensional case we just briefly present the essential formulas, postponing a more in-depth discussion of the relevant features of the κ-dS algebra to the following subsection focussing on the (2+1)-dimensional case.

5.2.1 The (1+1) κ-dS algebra

The Poisson version of $\mathfrak{g}_\Lambda^{1+1}$ (2.83) is the (undeformed) Poisson-Hopf $\mathcal{P}(\mathfrak{g}_\Lambda^{1+1})$ algebra in (1+1) dimensions defined by the brackets

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = P_0, \qquad \{P_0, P_1\} = -\Lambda\, K, \tag{5.16}$$

where K is the generator of boost transformations, P_0 and P_1 are the time and space translation generators and the (undeformed) coproduct is given by $\Delta_0(X) = X\otimes 1+1\otimes X$, with $X \in \{K, P_0, P_1\}$. The Poisson version $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda^{1+1}))$ of the (1+1) κ-dS quantum algebra [142] is a Hopf algebra deformation of (5.16), given by

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = \frac{\sinh(P_0/\kappa)}{1/\kappa}, \qquad \{P_0, P_1\} = -\Lambda\, K, \tag{5.17}$$

with deformed coproduct map

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_1) &= P_1 \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes P_1, \\ \Delta(K) &= K \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes K. \end{aligned} \tag{5.18}$$

The deformed Casimir function for (5.17) is

$$C_\kappa = \left(\frac{\sinh(P_0/2\kappa)}{1/2\kappa}\right)^2 - P_1^2 + \Lambda\, K^2. \tag{5.19}$$

The so-called bicrossproduct-type basis [64] for this algebra is given through the nonlinear change

$$P_0 \to P_0, \qquad P_1 \to e^{P_0/2\kappa}\, P_1, \qquad K \to e^{P_0/2\kappa}\, K, \tag{5.20}$$

so that the algebra becomes

$$\{K, P_0\} = P_1, \qquad \{K, P_1\} = \frac{1-\exp(-2P_0/\kappa)}{2/\kappa} - \frac{1}{2\kappa}\,(P_1^2 - \Lambda\, K^2), \qquad \{P_0, P_1\} = -\Lambda\, K, \tag{5.21}$$

with associated coproduct map

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_1) &= P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1, \\ \Delta(K) &= K \otimes 1 + e^{-P_0/\kappa} \otimes K. \end{aligned} \tag{5.22}$$

In this basis, the deformed Casimir reads

$$C_z = \left(\frac{\sinh(P_0/2\kappa)}{1/2\kappa}\right)^2 - e^{P_0/\kappa}(P_1^2 - \Lambda K^2). \tag{5.23}$$

We point out that for $\Lambda = 0$ (the κ-Poincaré case), the momentum sector given by P_0 and P_1 generates an abelian Hopf subalgebra, and the $\Lambda = 0$ Casimir function provides the well-known (1+1) κ-Poincaré deformed dispersion relation (see e.g. [87]). Note also that the coproduct (5.22) does not depend on Λ, although this property will not hold in higher dimensions.

5.2.2 The (2+1) κ-dS algebra

In (2+1) dimensions, the Poisson version $\mathcal{P}(\mathfrak{g}_\Lambda^{2+1})$ of $\mathfrak{g}_\Lambda^{1+1}$ (2.79) dS algebra takes the form

$$\begin{array}{lll} \{J, P_a\} = \epsilon_{ab} P_b, & \{J, K_a\} = \epsilon_{ab} K_b, & \{J, P_0\} = 0, \\ \{P_a, K_b\} = -\delta_{ab} P_0, & \{P_0, K_a\} = -P_a, & \{K_1, K_2\} = -J, \\ \{P_0, P_a\} = -\Lambda\, K_a, & \{P_1, P_2\} = \Lambda\, J. & \end{array} \tag{5.24}$$

The two quadratic Casimir functions for (5.57) are

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda(J^2 - \mathbf{K}^2), \qquad \mathcal{W} = -J P_0 + K_1 P_2 - K_2 P_1, \tag{5.25}$$

where $\mathbf{P}^2 = P_1^2 + P_2^2$ and $\mathbf{K}^2 = K_1^2 + K_2^2$. Recall that $\mathcal{C}$ comes from to the Killing–Cartan form and is related to the energy of a point particle, while $\mathcal{W}$ is the Pauli–Lubanski vector. The undeformed Poisson-Hopf algebra structure on $\mathcal{P}(\mathfrak{g}_\Lambda^{2+1})$ is given by $\Delta_0 : \mathcal{P}(\mathfrak{g}_\Lambda^{2+1}) \to \mathcal{P}(\mathfrak{g}_\Lambda^{2+1}) \otimes \mathcal{P}(\mathfrak{g}_\Lambda^{2+1}), X \to X \otimes 1 + 1 \otimes X$.

The (2+1) κ-dS Poisson-Hopf algebra $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda^{2+1}))$ in the bicrossproduct basis is the Hopf algebra deformation with parameter κ given by [143, 144, 68]

$$\begin{array}{lll} \{J, P_0\} = 0, & \{J, P_1\} = P_2, & \{J, P_2\} = -P_1, \\ \{J, K_1\} = K_2, & \{J, K_2\} = -K_1, & \{K_1, K_2\} = -\frac{\sin(2\sqrt{\Lambda} J/\kappa)}{2\sqrt{\Lambda}/\kappa}, \\ \{P_0, P_1\} = -\Lambda\, K_1, & \{P_0, P_2\} = -\Lambda\, K_2, & \{P_1, P_2\} = \Lambda\, \frac{\sin(2\sqrt{\Lambda} J/\kappa)}{2\sqrt{\Lambda}/\kappa}, \end{array}$$

$$\begin{aligned} &\{K_1, P_0\} = P_1, \qquad\qquad \{K_2, P_0\} = P_2, \\ &\{P_2, K_1\} = \frac{1}{\kappa}\left(P_1 P_2 - \Lambda K_1 K_2\right) \qquad \{P_1, K_2\} = \frac{1}{\kappa}\left(P_1 P_2 - \Lambda K_1 K_2\right), \\ &\{K_1, P_1\} = \frac{\kappa}{2}\left(\cos(2\sqrt{\Lambda} J/\kappa) - e^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}\left(P_2^2 - P_1^2\right) - \frac{\Lambda}{2\kappa}\left(K_2^2 - K_1^2\right), \\ &\{K_2, P_2\} = \frac{\kappa}{2}\left(\cos(2\sqrt{\Lambda} J/\kappa) - e^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}\left(P_1^2 - P_2^2\right) - \frac{\Lambda}{2\kappa}\left(K_1^2 - K_2^2\right), \end{aligned} \tag{5.26}$$

and with deformed coproduct map

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J) = J \otimes 1 + 1 \otimes J, \\ \Delta(P_1) &= P_1 \otimes \cos(\sqrt{\Lambda} J/\kappa) + e^{-P_0/\kappa} \otimes P_1 + \Lambda\, K_2 \otimes \frac{\sin(\sqrt{\Lambda} J/\kappa)}{\sqrt{\Lambda}}, \\ \Delta(P_2) &= P_2 \otimes \cos(\sqrt{\Lambda} J/\kappa) + e^{-P_0/\kappa} \otimes P_2 - \Lambda\, K_1 \otimes \frac{\sin(\sqrt{\Lambda} J/\kappa)}{\sqrt{\Lambda}}, \\ \Delta(K_1) &= K_1 \otimes \cos(\sqrt{\Lambda} J/\kappa) + e^{-P_0/\kappa} \otimes K_1 + P_2 \otimes \frac{\sin(\sqrt{\Lambda} J/\kappa)}{\sqrt{\Lambda}}, \\ \Delta(K_2) &= K_2 \otimes \cos(\sqrt{\Lambda} J/\kappa) + e^{-P_0/\kappa} \otimes K_2 - P_1 \otimes \frac{\sin(\sqrt{\Lambda} J/\kappa)}{\sqrt{\Lambda}}, \end{aligned} \tag{5.27}$$

which explicitly depends on the cosmological constant Λ. The deformed Casimir function for this Poisson-Hopf algebra reads

$$\begin{aligned}\mathcal{C}_\kappa =&2\kappa^2 \left[\cosh(P_0/\kappa)\cos(\sqrt{\Lambda}J/\kappa) - 1\right] - \\ &- e^{P_0/\kappa}\left(\mathbf{P}^2 - \Lambda\,\mathbf{K}^2\right)\cos(\sqrt{\Lambda}\,J/\kappa) - 2\,\Lambda\,e^{P_0/\kappa}\,\frac{\sin(\sqrt{\Lambda}J/\kappa)}{\sqrt{\Lambda}}\,R_3,\end{aligned} \tag{5.28}$$

with $R_3 = \epsilon_{3bc}K_bP_c$. Note that the projection to the κ-dS algebra in (1+1) dimensions is obtained by setting to zero the generators $\{P_2, K_2, J\}$.

The (2+1) κ-Poincaré Poisson-Hopf algebra $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_0^{2+1}))$ is smoothly recovered in the $\Lambda \to 0$ limit and in this 'flat' case the momentum sector $\{P_0, P_1, P_2\}$ generates an abelian Hopf subalgebra with coproduct

$$\begin{aligned}\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_1) &= P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1, \\ \Delta(P_2) &= P_2 \otimes 1 + e^{-P_0/\kappa} \otimes P_2.\end{aligned} \tag{5.29}$$

Such a nonlinear superposition law for momenta is the essential footprint of a curved momentum space, which can be explicitly constructed by following the procedure presented in [121].

Essentially, the κ-Poincaré momentum space is a three-dimensional manifold generated by the action on a certain ambient space of the three-dimensional dual Lie group G_Λ^* whose Lie algebra $\mathfrak{g}_\Lambda^*$,

$$\left[X^0, X^1\right] = -z\,X^1, \qquad \left[X^0, X^2\right] = -z\,X^2, \qquad \left[X^1, X^2\right] = 0, \tag{5.30}$$

is defined as the dual of the skew-symmetric part of the first order deformation in $1/\kappa$ of the coproducts (5.29) (see (5.32)). The Lie algebra (5.30) is the so-called (2+1) κ-Minkowski noncommutative spacetime [63, 64]. Moreover, when $\Lambda = 0$ the deformed Casimir function

$$\mathcal{C}_\kappa = 2\kappa^2\left[\cosh(P_0/\kappa) - 1\right] - e^{P_0/\kappa}\,(P_1^2 + P_2^2), \tag{5.31}$$

provides the κ-Poincaré deformed dispersion relation in (2+1) dimensions. The same construction can be straightforwardly generalized to the (3+1) κ-Poincaré algebra (see [121] and references therein).

The main obstruction to a similar construction when $\Lambda \neq 0$ is readily seen by inspection of (5.27). In fact, in the κ-dS case the momentum sector $\{P_0, P_1, P_2\}$ is no longer a Hopf subalgebra, since the coproduct of spatial momenta includes all the generators $\{J, K_1, K_2\}$ of the Lorentz sector (note that this is not the case in (1+1) dimensions, where the coproduct does not depend on Λ). Moreover, the deformed Casimir $\mathcal{C}_\kappa$ contains the Lorentz generators as well, and this feature is also present in the (1+1) case (see (5.23)). These two observations hold true also in the (3+1) κ-dS Poisson-Hopf algebra that has been explicitly presented in (4.45).

We already mentioned that the Hopf-algebraic deformations of spacetime symmetries can be endowed with a phenomenological interpretation. Specifically, the Casimir $\mathcal{C}_\kappa$ of the algebra determines the dispersion relation of free particles, while the coproduct of the translation generators determines the rules of conservation of energy and spatial momentum in interactions [87]. Therefore, when $\Lambda \neq 0$ we can say that both the conservation rules in interactions and the deformed dispersion relation involve an enlarged set of 'momenta', including also the angular momentum and the 'hyperbolic' momenta corresponding, respectively, to the rotation and to boost transformations (hyperbolic rotations). In this framework, it seems natural to propose that when $\Lambda \neq 0$ the (curved) momentum space is defined by an enlarged space parametrized by the six coordinates that are dual to the generators of the full quantum algebra. Nevertheless, a simple inspection at the coproducts (5.27) shows that the role of the J generator is somewhat different from that of K_1 and K_2, since the latter have coproducts which are formally equivalent to those of P_1 and P_2. All these aspects will have a clear interpretation once the explicit construction of the κ-dS momentum space is performed in the following section.

5.3 Momentum space for the κ-dS Poisson-Hopf algebra

As anticipated above, in this section the momentum space for the κ-dS Poisson algebra with non-vanishing cosmological constant will be constructed as the full dual Poisson-Lie group G^*_Λ, whose Lie algebra $\mathfrak{g}^*_\Lambda$ is provided by the dual of the cocommutator map δ generated by the coproduct of all the κ-dS generators in the bicrossproduct basis, including the Lorentz sector. This construction will be firstly illustrated in (1+1) dimensions. While this case is simpler, it does not allow us to appreciate the richness of structure characterising higher-dimensional models. The consistency and geometric features of our approach will be made fully explicit in the second subsection, where we demonstrate the full construction for the (2+1)-dimensional case.

5.3.1 The (1+1) case

The cocommutator map for the full κ-dS algebra can be read from the skew-symmetric part of the first order deformation in $1/\kappa$ of the coproduct (5.22), namely

$$\delta(P_0) = 0, \qquad \delta(P_1) = \frac{1}{\kappa} P_1 \wedge P_0, \qquad \delta(K) = \frac{1}{\kappa} K \wedge P_0. \tag{5.32}$$

If we denote by $\{X^0, X^1, L\}$ the generators dual to, respectively, $\{P_0, P_1, K\}$, the dual Lie algebra $\mathfrak{g}^*_\Lambda$ is given by the Lie brackets

$$[X^0, X^1] = -\frac{1}{\kappa} X^1, \qquad [X^0, L] = -\frac{1}{\kappa} L, \qquad [X^1, L] = 0. \tag{5.33}$$

A generic element Q of this Lie algebra for $\Lambda \neq 0$ can be written as the 4×4 matrix

$$Q = p_0 X^0 + p_1 X^1 + \chi L = \frac{1}{\kappa} \begin{pmatrix} 0 & p_1 & \sqrt{\Lambda}\chi & p_0 \\ p_1 & 0 & 0 & p_1 \\ \sqrt{\Lambda}\chi & 0 & 0 & \sqrt{\Lambda}\chi \\ p_0 & -p_1 & -\sqrt{\Lambda}\chi & 0 \end{pmatrix} \tag{5.34}$$

If we denote as $\{p_0, p_1, \chi\}$ the local group coordinates which are dual, respectively, to $\{X^0, X^1, L\}$, then the group element of the dual Lie group G^*_Λ is given by:

$$G^*_\Lambda = \exp\left(p_1 X^1\right) \exp\left(\chi L\right) \exp\left(p_0 X^0\right). \tag{5.35}$$

A straightforward computation leads to the following explicit matrix

$$G^*_\Lambda = \begin{pmatrix} \cosh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2 + \Lambda\chi^2) & \frac{p_1}{\kappa} & \frac{\sqrt{\Lambda}\chi}{\kappa} & \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2 + \Lambda\chi^2) \\ \frac{e^{p_0/\kappa} p_1}{\kappa} & 1 & 0 & \frac{e^{p_0/\kappa} p_1}{\kappa} \\ \frac{e^{p_0/\kappa}\sqrt{\Lambda}\chi}{\kappa} & 0 & 1 & \frac{e^{p_0/\kappa}\sqrt{\Lambda}\chi}{\kappa} \\ \sinh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2 + \Lambda\chi^2) & -\frac{p_1}{\kappa} & -\frac{\sqrt{\Lambda}\chi}{\kappa} & \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}(p_1^2 + \Lambda\chi^2) \end{pmatrix}, \tag{5.36}$$

from which the coproduct (see [205]) can be directly obtained by multiplying two group matrices, and reads

$$\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \qquad \Delta(p_1) = p_1 \otimes 1 + e^{-p_0/\kappa} \otimes p_1, \qquad \Delta(\chi) = \chi \otimes 1 + e^{-p_0/\kappa} \otimes \chi. \tag{5.37}$$

As the quantum duality principle indicates, this coproduct is just the one (5.22) for the κ-dS algebra once one identifies the dual group coordinates and the generators of the κ-dS Poisson-Hopf algebra as follows:

$$p_0 \equiv P_0, \quad p_1 \equiv P_1, \quad \chi \equiv K. \tag{5.38}$$

Moreover, by following the technique presented in [205] it can be shown that the unique Poisson-Lie structure on G^*_Λ that is compatible with the coproduct (5.37) and has the undeformed dS Lie algebra (5.16) as its linearization is given by the Poisson brackets

$$\{\chi, p_0\} = p_1, \qquad \{\chi, p_1\} = \frac{1 - \exp(-2p_0/\kappa)}{2/\kappa} - \frac{1}{2\kappa}\,(p_1^2 - \Lambda\,\chi^2), \qquad \{p_0, p_1\} = -\Lambda\,\chi, \tag{5.39}$$

which is exactly the κ-dS algebra (5.21) under the identification (5.38). The Casimir function for this Poisson bracket is

$$C_z = \left(\frac{\sinh\left(p_0/2\kappa\right)}{1/2\kappa}\right)^2 - e^{p_0/\kappa}(p_1^2 - \Lambda\chi^2). \tag{5.40}$$

In this way, the composition law for the momenta with κ-dS symmetry (5.22) has been reobtained as the group law (5.37) for the coordinates of the dual Poisson-Lie group G^*_Λ, and the κ-dS Casimir function (5.23) can be interpreted as an on-shell relation (5.40) for these coordinates.

We stress that the main novelty with respect to the κ-Poincaré case described in §3.8 (see also §4.1 for the (3+1)-dimensional case) is the fact that the dual Lie group G^*_Λ is now three-dimensional, and the momentum space associated to κ-dS is parametrized by the three coordinates $\{p_0, p_1, \chi\}$, and not only by the momenta associated to spacetime translations. Moreover, both in the coproduct (5.37) and the Casimir function (5.40) the role of the parameters χ and p_1 turns out to be identical, which supports the role of the former as an additional 'hyperbolic' momentum for quantum symmetries with non-vanishing cosmological constant.

An explicit geometric interpretation of this enlarged momentum space can be obtained along the same lines of [121] by observing that the entries of the fourth column in G^*_Λ, given by

$$\begin{aligned} S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\,(p_1^2 + \Lambda\chi^2), \\ S_1 &= \frac{e^{p_0/\kappa}\, p_1}{\kappa}, \\ S_2 &= \frac{e^{p_0/\kappa}\,\sqrt{\Lambda}\,\chi}{\kappa}, \\ S_3 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\,(p_1^2 + \Lambda\chi^2), \end{aligned} \tag{5.41}$$

satisfy the defining relation for the (2+1)-dimensional dS space,

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 = 1. \tag{5.42}$$

Moreover, if we consider a linear action of the Lie group G^*_Λ onto a four-dimensional ambient Minkowski space with coordinates (S_0, S_1, S_2, S_3), we have that

$$G^*_\Lambda \cdot (0,0,0,1)^T = (S_0, S_1, S_2, S_3)^T, \tag{5.43}$$

which means that the (2+1)-dimensional dS space is generated through G^*_Λ as the orbit that passes through the point $(0,0,0,1)$ in the ambient space, corresponding to the origin of the (generalized) momentum space. Note that the orbit passing through the point $(0,0,0,\alpha)$, with $\alpha \neq 0$, would satisfy $-S_0^2 + S_1^2 + S_2^2 + S_3^2 = \alpha^2$. Moreover, we have that the condition

$$S_0 + S_3 = e^{p_0/\kappa} > 0, \tag{5.44}$$

is automatically obeyed, so that only half of the (2+1)-dimensional dS space is generated as an orbit of the free action of G^*_Λ, and we will denote this manifold as M_{dS_3}. Finally, when $\Lambda = 0$ the ambient coordinate S_2 vanishes, as well as the realization $\rho(L)$ of the dual of the boost generator, thus recovering the well-known interpretation of the κ-Poincaré momentum space as (half of) a (1+1)-dimensional dS space, *i.e.*, M_{dS_2}.

5.3.2 The (2+1) case

The very same procedure described in the previous section can be applied to the construction of the momentum space associated to the (2+1) κ-dS Poisson-Hopf algebra $\mathcal{P}(\mathcal{U}_\kappa(\mathfrak{g}_\Lambda^{2+1}))$. The skew symmmetrized first order in $1/\kappa$ of the coproduct (5.27) is given by the cocommutator map

$$\begin{aligned} &\delta(P_0) = \delta(J) = 0, \\ &\delta(P_1) = \frac{1}{\kappa}(P_1 \wedge P_0 + \Lambda\, K_2 \wedge J), \\ &\delta(P_2) = \frac{1}{\kappa}(P_2 \wedge P_0 - \Lambda\, K_1 \wedge J), \\ &\delta(K_1) = \frac{1}{\kappa}(K_1 \wedge P_0 + P_2 \wedge J), \\ &\delta(K_2) = \frac{1}{\kappa}(K_2 \wedge P_0 - P_1 \wedge J). \end{aligned} \tag{5.45}$$

Denoting by $\{X^0, X^1, X^2, L^1, L^2, R\}$ the generators dual to, respectively, $\{P_0, P_1, P_2, K_1, K_2, J\}$, the Lie brackets defining the Lie algebra $\mathfrak{g}^*$ of the dual Poisson-Lie group G^*_Λ are

$$
\begin{array}{lll}
[X^0, X^1] = -\frac{1}{\kappa}\, X^1, & [X^0, X^2] = -\frac{1}{\kappa}\, X^2, & [X^1, X^2] = 0, \\
{[X^0, L^1]} = -\frac{1}{\kappa}\, L^1, & [X^0, L^2] = -\frac{1}{\kappa}\, L^2, & [L^1, L^2] = 0, \\
{[R, X^2]} = -\frac{1}{\kappa}\, L^1, & [R, L^1] = \frac{1}{\kappa}\, \Lambda\, X^2, & [L^1, X^2] = 0, \\
{[R, X^1]} = \frac{1}{\kappa}\, L^2, & [R, L^2] = -\frac{1}{\kappa}\, \Lambda\, X^1, & [L^2, X^1] = 0, \\
{[R, X^0]} = 0, & [L^1, X^1] = 0, & [L^2, X^2] = 0.
\end{array}
\tag{5.46}
$$

A general Lie algebra element Q for $\Lambda \neq 0$ can be represented as the 6×6 matrix

$$
\begin{aligned}
\rho(Q) =& p_0\rho(X^0) + p_1\rho(X^1) + p_2\rho(X^2) + \chi_1\rho(L^1) + \chi_2\rho(L^2) + \theta\rho(R) = \\
=& \frac{1}{\kappa}\begin{pmatrix}
0 & p_1 & p_2 & \sqrt{\Lambda}\chi_1 & \sqrt{\Lambda}\chi_2 & p_0 \\
p_1 & 0 & 0 & 0 & -\sqrt{\Lambda}\theta & p_1 \\
p_2 & 0 & 0 & \sqrt{\Lambda}\theta & 0 & p_2 \\
\sqrt{\Lambda}\chi_1 & 0 & -\sqrt{\Lambda}\theta & 0 & 0 & \sqrt{\Lambda}\chi_1 \\
\sqrt{\Lambda}\chi_2 & \sqrt{\Lambda}\theta & 0 & 0 & 0 & \sqrt{\Lambda}\chi_2 \\
p_0 & -p_1 & -p_2 & -\sqrt{\Lambda}\chi_1 & -\sqrt{\Lambda}\chi_2 & 0
\end{pmatrix}
\end{aligned}
\tag{5.47}
$$

If we denote as $\{p_0, p_1, p_2, \chi_1, \chi_2, \theta\}$ the local group coordinates which are dual, respectively, to $\{X^0, X^1, X^2, L^1, L^2, R\}$, then a Lie group element sufficiently closed to the identity G^*_Λ can be written as

$$
\begin{aligned}
G^*_\Lambda = &\exp\left(\theta\rho(R)\right)\exp\left(p_1\rho(X^1)\right)\exp\left(p_2\rho(X^2)\right)\times \\
&\times\exp\left(\chi_1\rho(L^1)\right)\exp\left(\chi_2\rho(L^2)\right)\exp\left(p_0\rho(X^0)\right),
\end{aligned}
\tag{5.48}
$$

and its explicit expression can be straightforwardly computed, although we omit it here for the sake of brevity. By multiplying two of these generic group elements, the group law for G^*_Λ can be directly derived and written as the following coproduct map for the six dual group coordinates:

$$
\begin{aligned}
\Delta(p_0) &= p_0 \otimes 1 + 1 \otimes p_0, \qquad \Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta, \\
\Delta(p_1) &= p_1 \otimes \cos(\sqrt{\Lambda}\,\theta/\kappa) + e^{-p_0/\kappa} \otimes p_1 + \Lambda\,\chi_2 \otimes \frac{\sin(\sqrt{\Lambda}\,\theta/\kappa)}{\sqrt{\Lambda}}, \\
\Delta(p_2) &= p_2 \otimes \cos(\sqrt{\Lambda}\,\theta/\kappa) + e^{-p_0/\kappa} \otimes p_2 - \Lambda\,\chi_1 \otimes \frac{\sin(\sqrt{\Lambda}\,\theta/\kappa)}{\sqrt{\Lambda}}, \\
\Delta(\chi_1) &= \chi_1 \otimes \cos(\sqrt{\Lambda}\,\theta/\kappa) + e^{-p_0/\kappa} \otimes \chi_1 + p_2 \otimes \frac{\sin(\sqrt{\Lambda}\,\theta/\kappa)}{\sqrt{\Lambda}}, \\
\Delta(\chi_2) &= \chi_2 \otimes \cos(\sqrt{\Lambda}\,\theta/\kappa) + e^{-p_0/\kappa} \otimes \chi_2 - p_1 \otimes \frac{\sin(\sqrt{\Lambda}\,\theta/\kappa)}{\sqrt{\Lambda}}.
\end{aligned}
\tag{5.49}
$$

Again, under the identification

$$
p_0 \equiv P_0, \quad p_1 \equiv P_1, \quad p_2 \equiv P_2, \quad \chi_1 \equiv K_1, \quad \chi_2 \equiv K_2, \quad \theta \equiv J,
\tag{5.50}
$$

this is exactly the coproduct for the κ-dS Poisson-Hopf algebra given in (5.27), and the unique Poisson-Lie structure on G^*_Λ that is compatible with (5.50) and has the Poisson version $\mathcal{P}(\mathfrak{g}^{2+1}_\Lambda)$ of the dS Lie algebra $\mathfrak{g}^{2+1}_\Lambda$ (5.24) as its linearization is the deformed Poisson algebra given by (5.27).

In order to provide a geometric interpretation of the six-dimensional generalized momentum space manifold, we proceed similarly to the (1+1) case and consider the action of G^*_Λ onto an ambient space. The entries of the sixth column in the matrix realization (5.48) are

$$
\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\, \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right), \\
S_1 &= \frac{e^{p_0/\kappa}}{\kappa}\left(\cos(\sqrt{\Lambda}\,\theta/\kappa)\, p_1 - \sqrt{\Lambda}\,\sin(\sqrt{\Lambda}\,\theta/\kappa)\,\chi_2\right), \\
S_2 &= \frac{e^{p_0/\kappa}}{\kappa}\left(\cos(\sqrt{\Lambda}\,\theta/\kappa)\, p_2 + \sqrt{\Lambda}\,\sin(\sqrt{\Lambda}\,\theta/\kappa)\,\chi_1\right), \\
S_3 &= \frac{e^{p_0/\kappa}}{\kappa}\left(-\sin(\sqrt{\Lambda}\,\theta/\kappa)\, p_2 + \sqrt{\Lambda}\,\cos(\sqrt{\Lambda}\,\theta/\kappa)\,\chi_1\right), \\
S_4 &= \frac{e^{p_0/\kappa}}{\kappa}\left(\sin(\sqrt{\Lambda}\,\theta/\kappa)\, p_1 + \sqrt{\Lambda}\,\cos(\sqrt{\Lambda}\,\theta/\kappa)\,\chi_2\right), \\
S_5 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}\, e^{p_0/\kappa}\, \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right),
\end{aligned}
\tag{5.51}
$$

and satisfy the condition

$$
-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1, \tag{5.52}
$$

which is the defining relation for the (4+1)-dimensional dS space. Therefore, by assuming that the space of generalized momenta is the group manifold for the dual group G^*_Λ, we can conclude that a linear action of the Lie group G^*_Λ onto a six-dimensional ambient Minkowski space with coordinates $(S_0, S_1, S_2, S_3, S_4, S_5)$ allows us to obtain a (4+1) dS space as the orbit that passes through the point in the ambient space with coordinates $(0,0,0,0,0,1)$, which is the origin of the (generalized) momentum space. Moreover, we have that $S_0 + S_5 = e^{p_0/\kappa} > 0$, so only half of the dS space is generated in this way, and we will denote this manifold as M_{dS_5}. Therefore, the (1+1) construction can be generalized to this (2+1) setting, although some distinctive features of the latter are worth to be stressed.

Firstly, given that in the (2+1) case one has six symmetry generators, one would naively expect that the generalized momentum space be a six dimensional manifold, given that in the (1+1) case the dimensionality of the manifold corresponds to the number of symmetry generators. Instead, we demonstrated the emergence of a five-dimensional orbit under the action of G^*_Λ. The reason for this is the completely different role that the dual rotation (R, θ) plays with respect to the dual boosts (L^i, χ_i), both in the coproduct and in the action (5.51). In particular, it is immediate to check that the isotropy subgroup of the point $(0,0,0,0,0,1)$ is just the one given by $G^*_0 = \exp\left(\theta\rho(R)\right)$. Therefore, the full momentum space for the κ-dS algebra in (2+1) dimensions is the six-dimensional manifold $M_{dS_5} \times S^1$, where the rotation coordinate θ is the one parametrizing S^1 while (p_i, χ_i) parametrize M_{dS_5}.

Secondly, under the identification (5.50) the deformed Casimir is written as the following function on the generalized momentum space:

$$\begin{aligned} \mathcal{C}_\kappa =& 2\kappa^2 \left[\cosh(p_0/\kappa)\cos(\sqrt{\Lambda}\theta/\kappa) - 1\right] - \\ & - e^{p_0/\kappa}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)\cos(\sqrt{\Lambda}\theta/\kappa) - 2\,\Lambda\, e^{p_0/\kappa}\frac{\sin(\sqrt{\Lambda}\theta/\kappa)}{\sqrt{\Lambda}} R_3, \end{aligned} \tag{5.53}$$

which involves all the translation and Lorentz momenta. Nevertheless, if we specialize this function onto the five-dimensional orbit M_{dS_5} by taking the S^1 coordinate $\theta = 0$, we get

$$\mathcal{C}_\kappa = 2\kappa^2 \left[\cosh(p_0/\kappa) - 1\right] - e^{p_0/\kappa}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right), \tag{5.54}$$

which is an on-shell relation that is just a higher dimensional generalization of the one obtained in the (1+1) κ-dS case, (5.40). In this way, the striking equivalence between the role played by the momenta associated to space translations and boosts is manifestly shown.

Finally, the (2+1) κ-Poincaré construction is again straightforwardly recovered in the limit $\Lambda \to 0$, where the action (5.51) provides $S_3 = S_4 = 0$ and the representation (5.47) is only defined for $\{X^0, X^1, X^2\}$, thus giving rise to (half of) a (2+1) dS space as an orbit under the action of the corresponding three-dimensional dual group. Summarizing, in (2+1) dimensions the momentum space for κ-dS is found to be the six-dimensional manifold $M_{dS_5} \times S^1$, while its κ-Poincaré limit was known to be the three-dimensional one M_{dS_3}.

5.4 Curved momentum spaces in (3+1) dimensions

In the previous section we have presented the construction of the curved momentum space corresponding to the (1+1) and (2+1)-dimensional κ-(A)dS quantum algebra, which was presented in [145]. However, the curved momentum space corresponding to the (3+1) κ-(A)dS quantum algebra [62, 74, 75, 76] deserves further attention, since completely new features arise when the cosmological constant $\Lambda \neq 0$, while for the vanishing cosmological constant the construction is completely analogous.

5.4.1 The (3+1) κ-(A)dS momentum space

By mimicking the procedure used in the previous Sections, the first step for the construction of the curved momentum spaces of the κ-(A)dS algebras is to obtain the cocommutator map δ associated to the κ-deformed coproduct map with non-vanishing cosmological constant. This can be found by extracting the first order deformation in $1/\kappa$ of the coproduct

(4.42)-(4.44), which has a skew-symmetric part given by [114]

$$
\begin{aligned}
&\delta(P_0) = 0, \qquad \delta(J_3) = 0, \\
&\delta(J_1) = \frac{1}{\kappa}\sqrt{-\Lambda} J_1 \wedge J_3, \qquad \delta(J_2) = \frac{1}{\kappa}\sqrt{-\Lambda} J_2 \wedge J_3, \\
&\delta(P_1) = \frac{1}{\kappa}\left(P_1 \wedge P_0 + \Lambda J_2 \wedge K_3 - \Lambda J_3 \wedge K_2 + \sqrt{-\Lambda} J_1 \wedge P_3\right), \\
&\delta(P_2) = \frac{1}{\kappa}\left(P_2 \wedge P_0 + \Lambda J_3 \wedge K_1 - \Lambda J_1 \wedge K_3 + \sqrt{-\Lambda} J_2 \wedge P_3\right), \\
&\delta(P_3) = \frac{1}{\kappa}\left(P_3 \wedge P_0 + \Lambda J_1 \wedge K_2 - \Lambda J_2 \wedge K_1 - \sqrt{-\Lambda} J_1 \wedge P_1 - \sqrt{-\Lambda} J_2 \wedge P_2\right), \\
&\delta(K_1) = \frac{1}{\kappa}\left(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2 + \sqrt{-\Lambda} J_1 \wedge K_3\right), \\
&\delta(K_2) = \frac{1}{\kappa}\left(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3 + \sqrt{-\Lambda} J_2 \wedge K_3\right), \\
&\delta(K_3) = \frac{1}{\kappa}\left(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1 - \sqrt{-\Lambda} J_1 \wedge K_1 - \sqrt{-\Lambda} J_2 \wedge K_2\right).
\end{aligned}
\tag{5.55}
$$

This expression is just (4.36) whit $\eta = \sqrt{-\Lambda}$. The differences between the (A)dS and Poicaré deformations leave their traces in the cocommutator map. In particular, we stress that $\delta(P)$ and $\delta(K)$ are structurally similar when $\Lambda \neq 0$, and $\delta(P)$ does not close a sub-Lie bialgebra since it includes the full Lorentz sector. As a consequence, the main idea introduced in the previous Sections for the construction of curved momentum spaces for the κ-(A)dS algebras in (2+1) dimensions becomes fully applicable: when $\Lambda \neq 0$ the momentum space has to be enlarged by including the angular momenta associated to the rotation symmetries and the 'hyperbolic' momenta associated to boosts.

This means that the momentum space arises as the orbit of an appropriate action of the dual Poisson-Lie group G^*_Λ, whose Lie algebra $\mathfrak{g}^*_\Lambda$ is obtained by dualizing δ. In terms of an algebraic basis for $\mathfrak{g}^*_\Lambda$ denoted by $\{X^0, X^1, X^2, X^3, L^1, L^2, L^3, R^1, R^2, R^3\}$ such that

$$
\langle X^\alpha, P_\beta\rangle = \delta^\alpha_\beta, \qquad \langle L^a, K_b\rangle = \delta^a_b, \qquad \langle R^a, J_a\rangle = \delta^a_b, \tag{5.56}
$$

the commutation relations for $\mathfrak{g}^*_\Lambda$ read

$$
\begin{array}{lll}
[R^1, R^2] = 0, & [R^1, R^3] = \frac{1}{\kappa}\sqrt{-\Lambda} R^1, & [R^2, R^3] = \frac{1}{\kappa}\sqrt{-\Lambda} R^2, \\
{}[R^1, X^1] = -\frac{1}{\kappa}\sqrt{-\Lambda} X^3, & [R^1, X^2] = \frac{1}{\kappa} L^3, & [R^1, X^3] = -\frac{1}{\kappa}(L^2 - \sqrt{-\Lambda} X^1), \\
{}[R^2, X^1] = -\frac{1}{\kappa} L^3, & [R^2, X^2] = -\frac{1}{\kappa}\sqrt{-\Lambda} X^3, & [R^2, X^3] = \frac{1}{\kappa}(L^1 + \sqrt{-\Lambda} X^2), \\
{}[R^3, X^1] = \frac{1}{\kappa} L^2, & [R^3, X^2] = -\frac{1}{\kappa} L^1, & [R^3, X^3] = 0, \\
{}[R^1, L^1] = -\frac{1}{\kappa}\sqrt{-\Lambda} L^3, & [R^1, L^2] = \frac{1}{\kappa}\Lambda X^3, & [R^1, L^3] = \frac{1}{\kappa}(\sqrt{-\Lambda} L^1 - \Lambda X^2), \\
{}[R^2, L^1] = -\frac{1}{\kappa}\Lambda X^3, & [R^2, L^2] = -\frac{1}{\kappa}\sqrt{-\Lambda} L^3, & [R^2, L^3] = \frac{1}{\kappa}(\sqrt{-\Lambda} L^2 + \Lambda X^1), \\
{}[R^3, L^1] = \frac{1}{\kappa}\Lambda X^2, & [R^3, L^2] = -\frac{1}{\kappa}\Lambda X^1, & [R^3, L^3] = 0, \\
{}[L^a, X^0] = \frac{1}{\kappa} L^a, & [L^a, K^b] = 0, & [L^a, X^b] = 0, \\
{}[X^a, X^0] = \frac{1}{\kappa} X^a, & [X^a, X^b] = 0, & [X^0, R^a] = 0\,.
\end{array}
\tag{5.57}
$$

From these expressions it is easy to see that in $\mathfrak{g}^*_\Lambda$ there exists a seven-dimensional solvable

Lie subalgebra generated by

$$[X^0, X^a] = -\frac{1}{\kappa}X^a, \quad [X^0, L^a] = -\frac{1}{\kappa}L^a, \quad [X^a, L^b] = 0, \quad [X^a, X^b] = 0, \quad [L^a, L^b] = 0. \tag{5.58}$$

This subalgebra is Λ–independent, and the dual of the rotation sector generates a three-dimensional solvable subgroup

$$\left[R^1, R^2\right] = 0, \quad \left[R^1, R^3\right] = \frac{1}{\kappa}\sqrt{-\Lambda}R^1, \quad \left[R^2, R^3\right] = \frac{1}{\kappa}\sqrt{-\Lambda}R^2\,. \tag{5.59}$$

In the limit $\Lambda \to 0$ this turns out to be abelian, which is the dual counterpart of the fact that $\lim_{\Lambda\to 0}\Delta_\kappa(J_a) = J_a \otimes 1 + 1 \otimes J_a$.

We stress that first order in $\eta = \sqrt{-\Lambda}$ noncommutative κ-(A)dS spacetime (4.62) would be given by the dual of the translations sector, namely

$$[X^0, X^a] = -\frac{1}{\kappa}\hat{X}^a, \qquad [X^a, X^b] = 0. \tag{5.60}$$

This is indeed Λ-independent but, as shown in the previous Chapter, when the all-orders quantum group is computed, the quantum space-time with non-vanishing Λ is a nonlinear algebra whose higher order contributions explicitly depend on the cosmological constant.

The κ-AdS curved momentum space

In the sequel we separately analyze the κ-AdS and κ-dS dual Poisson-Lie groups and construct the associated momentum spaces, since their geometric properties are different. A matrix representation ρ for the Lie algebra (5.57) when $\Lambda < 0$ can be found to be

$$\rho(Q) = \sum_\alpha p_\alpha \rho(X^\alpha) + \sum_a \chi_a\, L^a + \sum_a \theta_a R^a =$$
$$= \frac{1}{\kappa}\begin{pmatrix} 0 & p_1 & p_2 & p_3 & -\sqrt{-\Lambda}\chi_1 & \sqrt{-\Lambda}\chi_2 & \sqrt{-\Lambda}\chi_3 & p_0 \\ p_1 & 0 & 0 & \sqrt{-\Lambda}\theta_2 & 0 & -\sqrt{-\Lambda}\theta_3 & \sqrt{-\Lambda}\theta_1 & p_1 \\ p_2 & 0 & 0 & \sqrt{-\Lambda}\theta_1 & -\sqrt{-\Lambda}\theta_3 & 0 & -\sqrt{-\Lambda}\theta_2 & p_2 \\ p_3 & -\sqrt{-\Lambda}\theta_2 & -\sqrt{-\Lambda}\theta_1 & 0 & \sqrt{-\Lambda}\theta_1 & \sqrt{-\Lambda}\theta_2 & 0 & p_3 \\ \sqrt{-\Lambda}\chi_1 & 0 & -\sqrt{-\Lambda}\theta_3 & \sqrt{-\Lambda}\theta_1 & 0 & 0 & 0 & \sqrt{-\Lambda}\chi_1 \\ -\sqrt{-\Lambda}\chi_2 & -\sqrt{-\Lambda}\theta_3 & 0 & \sqrt{-\Lambda}\theta_2 & 0 & 0 & \sqrt{-\Lambda}\theta_1 & -\sqrt{-\Lambda}\chi_2 \\ -\sqrt{-\Lambda}\chi_3 & \sqrt{-\Lambda}\theta_1 & -\sqrt{-\Lambda}\theta_2 & 0 & \sqrt{-\Lambda}\theta_2 & -\sqrt{-\Lambda}\theta_1 & 0 & -\sqrt{-\Lambda}\chi_3 \\ p_0 & -p_1 & -p_2 & -p_3 & \sqrt{-\Lambda}\chi_1 & -\sqrt{-\Lambda}\chi_2 & -\sqrt{-\Lambda}\chi_3 & 0 \end{pmatrix} \tag{5.61}$$

If we denote as $\{p_\alpha, p_a, \chi_a, \theta_a,\}$ the local dual group coordinates that correspond, respectively, to $\{X^\alpha, L^a, R^a\}$, a representation of the Lie group G^*_Λ can be explicitly obtained as:

$$\begin{aligned} G^*_\Lambda(\theta, p, \chi) = e^{\theta_3 D(R^3)} e^{\theta_2 D(R^2)} e^{\theta_1 D(R^1)} e^{p_1 D(X^1)} e^{p_2 D(X^2)} e^{p_3 D(X^3)} \times \\ \times\, e^{\chi_1 D(L^1)} e^{\chi_2 D(L^2)} e^{\chi_3 D(L^3)} e^{p_0 D(X^0)}. \end{aligned} \tag{5.62}$$

Moreover, a long but straightforward computation shows that if we multiply two G^*_Λ elements:

$$G^*_\Lambda(\theta'', p'', \chi'') = G^*_\Lambda(\theta, p, \chi) \cdot G^*_\Lambda(\theta', p', \chi'), \tag{5.63}$$

the group law

$$\theta'' = f(\theta, \theta', p, p', \chi, \chi'), \qquad p'' = g(\theta, \theta', p, p', \chi, \chi'), \qquad \chi'' = h(\theta, \theta', p, p', \chi, \chi'), \tag{5.64}$$

can be explicitly obtained and it can be exactly written as the coproduct (4.42)-(4.44) for $\Lambda > 0$, provided the identification

$$\theta_a \equiv J_a, \qquad p_\alpha \equiv P_\alpha, \qquad \chi_a \equiv K_a, \tag{5.65}$$

is assumed and by following the convention (5.9).

Now, the κ-AdS momentum space can be constructed by considering the left action of the group element $G^*_\Lambda(\theta, p, \chi)$ on an 8-dimensional ambient space. The points that can be reached from the origin $\mathcal{O} \equiv (0,0,0,0,0,0,0,1)$ under such action are those with coordinates $(S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7)$ given by:

$$G^*_\Lambda \cdot (0,0,0,0,0,0,0,1)^T = (S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7)^T \,. \tag{5.66}$$

These can explicitly be written as:

$$\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2} e^{p_0/\kappa} \left(\mathbf{p}^2 + \Lambda \boldsymbol{\chi}^2\right), \\
S_1 &= A \Big(p_1\, B^+_{21} + \sqrt{-\Lambda}\left(C + \chi_2\, B^-_{21}\right)\Big), \\
S_2 &= A \Big(p_2\, B^+_{12} + \sqrt{-\Lambda}\left(D - \chi_1\, B^-_{12}\right)\Big), \\
S_3 &= \frac{e^{p_0/\kappa}}{\kappa} \Big(p_3 - z\sqrt{-\Lambda}\Big(\theta_1\, p_1 + \theta_2\, p_2 + \sqrt{-\Lambda}\left(\theta_1\, \chi_2 - \theta_2\, \chi_1\right)\Big)\Big), \\
S_4 &= A \Big(-p_2\, B^-_{21} + \sqrt{-\Lambda}\left(D + \chi_1\, B^+_{21}\right)\Big), \\
S_5 &= A \Big(-p_1\, B^-_{12} + \sqrt{-\Lambda}\left(C - \chi_2\, B^+_{12}\right)\Big), \\
S_6 &= -\frac{\sqrt{-\Lambda}\, e^{p_0/\kappa}}{\kappa} \Big(\chi_3 - z\Big(\theta_2\, p_1 - \theta_1\, p_2 + \sqrt{-\Lambda}\left(\theta_1\, \chi_1 + \theta_2\, \chi_2\right)\Big)\Big), \\
S_7 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2} e^{p_0/\kappa} \left(\mathbf{p}^2 + \Lambda \boldsymbol{\chi}^2\right),
\end{aligned} \tag{5.67}$$

where we have defined:

$$\begin{aligned}
A &= \frac{1}{2\kappa} e^{(p_0 - \theta_3\sqrt{-\Lambda})/\kappa}, \\
B^\pm_{ij} &= -\frac{\Lambda}{\kappa^2}\,(\theta_i^2 - \theta_j^2) + e^{2\sqrt{-\Lambda}\theta_3/\kappa} \pm 1, \quad i \in \{1,2\}, \\
C &= \frac{2}{\kappa}\left(\theta_2\sqrt{-\Lambda}\left(\frac{\theta_1}{\kappa}\left(-p_2 + \sqrt{-\Lambda}\,\chi_1\right) - \chi_3\right) + \theta_1\, p_3\right), \\
D &= \frac{2}{\kappa}\left(\theta_1\sqrt{-\Lambda}\left(\frac{\theta_2}{\kappa}\left(-p_1 - \sqrt{-\Lambda}\,\chi_2\right) + \chi_3\right) + \theta_2\, p_3\right).
\end{aligned} \tag{5.68}$$

Note that, when evaluated at $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$, the last four functions give $A \to \frac{1}{2\kappa} e^{p_0/\kappa}$, $B^+_{ij} \to 2$, $B^-_{ij} \to 0$, $C \to 0$ and $D \to 0$.

We would like to stress that the $\Lambda \to 0$ limit of these expressions makes S_4, S_5 and S_6 vanish, and for the remaining ambient coordinates we get exactly the κ-Poincaré curved momentum space (5.12). In other words, this means that for $\Lambda = 0$ the matrix (5.62) is a reducible representation of the dual κ-Poincaré group, which is consistent with the fact that the ambient space has been enlarged when the cosmological constant has been introduced.

From (5.67) we can deduce the geometrical properties of the κ-AdS momentum space. In fact, it is straightforward to check that the following relations hold:

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 - S_4^2 - S_5^2 - S_6^2 + S_7^2 = 1, \qquad S_0 + S_7 = e^{p_0/\kappa} > 0. \tag{5.69}$$

This means that, if we consider an $\mathbb{R}^{4,4}$ ambient space, the κ-AdS momentum space is (half of a) $SO(4,4)$ quadric. From the expressions (5.67) it is also straightforward to check that the subgroup of dual rotations,

$$G_0 = e^{\theta_3 D(R^3)} e^{\theta_2 D(R^2)} e^{\theta_1 D(R^1)}\,, \tag{5.70}$$

leaves the point $\mathcal{O}$ invariant. The action of the remaining 7-parameter subgroup (generated by the Lie subalgebra (5.58)) is obtained by evaluating (5.67) at $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$:

$$\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2} e^{p_0/\kappa} \left(\mathbf{p}^2 + \Lambda \boldsymbol{\chi}^2\right), \\
S_1 &= \frac{p_1}{\kappa}\, e^{p_0/\kappa}, \\
S_2 &= \frac{p_2}{\kappa}\, e^{p_0/\kappa}, \\
S_3 &= \frac{p_3}{\kappa}\, e^{p_0/\kappa}, \\
S_4 &= \frac{\sqrt{-\Lambda}\,\chi_1}{\kappa}\, e^{p_0/\kappa}, \\
S_5 &= -\frac{\sqrt{-\Lambda}\,\chi_2}{\kappa}\, e^{p_0/\kappa}, \\
S_6 &= -\frac{\sqrt{-\Lambda}\,\chi_3}{\kappa}\, e^{p_0/\kappa}, \\
S_7 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2} e^{p_0/\kappa} \left(\mathbf{p}^2 + \Lambda \boldsymbol{\chi}^2\right).
\end{aligned} \tag{5.71}$$

These expressions encode the essential information concerning the non-vanishing cosmological constant generalization of (5.12), since dual rotations leave the point $\mathcal{O}$ invariant. Therefore, we can think of the κ-AdS momentum space (5.69) as the 7-dimensional orbit in $\mathbb{R}^{4,4}$ that can be parametrized through (5.71) in terms of the dual translation and boost coordinates, while the dual rotation coordinates θ do not play any role in the description of the curved momentum space.

We recall that the deformed Poisson brackets for the κ-AdS algebra would be (4.45) for $\Lambda < 0$, and (5.65) allows them to be interpreted as a Poisson–Lie structure on the dual Lie group G^*_Λ for which the multiplication on G^*_Λ (*i.e.* the coproduct (4.42)-(4.44) for the κ-AdS algebra) is a Poisson map. If we now apply the identification (5.65) onto the deformed Casimir function (4.46) and afterwards we project it onto the curved momentum space parametrized by the p and χ coordinates by setting $\theta_i \to 0$, we obtain

$$\mathcal{C}_\kappa = 2\kappa^2 \left(\cosh(p_0/\kappa) - 1\right) - e^{p_0/\kappa} \left(\mathbf{p}^2 - \Lambda \boldsymbol{\chi}^2\right), \tag{5.72}$$

which could be considered as the deformed dispersion relation that corresponds to the (3+1) κ-AdS momentum space.

The κ-dS curved momentum space

As it could be expected, if we apply the construction presented in the previous Section to the case $\Lambda > 0$ we obtain the same kind of geometric construction for the κ-dS momentum space, that should generalize the (2+1) results presented in the previous Section. The only aspect we have to be careful about is the appearance of complex quantities when $\Lambda > 0$, due to the presence of $\sqrt{-\Lambda}$ in some of the expressions (for instance, see (5.55)). This is not a major obstacle to the construction of the momentum space, since, as we are going to show, all the complex contributions are linked to the dual of the rotation subgroup, which is again the isotropy subgroup of the origin of the momentum space. So they disappear when we consider the projection to the submanifold parametrized by momenta and boost coordinates.

The matrix representation of the algebra (5.57) when $\Lambda > 0$ is to be

$$Q = p_0 X^0 + p_1 X^1 + p_2 X^2 + p_3 X^3 + \chi_1 L^1 + \chi_2 L^2 + \chi_3 L^3 + \theta_1 T^1 + \theta_2 R^2 + \theta_3 R^3 =$$
$$= z \begin{pmatrix} 0 & p_1 & p_2 & p_3 & \sqrt{\Lambda}\chi_1 & \sqrt{\Lambda}\chi_2 & \sqrt{\Lambda}\chi_3 & p_0 \\ p_1 & 0 & 0 & i\sqrt{\Lambda}\theta_1 & 0 & -\sqrt{\Lambda}\theta_3 & \sqrt{\Lambda}\theta_2 & p_1 \\ p_2 & 0 & 0 & i\sqrt{\Lambda}\theta_2 & \sqrt{\Lambda}\theta_3 & 0 & -\sqrt{\Lambda}\theta_1 & p_2 \\ p_3 & -i\sqrt{\Lambda}\theta_1 & -i\sqrt{\Lambda}\theta_2 & 0 & -\sqrt{\Lambda}\theta_2 & \sqrt{\Lambda}\theta_1 & 0 & p_3 \\ \sqrt{\Lambda}\chi_1 & 0 & -\sqrt{\Lambda}\theta_3 & \sqrt{\Lambda}\theta_2 & 0 & 0 & i\sqrt{\Lambda}\theta_1 & \sqrt{\Lambda}\chi_1 \\ \sqrt{\Lambda}\chi_2 & \sqrt{\Lambda}\theta_3 & 0 & -\sqrt{\Lambda}\theta_1 & 0 & 0 & i\sqrt{\Lambda}\theta_2 & \sqrt{\Lambda}\chi_2 \\ \sqrt{\Lambda}\chi_3 & -\sqrt{\Lambda}\theta_2 & \sqrt{\Lambda}\theta_1 & 0 & -i\sqrt{\Lambda}\theta_1 & -i\sqrt{\Lambda}\theta_2 & 0 & \sqrt{\Lambda}\chi_3 \\ p_0 & -p_1 & -p_2 & -p_3 & \sqrt{\Lambda}\chi_1 & \sqrt{\Lambda}\chi_2 & \sqrt{\Lambda}\chi_3 & 0 \end{pmatrix} \tag{5.73}$$

Again, the left linear action of (5.62) onto the point $\mathcal{O} = (0,0,0,0,0,0,0,1)$ gives rise to an orbit whose points have ambient coordinates in $R^{1,7}$ given by:

$$\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}e^{p_0/\kappa}\left(\bar{p}^2 + \Lambda\bar{\chi}^2\right),\\
S_1 &= A\left(p_1\, B_{21}^+ + i\sqrt{\Lambda}\left(C + \chi_2\, B_{21}^-\right)\right),\\
S_2 &= A\left(p_2\, B_{12}^+ + i\sqrt{\Lambda}\left(D - \chi_1\, B_{12}^-\right)\right),\\
S_3 &= \frac{1}{\kappa}e^{p_0/\kappa}\left(p_3 - i\,\frac{\sqrt{\Lambda}}{\kappa}\left(\theta_1\, p_1 + \theta_2\, p_2 + i\sqrt{\Lambda}\left(\theta_1\,\chi_2 - \theta_2\,\chi_1\right)\right)\right),\\
S_4 &= A\left(i\, p_2\, B_{21}^- + \sqrt{\Lambda}\left(D + \chi_1\, B_{21}^+\right)\right),\\
S_5 &= A\left(-i\, p_1\, B_{12}^- - \sqrt{\Lambda}\left(C - \chi_2\, B_{12}^+\right)\right),\\
S_6 &= \sqrt{\Lambda}\, z\, e^{p_0/\kappa}\left(\chi_3 - \frac{1}{\kappa}\left(\theta_2\, p_1 - \theta_1\, p_2 + i\,\sqrt{\Lambda}\left(\theta_1\,\chi_1 + \theta_2\,\chi_2\right)\right)\right),\\
S_7 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}e^{p_0/\kappa}\left(\bar{p}^2 + \Lambda\bar{\chi}^2\right),
\end{aligned} \tag{5.74}$$

where $A, B_{ij}^{\pm}, C, D$ are the same functions appearing in (5.68). It is straightforward to check that such coordinates obey the constraints:

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 + S_6^2 + S_7^2 = 1, \qquad S_0 + S_7 = e^{p_0/\kappa} > 0, \tag{5.75}$$

so that we obtain (half of) the (6+1) dS space as the curved momentum space for the κ-dS quantum algebra.

Again, the isotropy subgroup for $\mathcal{O}$ is generated by the subgroup of dual rotations (5.70), and each point of the curved momentum space can be charaterized by the seven momenta and rapidities by evaluating (5.74) at $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$:

$$\begin{aligned}
S_0 &= \sinh(p_0/\kappa) + \frac{1}{2\kappa^2}e^{p_0/\kappa}\left(\bar{p}^2 + \Lambda\bar{\chi}^2\right),\\
S_1 &= \frac{p_1}{\kappa}\, e^{p_0/\kappa},\\
S_2 &= \frac{p_2}{\kappa}\, e^{p_0/\kappa},\\
S_3 &= \frac{p_3}{\kappa}\, e^{p_0/\kappa},\\
S_4 &= \frac{\sqrt{\Lambda}\,\chi_1}{\kappa}\, e^{p_0/\kappa},\\
S_5 &= \frac{\sqrt{\Lambda}\,\chi_2}{\kappa}\, e^{p_0/\kappa},\\
S_6 &= \frac{\sqrt{\Lambda}\,\chi_3}{\kappa}\, e^{p_0/\kappa},\\
S_7 &= \cosh(p_0/\kappa) - \frac{1}{2\kappa^2}e^{p_0/\kappa}\left(\bar{p}^2 + \Lambda\bar{\chi}^2\right).
\end{aligned} \tag{5.76}$$

Note that these ambient space coordinates (5.76) are all real, since all the complex contributions in (5.74) are linked to the action of the dual rotation subgroup.

Finally, the projection of the deformed Casimir onto the curved momentum space reads:

$$\mathcal{C}_\kappa = 2\kappa^2 \left[\cosh(p_0/\kappa) - 1\right] - e^{p_0/\kappa} \left(\bar{p}^2 - \Lambda \bar{\chi}^2\right) , \tag{5.77}$$

which can be interpreted as the dispersion relation for the $\Lambda > 0$ case. Also, the $\Lambda \to 0$ Poincaré limit of all of these expressions is straightforward, and leads to the results presented in §5.1.

5.5 Remarks

As mentioned in the Introduction, deformed special relativity (DSR) theories are characterized by the presence of an energy scale that plays the role of a second relativistic invariant besides the speed of light. Such an energy scale allows the geometry of momentum space to be nontrivial, and in fact it is a general feature of DSR models that the manifold of momenta has nonzero curvature.

In this Chapter we have shown that the curved momentum space construction can be extended to cases where also a non-vanishing spacetime cosmological constant is present. We explored in particular the momentum space of the κ-deformation of the (A)dS algebra, and we showed that a curved generalized-momentum space can be constructed, that includes not only the momenta associated to spacetime translations but also the hyperbolic momenta associated to boost transformations. The procedure is an adaptation of the one that was successfully used to show that the momentum space of the κ-Poincaré algebra has the geometry of (half of) a dS manifold and is generated by the orbits of the dual Poisson-Lie group. The construction here presented can be applied to any other Hopf algebra deformation of kinematical symmetries with non-vanishing Λ, although the orbit structure of the momentum space so obtained will indeed depend on the chosen quantum deformation.

The construction in (1+1) dimensions is quite straightforward once one realizes that the boosts and spatial translations play a very similar role in the structure of the algebra and coalgebra. We indeed found that the generalized-momentum manifold is a (2+1)-dimensional dS manifold, whose coordinates are the local group coordinates associated to spacetime translations and boosts.

The situation in (2+1) dimensions is more intricate, due to the presence of a rotation generator in the algebra, that significantly complicates its structure. However the rotation generator has a peculiar role in the structure of the algebra and coalgebra, while boosts still behave similarly to spatial translations. We were indeed able to construct the generalized momentum space of the (2+1) κ-dS algebra whose coordinates are the local group coordinates associated to spacetime translations and boosts, and we showed that this is half of a (4+1)-dimensional dS manifold, for which the dual rotation generates the isotropy subgroup of the origin.

It is worth mentioning that the formalism here presented, in which Λ is considered as an explicit 'classical' deformation parameter (and this fact is connected with the so-called 'semidualization' approaches in (2+1) quantum gravity [159, 224]), suggests the possibility of performing the same construction of the generalized momentum space for the κ-AdS

(Anti de Sitter) algebra by taking $\Lambda < 0$. It turns out that one can indeed work out fully the κ-AdS counterpart of the results described above. The main difference between the κ-dS and κ-AdS cases arises from the dual group representation (5.47), which has to be modified in the $\Lambda < 0$ case in order to have a real representation of the corresponding dual Lie group G^*_Λ. The latter can be explicitly constructed and leads to an action on the point $(0, 0, 0, 0, 0, 1)$ that generates the quadric

$$-S_0^2 + S_1^2 + S_2^2 - S_3^2 - S_4^2 + S_5^2 = 1, \tag{5.78}$$

which is no longer the M_{dS_5} momentum space, but a pseudosphere. Nevertheless, the $\Lambda \to 0$ limit of this action annihilates the S_3 and S_4 coordinates, thus giving rise to the same κ-Poincaré limit as the one previously obtained from the κ-dS algebra, as it should be. This analysis provides the first example where quantum effects do not produce a momentum space with dS geometry, but something different - we found that the κ-AdS algebra has a momentum manifold with $SO(3, 3)$ invariance.

Going from the (2+1)-dimensional case to the one with (3+1) dimensions entails dealing with a deformed rotation sector, which is still classical in lower dimensional models. Specifically, the coalgebra of the rotations is modified in (3+1) dimensions, in such a way that one of the rotation generators takes a special role compared to the others (see Chapter 4). This might raise worries that the model breaks spatial isotropy. However, just as the deformed boost transformations do not break relativistic invariance, but simply deform the laws of transformation between inertial frames, the deformed rotations could imply that the concept of isotropy has to be adapted to fit within the new transformation rules. What the observational consequences of this deformed isotropy could be is still a matter of investigation.

Despite these novel features, the analysis of the generalized momentum space of the κ-dS algebra in (3+1) dimensions led to a higher dimensional version of the results for (1+1) and (2+1) dimensions: the momentum space is half of a $6 + 1$-dimensional dS manifold and the rotations are the isotropy group of its origin. The lower-dimensional results are recovered via canonical projection from this construction.

Finally, we generalized our construction to the case of the κ-AdS algebra, which can be defined starting from the κ-dS algebra and changing the sign to the cosmological constant parameter. While the difference between the two models is minimal at the level of the algebra and coalgebra, we found that the momentum space is characterized by a qualitatively different geometry. This is because the change of sign of the cosmological constant produces the appearance of complex quantities due to the presence of $\sqrt{-\Lambda}$ factors. This (3+1)-dimensional case confirms the results for (2+1) dimensions, in the sense that the κ-AdS algebra has a momentum manifold with $SO(4, 4)$ invariance, confirming that quantum deformation effects do not necessarily produce a momentum space with dS geometry, as it seemed to be in the literature available so far.

Chapter 6

Poisson Minkowski spacetimes from Drinfel'd doubles

The two preceding chapters of this Thesis have been devoted to the κ-(A)dS deformation. Let us now change our point of view and consider different deformations. Among all the possible different deformations of the isometry groups of maximally symmetric spacetimes of constant curvature, the ones coming from Drinfel'd double structures of these groups are specially interesting, because they provide quasitriangular r-matrices compatible with the Fock-Rosly approach to quantization of $(2+1)$-gravity [48, 110].

In this Chapter we present all the possible different Drinfel'd double (DD) structures for the Poincaré and Euclidean Lie groups, and we show that the plurality of DD structures for the Poincaré group is completely lost when considering the Euclidean group instead. In fact, we will show how whereas there are eight non-isomorphic DD structures for the Poincaré Lie group, there is only one for the Euclidean group. Moreover, the only DD structure for the Euclidean group is the trivial one, whose existence is guaranteed from the semi-direct product structure (see Chapter 3). In addition, we also study the centrally extended $(1+1)$-Poincaré group and we find that it posses two non-isomorphic DD structures.

The corresponding analysis for the other two Lorentzian groups (dS and (A)dS) was performed in [110], and we will also study here the corresponding contraction procedure, leading to Poincaré and Euclidean structures. In addition to that, we will obtain the quasitriangular r-matrix associated to any of this DD structures and wee will construct their associated noncommutative spacetimes.

With all the above in mind, the main objectives of this Chapter are:

- Firstly, to fill the gap concerning Lorentzian DD structures by constructing explicitly the full set of DD structures for the (2+1)-dimensional Poincaré group. This will be based on the classifications given in [225] and [226], and thus completing in this way the previous works [110, 109, 112, 111] in which all the DD structures for the (2+1) (anti-)de Sitter Lie algebras have been presented. The connection between the latter results and the Poincaré DD structures here presented will also be analysed through Lie bialgebra contraction techniques.

- To construct the Poisson Minkowski spacetimes corresponding to the five coisotropic Lie bialgebras that come from the Poincaré DD structures that we have previously obtained. Two of them will be of Poisson subgroup type, and the features of their associated noncommutative Minkowski spacetimes will be analysed.

- To address the (1+1)-dimensional case by enlarging the (1+1) Poincaré algebra with a (non-trivial) central generator in order to have an extended even-dimensional Lie algebra. Surprisingly enough, this extended algebra can be endowed with two different DD structures, whose Poisson Minkowski spacetimes are also constructed. This completes the study of DDs for the Poincaré group, since it is well-known that in (3+1) and higher dimensions the Poincaré Lie algebra does not admit any DD structure due to the lack of a nondegenerate symmetric bilinear form, which is essential for the definition of an appropriate pairing.

- Finally, to perform a similar analysis for the (2+1)-dimensional Euclidean group, that could clarify the differences of Euclidean and Lorentzian theories in which DD structures play a prominent role, as it is the case for (2+1) gravity.

The structure of the Chapter is as follows: in §6.1 we present the trivial DD structure $D(\mathfrak{sl}(2,\mathbb{R}))$ and its associated PHS. In §6.2 we introduce the eight non-isomorphic DD structures for $\mathfrak{g}_0^{2+1} = \mathfrak{p}(2+1)$, while in §6.3 study each of these DD structures together with their respective PHS. In §6.4 we study the contraction of the DD r-matrices previously obtained from the ones for the DD structure of the (A)dS groups. The analogous analysis is performed in §6.5 for the case of the extended (1+1)-Poincaré group. In §6.6 we study the DD structure for the (2+1) Euclidean group, and in §6.7 the contraction of r-matrices from the ones coming from DD structures for the isometry group of the group of isometries of the hyperbolic space is performed. In §6.8 we construct all the Euclidean PHS from the classification in Appendix A and identify the one coming from a DD structure. Finally, we conclude with some remarks in §6.9. All these new results are contained in the papers [169, 170].

6.1 The (2+1) Poincaré algebra as $D(\mathfrak{sl}(2,\mathbb{R}))$

To the best of our knowledge, the only DD structure for the (2+1) Poincaré algebra $\mathfrak{g}_0^{2+1} \equiv \mathfrak{p}(2+1)$ that has been studied so far in the literature is the 'trivial' one, *i.e.* the one that comes from the trivial ($\delta = 0$) Lie bialgebra structure of the three-dimensional $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$ algebra [112, 123] (for its Euclidean counterpart coming from $\mathfrak{su}(2)$ see [159, 227, 228, 229]).

Let us write down explictly the (2+1) Poincaré Lie algebra $\mathfrak{g}_0^{2+1}$, which is the particular case $\Lambda = 0$ of (2.79), with commutators given by

$$
\begin{aligned}
&[J, P_a] = \epsilon_{ab} P_b, \qquad &&[J, K_a] = \epsilon_{ab} K_b, \qquad &&[J, P_0] = 0, \\
&[K_a, P_b] = \delta_{ab} P_0, \qquad &&[K_a, P_0] = P_a, \qquad &&[K_1, K_2] = -J, \\
&[P_0, P_a] = 0, \qquad &&[P_1, P_2] = 0,
\end{aligned}
\tag{6.1}
$$

The two quadratic Casimir elements for this algebra are

$$C_1 = P_0^2 - P_1^2 - P_2^2, \qquad C_2 = \tfrac{1}{2}\big(J\,P_0 + P_0\,J + K_2\,P_1 + P_1\,K_2 - (K_1\,P_2 + P_2\,K_1)\big), \tag{6.2}$$

where $C_1 = \mathcal{C}$ (2.80) and $C_2 = \mathcal{W}$ (2.81) written in a symmetric form.

It is well-known that all the Lie bialgebra structures for the (2+1) Poincaré algebra $\mathfrak{g}_0^{2+1}$ are coboundary ones [230]. The complete classification of nonisomorphic classes of r-matrices for $\mathfrak{p}(2+1)$, which is reviewed in Appendix A, is due to Stachura [166] (recall that the (3+1) classification was done by Zakrzewski in [230]).

In this section we recall the construction in detail of this DD structure, which is usually called $D(\mathfrak{sl}(2,\mathbb{R})) = D(\mathfrak{so}(2,1))$, and we will also provide all the technical aspects of the (2+1) Poincaré group that will be needed in the rest of the Chapter.

6.1.1 The 'trivial' Drinfel'd double structure

Let us consider the trivial $\delta = 0$ Lie bialgebra structure for the $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ algebra, which means that $\mathfrak{g}^*$ is the three-dimensional abelian algebra. If we take the following basis for $\mathfrak{sl}(2,\mathbb{R})$

$$[Y_0, Y_1] = 2Y_1, \qquad [Y_0, Y_2] = -2Y_2, \qquad [Y_1, Y_2] = Y_0, \tag{6.3}$$

together with a vanishing cocommutator map $\delta(Y_i) = 0$, then the DD relations (3.51) lead to the 6-dimensional Lie algebra $\mathfrak{a}$ with brackets

$$\begin{array}{lll}
[Y_0, Y_1] = 2Y_1, & [Y_0, Y_2] = -2Y_2, & [Y_1, Y_2] = Y_0, \\
[y^0, y^1] = 0, & [y^0, y^2] = 0, & [y^1, y^2] = 0, \\
[y^0, Y_0] = 0, & [y^0, Y_1] = y^2, & [y^0, Y_2] = -y^1, \\
[y^1, Y_0] = 2y^1, & [y^1, Y_1] = -2y^0, & [y^1, Y_2] = 0, \\
[y^2, Y_0] = -2y^2, & [y^2, Y_1] = 0, & [y^2, Y_2] = 2y^0.
\end{array} \tag{6.4}$$

The change of basis

$$\begin{array}{lll}
J = -\tfrac{1}{2}(Y_1 - Y_2), & K_1 = \tfrac{1}{2}(Y_1 + Y_2), & K_2 = -\tfrac{1}{2}Y_0, \\
P_0 = y^1 - y^2, & P_1 = 2y^0, & P_2 = y^1 + y^2,
\end{array} \tag{6.5}$$

shows that this algebra is isomorphic to the (2+1) Poincaré algebra (6.1), and the canonical pairing (3.48) for the kinematical generators is given by

$$\langle J, P_0\rangle = -1, \qquad \langle K_1, P_2\rangle = 1, \qquad \langle K_2, P_1\rangle = -1, \tag{6.6}$$

with all other entries equal to zero. Notice that the pairing is directly related with the Casimir C_2 (6.2), and so it is just the bilinear form induced by the Killing-Cartan form that defines the Lorentzian metric on Minkowski spacetime.

Therefore, $\mathfrak{p}(2+1)$ can be thought of as a DD Lie algebra $\mathfrak{a}$, which under the inverse change of basis

$$\begin{array}{lll}
Y_0 = -2K_2, & Y_1 = -J + K_1, & Y_2 = J + K_1, \\
y^0 = \tfrac{1}{2}P_1, & y^1 = \tfrac{1}{2}(P_0 + P_2), & y^2 = \tfrac{1}{2}(-P_0 + P_2),
\end{array} \tag{6.7}$$

provides the canonical classical r-matrix (3.53):

$$r = \sum_{i=0}^{2} y^i \otimes Y_i = -P_0 \otimes J - P_1 \otimes K_2 + P_2 \otimes K_1. \tag{6.8}$$

By adding the tensorised Casimir C_2 (6.2), the r-matrix can be skew-symmetrized and yields

$$r' = \tfrac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1), \tag{6.9}$$

which is just Class (IV) in the Stachura classification [166] of (2+1) Poincaré r-matrices (see (A.11) in Appendix A where the translation of this classification in terms of the kinematical basis we are using throughout the paper is presented).

The r-matrix (6.9) is composed by three non-commuting twists, and the DD structure above described induces a quantum Poincaré algebra whose cocommutator map induced by r' is given by 3.4.1 and reads

$$\delta_D(J) = \delta_D(K_1) = \delta_D(K_2) = 0, \tag{6.10}$$

$$\delta_D(P_0) = P_1 \wedge P_2, \qquad \delta_D(P_1) = P_0 \wedge P_2, \qquad \delta_D(P_2) = P_1 \wedge P_0. \tag{6.11}$$

Consequently, the corresponding quantum Poincaré algebra for which $\delta_D(J)$ gives the first order deformation will have a non-deformed coproduct for the Lorentz sector, and the quantum deformation will be concentrated in the addition law for the translations sector.

It is also immediate to check that this Lie bialgebra is trivially coisotropic with respect to the Lorentz subalgebra $\mathfrak{h} = \operatorname{span}\{J, K_1, K_2\}$, since $\delta_D(\mathfrak{h}) = 0$ and the Lorentz subgroup is a (trivial) Poisson subgroup. Therefore, the canonical projection of the PL structure on $P(2+1)$ generated by r' (6.9) onto the Minkowski spacetime M_0^{2+1} will give rise to a Poisson homogeneous Minkowski spacetime of Poisson subgroup type, whose quantisation will provide the noncommutative spacetime associated with this DD structure.

6.1.2 An $\mathfrak{so}(2,1)$ noncommutative Minkowski spacetime

Note that the DD structure presented above is the one associated to the semidirect product (associated to the coadjoint action of the Lorentz algebra $\mathfrak{h}^{2+1} \simeq \mathfrak{so}(2,1)$) structure for the Lie algebra $\mathfrak{p}(2+1)$, and so we can apply Theorem 3.7. In this way we know that the resulting Poisson homogeneous space defined by the Sklyanin bracket for the r-matrix (6.9) will be Lie algebraic and indeed isomorphic to $\mathfrak{so}(2,1)$. It could be also explicitly computed by projecting the Poisson structure Π_D on the Poincaré group G_0^{2+1} to Minkowski space M_0^{2+1}, as usual. In this way we obtain the Poisson structure Π_D on G_0^{2+1}

$$\{x^0, x^1\} = -x^2, \qquad \{x^0, x^2\} = x^1, \qquad \{x^1, x^2\} = x^0, \tag{6.12}$$

while the remaining Poisson brackets vanish. Hence, the canonical projection of the Π_D brackets to the spacetime coordinates $\{x^0, x^1, x^2\}$ gives rise to the Poisson Minkowski spacetime π_D associated with this DD structure. Thus, the relations (6.12) define the

Poisson Minkowski spacetime (M^{2+1}, π_D), which is a Lie-algebraic Poisson spacetime isomorphic to the $\mathfrak{so}(2,1)$ algebra, as commented before. By construction, this spacetime is covariant under the co-action defined by the Poincaré group element G_0^{2+1} (2.114) and can be straightforwardly quantized, since no ordering ambiguities appear either in the Poisson bracket (6.12) or in the coproduct induced by the group multiplication of two G_0^{2+1} matrix elements. Note also that the Poisson brackets for the Lorentz coordinates vanish, in accordance with the fact that we have a trivial Lorentz Poisson subgroup with cocommutator $\delta_D(\mathfrak{h}) = 0$.

This DD spacetime, together with its Euclidean E^3 counterpart giving rise to an $\mathfrak{so}(3)$ algebra, have been previously studied in the literature (see [159, 123, 227, 228, 229]). Both of them are Lie-algebraic spacetimes, and the representation theory of the corresponding algebra ($\mathfrak{so}(2,1)$ in the M^{2+1} case and $\mathfrak{so}(3)$ in the E^3 one) characterises their physical properties. On the other hand, we recall that the very same DD construction applied to the Drinfel'd-Jimbo Lie bialgebra structure for $\mathfrak{sl}(2,\mathbb{R})$ was shown in [112] to give rise to a DD which is isomorphic to the (2+1)-dimensional anti-de Sitter algebra in which the deformation parameter η defining a non-trivial Lie bialgebra structure δ is related to the cosmological constant in the form $\Lambda = -\eta^2$.

6.2 Nonisomorphic Drinfel'd double structures for $\mathfrak{p}(2+1)$

The DD structure studied in the previous section is by no means the unique one, and the three following sections will be devoted to the other seven DD structures that can be found for the (2+1) Poincaré algebra, together with their associated Poisson Minkowski spacetimes. The existence of all these DD structures can be traced back to the work [225], where all the non-isomorphic three-dimensional real Lie bialgebra structures are classified, and to [226] where all six-dimensional non-isomorphic real DD structures were also classified.

In the notation from [225] we will be interested in the Lie bialgebra structures for three-dimensional real Lie algebras whose double Lie algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{so}(2,1) \ltimes_{ad^*} \mathbb{R}^3 \simeq \mathfrak{p}(2+1)$. There the cocommutator δ (3.50) for each three-dimensional Lie bialgebra is given by a classical r-matrix (which is denoted by χ) for a given three-dimensional real Lie algebra $\mathfrak{g}$, plus a non-coboundary contribution to the cocommutator which is denoted by $\tilde{\delta}$, namely

$$\delta(e_a) = [1 \otimes e_a + e_a \otimes 1, \chi] + \tilde{\delta}(e_a), \qquad a = 0, 1, 2. \tag{6.13}$$

In this notation, the DD structure described in the previous section (which is not included in [225] since the cases with trivial three-dimensional cocommutator are not considered) would be of the form $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{sl}_2, \mathrm{Abelian}_3)$ with $\chi = 0$, $\tilde{\delta} = 0$. Also, this case would correspond in [226] to the DD denoted as $(8|1)$, and in the rest of the Chapter we will call it 'Case 0'.

A very careful inspection and comparison of both classifications leads to the following seven additional non-trivial three-dimensional Lie bialgebra structures having $\mathfrak{p}(2+1)$ as double Lie algebra:

- Case 1: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{r}_3(1), \mathfrak{sl}_2)$ with $\chi = \alpha e_0 \wedge e_1$, $\tilde{\delta}(e_1) = e_0 \wedge e_2$ (which corresponds to Nr. (3) from [225] and $f(8|5.iii)$ from [226], where the notation $f()$ denotes the dual DD structure).

- Case 2: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{r}_3(1), \mathfrak{n}_3)$ with $\chi = 0$, $\tilde{\delta}(e_1) = e_0 \wedge e_2$ (Nr. 10 from [225] and $(5|2.ii)$ from [226]).

- Case 3: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{r}'_3(1), \mathfrak{n}_3)$ with $\chi = 0$, $\tilde{\delta}(e_2) = \lambda e_0 \wedge e_1$ (Nr. 13 from [225] and $(4|2.iii|b)$ from [226]). Note that in [225] there is a misprint stating that $\tilde{\delta}(e_1) = \lambda e_0 \wedge e_2$.

- Case 4: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{s}_3(0), \mathfrak{r}'_3(1))$ with $\chi = \alpha e_0 \wedge e_1$, $\tilde{\delta}(e_0) = \lambda e_1 \wedge e_2$ (Nr. (14') from [225] and $(7_0|4|b)$ from [226]).

- Case 5: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{r}'_3(1), \mathfrak{r}_3(-1))$ with $\chi = \omega e_1 \wedge e_2$, $\tilde{\delta}(e_2) = \lambda e_0 \wedge e_1$ $(\omega\lambda > 0)$ (Nr. 14 from [225] and $f(6_0|4.i|b)$ from [226]). Note that in [225] there is a misprint stating that $\tilde{\delta}(e_1) = \lambda e_0 \wedge e_2$.

- Case 6: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{r}_3(1), \mathfrak{r}_3(-1))$ with $\chi = \omega e_1 \wedge e_2$ $(\omega > 0)$, $\tilde{\delta}(e_1) = e_0 \wedge e_2$ (Nr. 11 from [225] and $f(6_0|5.i)$ from [226]).

- Case 7: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{s}_3(0), \mathfrak{r}_3(1))$ with $\chi = e_0 \wedge e_1$, $\tilde{\delta} = 0$ (Nr. (11') from [225] and $(7_0|5.i)$ from [226]).

Thus, we have in total eight different DD structures whose commutation rules are displayed in Table 6.1. Notice that the double constructed from $(\mathfrak{g}, \mathfrak{g}^*)$ is always isomorphic to the one arising from $(\mathfrak{g}^*, \mathfrak{g})$. These eight DD structures for the (2+1) Poincaré algebra are nonisomorphic in the sense that for any pair of them there does not exist an algebra isomorphism that leaves the pairing (3.48) (and, therefore, the canonical classical r-matrix (3.53)) invariant. In the following section we will write all these DD structures in the kinematical basis (6.1), where the expression of each canonical r-matrix will be different, and will fall into a given class within the Stachura classification described in Appendix A. This reflects the fact that the inequivalence of 6-dimensional DD structures is translated into the inequivalence of the associated three-dimensional Lie bialgebra structures.

6.3 Drinfel'd double Poincaré r-matrices and Poisson Minkowski spacetimes

In the sequel we present, for each DD structure of the (2+1) Poincaré group given in Table 6.1, an invertible transformation to the kinematical basis (6.1) in which the canonical quasitriangular r-matrix (3.53) is given. Also, the Poisson homogeneous Minkowski spacetime arising from the DD r-matrices is explicitly computed for the five cases in which the Lie bialgebra is coisotropic with respect to the Lorentz subalgebra.

Table 6.1: The eigth non-equivalent DD Lie algebras which are isomorphic to the (2+1) Poincaré algebra. The parameter ω can be rescaled to any non-zero real number of the same sign, while λ is an essential parameter different from zero. In Case 5 they must obey $\omega\lambda > 0$. In Case 6 we have $\omega > 0$.

	Case 0	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
$[Y_0, Y_1]$	$2Y_1$	Y_1	Y_1	Y_1	$-Y_2$	Y_1	Y_1	$-Y_2$
$[Y_0, Y_2]$	$-2Y_2$	Y_2	Y_2	$Y_1 + Y_2$	Y_1	$Y_1 + Y_2$	Y_2	Y_1
$[Y_1, Y_2]$	Y_0	0	0	0	0	0	0	0
$[y^0, y^1]$	0	y^0	0	λy^2	0	λy^2	0	0
$[y^0, y^2]$	0	y^1	y^1	0	$-y^0$	0	y^1	$-y^0$
$[y^1, y^2]$	0	y^2	0	0	$\lambda y^0 - y^1$	$2\omega y^0$	$2\omega y^0$	$-y^1$
$[y^0, Y_0]$	0	$-Y_1$	0	0	$-Y_2$	0	0	Y_2
$[y^0, Y_1]$	y^2	$-Y_2$	$-Y_2$	0	$\lambda Y_2 - y^2$	0	$-Y_2$	0
$[y^0, Y_2]$	$-y^1$	0	0	$-\lambda y^1$	$Y_0 - \lambda Y_1 + y^1$	$-\lambda Y_1$	0	0
$[y^1, Y_0]$	$2y^1$	$Y_0 + y^1$	y^1	$y^1 + y^2$	0	$-2\omega Y_2 + y^1 + y^2$	$-2\omega Y_2 + y^1$	y^2
$[y^1, Y_1]$	$-2y^0$	$-y^0$	$-y^0$	$-y^0$	$-Y_2$	$-y^0$	$-y^0$	Y_2
$[y^1, Y_2]$	0	$-Y_2$	0	$\lambda Y_0 - y^0$	$Y_1 - y^0$	$\lambda Y_0 - y^0$	0	$-y^0$
$[y^2, Y_0]$	$-2y^2$	y^2	y^2	y^2	0	$2\omega Y_1 + y^2$	$2\omega Y_1 + y^2$	$-Y_0 - y^1$
$[y^2, Y_1]$	0	Y_0	Y_0	0	y^0	0	Y_0	$-Y_1 + y^0$
$[y^2, Y_2]$	$2y^0$	$Y_1 - y^0$	$-y^0$	$-y^0$	0	$-y^0$	$-y^0$	0

6.3.1 Case 1

An isomorphism between the DD Lie algebra and $\mathfrak{p}(2+1)$ in terms of the kinematical basis (6.1) reads

$$
\begin{aligned}
&J = y^0 + y^1 + y^2, && K_1 = y^0 + y^1, && K_2 = -y^1 - y^2, \\
&P_0 = y^0 + y^1 + Y_0 - Y_1 + Y_2, && P_1 = y^0 + y^1 + Y_0 - Y_1, && P_2 = y^0 - Y_1 + Y_2.
\end{aligned}
\tag{6.14}
$$

From the canonical pairing (3.48) we get the same (6.6) up to a global sign:

$$
\langle J, P_0 \rangle = 1, \qquad \langle K_1, P_2 \rangle = -1, \qquad \langle K_2, P_1 \rangle = 1. \tag{6.15}
$$

By inserting the inverse of the basis transformation (6.14) into (3.53), the following classical r-matrix is found

$$
r_1 = \sum_{i=0}^{2} y^i \otimes Y_i = K_1 \wedge J + K_1 \wedge K_2 + J \otimes P_0 + K_2 \otimes P_1 - K_1 \otimes P_2. \tag{6.16}
$$

And by subtracting the tensorized Casimir C_2 (6.2) from (6.16), one obtains the skew-symmetric r-matrix

$$
r_1' = K_1 \wedge J + K_1 \wedge K_2 + \big(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1\big), \tag{6.17}
$$

which is a solution of the modified CYBE, that belongs to Class (I) in [166]. In particular, if we apply the automorphism given by

$$J \to J, \qquad K_1 \to K_1, \qquad K_2 \to K_2, \qquad P_i \to \sqrt{2}P_i, \quad i = 0, 1, 2, \tag{6.18}$$

to r_1', we recover (A.5) with $\alpha = 1$ (up to a global constant $\sqrt{2}$). The cocommutator derived from (3.57) is

$$\begin{aligned}
&\delta_D(J) = K_2 \wedge J, \\
&\delta_D(K_1) = J \wedge K_1 + K_2 \wedge K_1, \\
&\delta_D(K_2) = J \wedge K_2, \\
&\delta_D(P_0) = J \wedge P_1 + P_2 \wedge K_1 + K_2 \wedge P_1 + 2P_1 \wedge P_2, \\
&\delta_D(P_1) = J \wedge P_0 + K_2 \wedge P_0 + P_2 \wedge K_1 + 2P_0 \wedge P_2, \\
&\delta_D(P_2) = P_0 \wedge K_1 + K_1 \wedge P_1 + 2P_1 \wedge P_0.
\end{aligned} \tag{6.19}$$

Therefore, from the viewpoint of the construction of Poisson homogeneous Minkowski spacetimes, the Poincaré deformation induced by r_1' is of Poisson subgroup type, since $\delta_D(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$ and the Lorentz subalgebra closes a sub-Lie bialgebra structure.

By making use of the results given in [160], the DD r-matrix (6.17) can be thought of as a particular case of a more general solution of the modified CYBE that contains two independent real parameters α_1, β_1, namely

$$r'_{1,(\alpha_1,\beta_1)} = \alpha_1 \left(J \wedge K_1 + K_2 \wedge K_1\right) + \beta_1 \left(P_0 \wedge J + P_1 \wedge K_2 + K_1 \wedge P_2\right), \tag{6.20}$$

although only for the case with $\alpha_1 = \beta_1$ the DD structure is recovered. This embedding allows for a more clear interpretation of the contributions coming from each term of the r-matrix. In particular, the associated Poisson Minkowski spacetime can be obtained by computing the Sklyanin bracket for (6.20) and afterwards by projecting to the Poisson subalgebra generated by the spacetime coordinates $\{x^0, x^1, x^2\}$. A straightforward computation shows that the final result is

$$\begin{aligned}
&\{x^0, x^1\} = -\alpha_1 x^2(x^0 + x^1) + 2\beta_1 x^2, \\
&\{x^0, x^2\} = \alpha_1 x^1(x^0 + x^1) - 2\beta_1 x^1, \\
&\{x^1, x^2\} = \alpha_1 x^0(x^0 + x^1) - 2\beta_1 x^0,
\end{aligned} \tag{6.21}$$

which is a noncommutative *quadratic* Poisson Minkowski spacetime, whose linear part is ruled by the parameter β_1 (and is just proportional to the one in Case 0 (6.9)) and comes from the dual of the cocommutator (6.19). The quadratic part of the bracket comes from the α_1-contribution to the r-matrix, which is a triangular solution of the CYBE (a twist) of the type considered in [231] in (3+1) dimensions. Moreover, this twist is the one that generates the non-zero cocommutator for the Lorentz sector, thus being responsible of the Poisson subgroup structure of the isotropy subgroup. Hence, the Poisson Minkowski spacetime (6.21) can be regarded as a quadratic generalization of the Lie-algebraic one (6.12). As it is detailed in the Appendix, note that this is the only r-matrix for $\mathfrak{p}(2+1)$ having a term living in $\mathfrak{h} \wedge \mathfrak{h}$. We also remark that the quantization of this Poisson structure is far from being trivial, due to the ordering problems arising from the quadratic terms.

6.3.2 Case 2

Now the Lie algebra isomorphism is given by

$$
\begin{aligned}
&J = y^2 + Y_0 + Y_1, && K_1 = -y^2 - Y_0, && K_2 = Y_0 + Y_1, \\
&P_0 = y^0 - y^1 - Y_2, && P_1 = -y^0 + Y_2, && P_2 = -y^0 + y^1,
\end{aligned} \tag{6.22}
$$

and the canonical pairing is again (6.6). The inverse of the basis transformation into (3.53) leads to

$$
r_2 = P_2 \wedge J + K_2 \wedge P_0 + K_2 \wedge P_2 + P_2 \otimes K_1 - P_1 \otimes K_2 - J \otimes P_0. \tag{6.23}
$$

This r-matrix can be straightforwardly skew-symmetrized through the tensorized Casimir C_2 (6.2) yielding

$$
r_2' = P_2 \wedge J - P_0 \wedge K_2 - P_2 \wedge K_2 + \tfrac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1), \tag{6.24}
$$

and it can be shown to belong to Class (IIa) in [166] by applying the following automorphism to r_2'

$$
\begin{aligned}
&J \to -2J - K_1 + \sqrt{2}\, K_2, && P_0 \to -2\big(2P_0 + \sqrt{2}\, P_1 + P_2\big), \\
&K_1 \to \big(1 + \tfrac{1}{\sqrt{2}}\big) J + K_1 - \big(1 + \tfrac{1}{\sqrt{2}}\big) K_2, && P_1 \to \big(2 - \sqrt{2}\big) P_0 - \big(2 - \sqrt{2}\big) P_1 + 2P_2, \\
&K_2 \to -\big(1 - \tfrac{1}{\sqrt{2}}\big) J - K_1 - \big(1 - \tfrac{1}{\sqrt{2}}\big) K_2, && P_2 \to \big(2 + \sqrt{2}\big) P_0 + \big(2 + \sqrt{2}\big) P_1 + 2P_2,
\end{aligned} \tag{6.25}
$$

which leads to the r-matrix (A.14) with parameters $\rho = \alpha = 1$ and term $a = 0$, namely

$$
r_2' = K_2 \wedge P_0 + J \wedge P_1 - K_1 \wedge P_2. \tag{6.26}
$$

In this form, Case 2 can be clearly interpreted as the superposition of the ‘space-like’ κ-Poincaré deformation [80, 143], coming from the r-matrix $K_2 \wedge P_0 + J \wedge P_1$, along with a twist $K_1 \wedge P_2$.

Next, by computing δ_D, it can be shown that this DD structure generates a Poisson Minkowski spacetime fulfilling the coisotropy condition $\delta_D(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$. By taking into account [160], we find that the embedding of (6.24) into a more general solution of the modified CYBE is given by

$$
r'_{2,(\alpha_2,\beta_2)} = \alpha_2 \left(P_0 \wedge K_2 + P_2 \wedge K_2\right) + \beta_2 \left(J \wedge P_2 + \tfrac{1}{2}\left(J \wedge P_0 + P_1 \wedge K_2 + K_1 \wedge P_2\right)\right), \tag{6.27}
$$

with $\alpha_2, \beta_2 \in \mathbb{R}$ (the DD case corresponds to $\alpha_2 = \beta_2$). The Poisson Minkowski spacetime turns out to be

$$
\{x^0, x^1\} = 0, \qquad \{x^0, x^2\} = -\alpha_2 \left(x^0 - x^2\right), \qquad \{x^1, x^2\} = -\beta_2 \left(x^0 - x^2\right), \tag{6.28}
$$

which is linear and thus can be straightforwardly quantized.

6.3.3 Case 3

The Lie algebra isomorphism reads

$$
\begin{aligned}
&J=\frac{1}{\lambda}(y^0+y^1)-Y_0+Y_2, && K_1=-\frac{1}{\lambda}y^1+Y_0-Y_1, && K_2=-\frac{1}{\lambda}y^2+Y_0-Y_2,\\
&P_0=y^0+y^2+\lambda Y_1, && P_1=y^0+\lambda Y_1, && P_2=-y^0-y^2,
\end{aligned}
\tag{6.29}
$$

and leads again to (6.15). The DD r-matrix is found to be

$$
\begin{aligned}
r_3 =&J\wedge P_2+K_2\wedge P_0+K_2\wedge P_2-P_2\otimes K_1+P_1\otimes K_2+J\otimes P_0+\\
&+\frac{1}{\lambda}\big(P_0\wedge P_1+2(P_0\wedge P_2+P_2\wedge P_1)+P_0\otimes P_0-P_1\otimes P_1-P_2\otimes P_2\big),
\end{aligned}
\tag{6.30}
$$

which by making use of the tensorised version of both Casimirs C_1 and C_2 (6.2) can be transformed into:

$$
\begin{aligned}
r_3' =& -P_2\wedge J-P_0\wedge K_2-P_2\wedge K_2+\frac{1}{2}(-P_0\wedge J+P_1\wedge K_2-P_2\wedge K_1)+\\
&+\frac{1}{\lambda}\big(P_0\wedge P_1+2(P_0\wedge P_2+P_2\wedge P_1)\big).
\end{aligned}
\tag{6.31}
$$

This r-matrix is shown to belong to Class (IIa) in [166] by applying to r_3' the composition of the automorphism (6.25) and

$$
J\to -J,\qquad K_1\to -K_1,\qquad K_2\to K_2,\qquad P_0\to P_0,\qquad P_1\to -P_1,\qquad P_2\to P_2.
\tag{6.32}
$$

In this way we obtain (A.14) with $\rho=\alpha=1$ but now with a term proportional to $1/\lambda$. Hence the difference between Cases 2 and 3 relies on the $1/\lambda$ term, which precludes the coisotropy condition $\delta_D(\mathfrak{h})\subset\mathfrak{h}\wedge\mathfrak{g}$ to hold since, for instance,

$$
\delta_D(J)=P_0\wedge K_1+P_1\wedge J+P_1\wedge K_2+P_2\wedge K_1+\frac{1}{\lambda}(P_0\wedge P_2+2P_1\wedge P_0).
\tag{6.33}
$$

Therefore, this condition is only fulfilled in the limit $\lambda\to\infty$, which leads to the previous Case 2. Therefore the Poisson Minkowski spacetime for this case will not be constructed.

6.3.4 Case 4

The isomorphism

$$
\begin{aligned}
&J=\lambda y^0-Y_0, && K_1=\lambda y^0+y^1+Y_0, && K_2=y^2+\lambda Y_2,\\
&P_0=y^0-Y_1, && P_1=-Y_2, && P_2=Y_1,
\end{aligned}
\tag{6.34}
$$

leads again to the (2+1) Poincaré algebra with pairing (6.6). The classical r-matrix is now

$$
r_4=P_2\wedge J-J\otimes P_0-P_1\otimes K_2+P_2\otimes K_1+\lambda(P_0\otimes P_0-P_1\otimes P_1-P_2\otimes P_2+P_0\wedge P_2),
\tag{6.35}
$$

which can be fully skew-symmetrized by making use of both Casimirs C_1 and C_2 (6.2) yielding

$$r_4' = P_2 \wedge J + \tfrac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1) + \lambda P_0 \wedge P_2. \tag{6.36}$$

This r-matrix belongs to Class (IIIb) in [166], since r_4' turns out to be proportional to the r-matrix (A.10) with $\rho = 1$ and term $a \neq 0$ under the automorphism

$$J \to iK_2, \qquad K_1 \to iJ, \qquad K_2 \to -K_1, \qquad P_0 \to -iP_1, \qquad P_1 \to P_2, \qquad P_2 \to iP_0. \tag{6.37}$$

Again, due to the presence of the non-zero essential parameter λ (i.e., $a \neq 0$), this case does not fulfil the coisotropy condition (3.79).

6.3.5 Case 5

Here we have two different subcases due to the constraint $\omega\lambda > 0$ (see Table 6.1): either $\omega > 0$ and $\lambda > 0$ or $\omega < 0$ and $\lambda < 0$. Although the isomorphism is different for each subcase, we shall show that the resulting r-matrices are the same in both of them.

As stated in [225], ω can be rescaled to any non-zero value of the same sign and λ is an essential parameter. Therefore, if we set $\omega = 1/2$ and hence $\lambda > 0$, the isomorphism is given by

$$\begin{aligned} &J = \frac{1}{\sqrt{\lambda}}(-y^1 + Y_1), && K_1 = -\frac{1}{\lambda}y^0 - Y_0, && K_2 = \frac{1}{\sqrt{\lambda}}(y^1 + Y_1 - Y_2), \\ &P_0 = -\sqrt{\lambda}\,(y^2 + Y_1), && P_1 = -\sqrt{\lambda}\,y^2, && P_2 = y^0, \end{aligned} \tag{6.38}$$

and the pairing is (6.15). On the other hand, if $\omega = -1/2$ and $\lambda < 0$, then

$$\begin{aligned} &J = -\frac{1}{\sqrt{-\lambda}}(y^1 + Y_1), && K_1 = -\frac{1}{\lambda}y^0 - Y_0, && K_2 = \frac{1}{\sqrt{-\lambda}}(y^1 - Y_1 + Y_2), \\ &P_0 = \sqrt{-\lambda}\,(y^2 - Y_1), && P_1 = \sqrt{-\lambda}\,y^2, && P_2 = y^0, \end{aligned} \tag{6.39}$$

together with the same pairing (6.15).

Both subcases lead to the same classical r-matrix

$$r_5 = P_1 \wedge J - P_2 \otimes K_1 + P_1 \otimes K_2 + J \otimes P_0 + \frac{1}{\lambda}(P_1 \wedge P_0 + P_0 \otimes P_0 - P_1 \otimes P_1 - P_2 \otimes P_2)\,, \tag{6.40}$$

which, by making use of both Casimirs, can be written in the skew-symmetric form

$$r_5' = P_1 \wedge J + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1) + \frac{1}{\lambda} P_1 \wedge P_0, \tag{6.41}$$

which is proportional to the r-matrix (A.10) with $\rho = -1$ and term $a \neq 0$, so belonging to Class (IIIb) in [166]. As in Case 4, the parameter λ precludes the coisotropy condition (3.79) to be satisfied.

6.3.6 Case 6

As the parameter $\omega > 0$ can be rescaled to any positive real number, hereafter we take $\omega = 1/2$. An isomorphism between the kinematical basis and the DD one is given by

$$\begin{array}{lll} J = Y_1 + y^2, & K_1 = Y_0, & K_2 = y^2, \\ P_0 = -y^1, & P_1 = -Y_2 + y^1, & P_2 = y^0, \end{array} \tag{6.42}$$

with pairing (6.6). The corresponding inverse isomorphism gives rise to the classical r-matrix

$$r_6 = P_0 \wedge K_2 + P_2 \otimes K_1 - K_2 \otimes P_1 - P_0 \otimes J, \tag{6.43}$$

and with the aid of C_2 we obtain

$$r_6' = P_0 \wedge K_2 + \tfrac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 + P_2 \wedge K_1). \tag{6.44}$$

By making use of the automorphism

$$J \to J, \qquad K_1 \to -K_1, \qquad K_2 \to -K_2, \qquad P_0 \to P_0, \qquad P_1 \to -P_1, \qquad P_2 \to -P_2, \tag{6.45}$$

we find that r_6' coincides (up to a factor 1/2) with the r-matrix (A.10) of Class (IIIb) with $\rho = 1$ but now with the term $a = 0$. This is a coisotropic Lie bialgebra, whose Poisson Minkowski spacetime reads

$$\{x^0, x^1\} = 0, \qquad \{x^0, x^2\} = -x^0 + x^1, \qquad \{x^1, x^2\} = 0, \tag{6.46}$$

and its quantization is straightforward.

Notice that Cases 4 and 6 are, obviously, related since they are within the same Class (IIIb) with parameter $\rho = 1$. In fact, if we write r_4' (6.36) under the automorphism (6.37), its limit $\lambda \to 0$ leads to r_6' (6.44) expressed under the map (6.45).

6.3.7 Case 7

Finally, the kinematical and DD basis are now related through

$$\begin{array}{lll} J = y^0, & K_1 = -Y_2, & K_2 = -Y_1 - y^0, \\ P_0 = Y_0 - y^1, & P_1 = -y^1, & P_2 = y^2, \end{array} \tag{6.47}$$

with pairing (6.15). The inverse isomorphism provides the classical r-matrix

$$r_7 = P_2 \wedge J + K_1 \otimes P_2 - K_2 \otimes P_1 - P_0 \otimes J. \tag{6.48}$$

By subtracting C_2 we find

$$r_7' = P_2 \wedge J + \tfrac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1), \tag{6.49}$$

Table 6.2: The (2+1) Poincaré r-matrices and Poisson subgroup/coisotropy condition for each of the eight DD structures on $\mathfrak{p}(2+1)$ as well as the corresponding class in the Stachura classification.

Case	Classical r-matrix r'_i	$\delta_D\,(\mathfrak{h})$	Class [166]
0	$\frac{1}{2}(-P_0\wedge J - P_1\wedge K_2 + P_2\wedge K_1)$	$=0$	(IV)
1	$K_1\wedge J + K_1\wedge K_2 + (-P_0\wedge J - P_1\wedge K_2 + P_2\wedge K_1)$	$\subset \mathfrak{h}\wedge\mathfrak{h}$	(I)
2	$P_2\wedge J - P_0\wedge K_2 - P_2\wedge K_2 + \frac{1}{2}(P_0\wedge J - P_1\wedge K_2 + P_2\wedge K_1)$	$\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIa)
3	$-P_2\wedge J - P_0\wedge K_2 - P_2\wedge K_2 + \frac{1}{2}(-P_0\wedge J + P_1\wedge K_2 - P_2\wedge K_1)$ $+\frac{1}{\lambda}\big(P_0\wedge P_1 + 2(P_0\wedge P_2 + P_2\wedge P_1)\big)$	$\not\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIa)
4	$P_2\wedge J + \frac{1}{2}(P_0\wedge J - P_1\wedge K_2 + P_2\wedge K_1) + \lambda P_0\wedge P_2$	$\not\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIIb)
5	$P_1\wedge J + \frac{1}{2}(-P_0\wedge J + P_1\wedge K_2 - P_2\wedge K_1) + \frac{1}{\lambda}P_1\wedge P_0$	$\not\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIIb)
6	$P_0\wedge K_2 + \frac{1}{2}(-P_0\wedge J + P_1\wedge K_2 + P_2\wedge K_1)$	$\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIIb)
7	$P_2\wedge J + \frac{1}{2}(-P_0\wedge J + P_1\wedge K_2 - P_2\wedge K_1)$	$\subset \mathfrak{h}\wedge\mathfrak{g}$	(IIIb)

which, under the automorphism

$$J\to J,\qquad K_1\to -K_2,\qquad K_2\to K_1,\qquad P_0\to P_0,\qquad P_1\to -P_2,\qquad P_2\to P_1, \tag{6.50}$$

turns out to correspond again to Case (IIIb) with r-matrix (A.10) such that $\rho=-1$ and the a term vanishes. Note that r'_7 (6.49), written under the automorphism (6.50), is recovered from r'_5 (6.41) by taking the limit $\lambda\to\infty$. The coisotropy condition is fulfilled and the associated Poisson Minkowski spacetime is given by

$$\{x^0,x^1\}=0,\qquad \{x^0,x^2\}=0,\qquad \{x^1,x^2\}=-(x^0+x^2), \tag{6.51}$$

whose quantization is also straightforward.

Therefore, by starting from the classification of Lie bialgebras given in [225], we have obtained eight DD for the (2+1) Poincaré group. Table 6.2 summarizes our results: five of the classical r-matrices give rise to coboundary Lie bialgebras compatible with our algebraic conditions for $\delta_D(\mathfrak{h})$ (3.79), which guarantee that the Poisson bracket between Minkowski coordinates close a Poisson subalgebra. Among them, only Case 0 (trivially) and Case 1 turn out to be of Poisson subgroup type. The noncommutative spacetimes obtained from these DD structures are summarized in Table 6.3.

6.4 Contraction from (A)dS r-matrices

Since the complete study of DD structures for the (A)dS Lie algebras $\mathfrak{g}_\Lambda^{2+1}$ in (2+1) dimensions was performed in [110], it is therefore natural to study the behavior of the associated DD r-matrices under the Lie algebra contraction to the Poincaré Lie algebra, that corresponds in kinematical terms to the $\Lambda\to 0$ limit. Note that the classification of classical r-matrices for all real forms of $\mathfrak{o}(4;\mathbb{C})$, in particular for $\mathfrak{o}(4)$, $\mathfrak{o}(3,1)$ and $\mathfrak{o}(2,2)$, has been given in [167, 168].

Table 6.3: The (2+1) Poisson Minkowski spacetimes arising from coisotropic DD structures [169].

Case	$\{x^0, x^1\}$	$\{x^0, x^2\}$	$\{x^1, x^2\}$
0	$-x^2$	x^1	x^0
1	$-\alpha_1 x^2(x^0 + x^1) + 2\beta_1 x^2$	$\alpha_1 x^1(x^0 + x^1) - 2\beta_1 x^1$	$\alpha_1 x^0(x^0 + x^1) - 2\beta_1 x^0$
2	0	$-\alpha_2(x^0 - x^2)$	$-\beta_2(x^0 - x^2)$
6	0	$-x^0 + x^1$	0
7	0	0	$-(x^0 + x^2)$

We recall that the classification of (A)dS DD r-matrices in [110] was carried out in a basis with generators denoted $\{J_0, J_1, J_2, P_0, P_1, P_2\}$. There are four DD structures for $\mathfrak{so}(3,1)$ (cases A, B, C and D), and three for AdS $\mathfrak{so}(2,2)$ (cases E, F and G). Depending on the case considered, the relationship of this basis with the kinematical one used throughout the present paper is established by means of the following isomorphisms:

$$\text{Cases A, C, E, F, G:} \quad J_0 \to J, \quad J_1 \to -K_2, \quad J_2 \to K_1, \quad P_a \to P_a, . \tag{6.52}$$

$$\begin{aligned} \text{Cases B, D:} \quad J_0 \to J, \quad J_1 \to \frac{1}{\eta} P_2, \quad J_2 \to -\frac{1}{\eta} P_1, \\ P_0 \to -P_0, \quad P_1 \to \eta K_1, \quad P_2 \to \eta K_2, \quad \eta = \sqrt{\Lambda}. \end{aligned} \tag{6.53}$$

By applying these transformations onto the brackets (2.3) in [110], we find that the commutation relations for the (2+1) (A)dS Lie algebras adopt the form

$$\begin{array}{lll} [J, K_1] = K_2, & [J, K_2] = -K_1, & [K_1, K_2] = -J, \\ [J, P_0] = 0, & [J, P_1] = P_2, & [J, P_2] = -P_1, \\ [K_1, P_0] = P_1, & [K_1, P_1] = P_0, & [K_1, P_2] = 0, \\ [K_2, P_0] = P_2, & [K_2, P_1] = 0, & [K_2, P_2] = P_0, \\ [P_0, P_1] = -\Lambda K_1, & [P_0, P_2] = -\Lambda K_2, & [P_1, P_2] = \Lambda J. \end{array} \tag{6.54}$$

Therefore, when $\Lambda < 0$ we recover the AdS Lie algebra $\mathfrak{so}(2,2)$, for $\Lambda > 0$ the dS algebra $\mathfrak{so}(3,1)$, and the contraction $\Lambda \to 0$ gives the Poincaré Lie algebra in a basis which is just (6.1). Now, by using (6.83) and (6.84), we rewrite in the kinematical basis (6.54) all the

DD (A)dS r-matrices obtained in [110], namely

$$
\begin{aligned}
\mathrm{dS} \equiv \mathfrak{so}(3,1) \quad (\Lambda > 0): \quad & r'_{\mathrm{A}} = \sqrt{\Lambda}\, K_1 \wedge K_2 + \tfrac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1),\\
& r'_{\mathrm{B}} = \tfrac{1}{\sqrt{\Lambda}}\, P_2 \wedge P_1 + \tfrac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1),\\
& r'_{\mathrm{C}} = \tfrac{1}{2}(P_0 \wedge K_2 + P_1 \wedge J - P_2 \wedge K_1),\\
& r'_{\mathrm{D}} = \sqrt{\Lambda}\, J \wedge K_1 + \tfrac{1}{\sqrt{\Lambda}}\, P_2 \wedge P_0 + \tfrac{(1+\mu^2)}{2\mu}\, P_1 \wedge K_2 +\\
& \qquad + \tfrac{(\mu^2-1)}{2\mu}\,(-P_2 \wedge J + P_0 \wedge K_1), \quad \mu > 0.\\
\mathrm{AdS} \equiv \mathfrak{so}(2,2) \quad (\Lambda < 0): \quad & r'_{\mathrm{E}} = \sqrt{-\Lambda}\, J \wedge K_1 + \tfrac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1),\\
& r'_{\mathrm{F}} = \tfrac{1}{2}(P_0 \wedge K_2 + P_1 \wedge J - P_2 \wedge K_1),\\
& r'_{\mathrm{G}} = \tfrac{(1+\rho^2)}{4}\,(P_0 \wedge K_2 + P_1 \wedge J) - \tfrac{\rho}{2}\, P_2 \wedge K_1 +\\
& \qquad + \tfrac{(1-\rho^2)}{4\sqrt{-\Lambda}}\,(\Lambda\, J \wedge K_2 + P_0 \wedge P_1), \quad -1 < \rho < 1.
\end{aligned} \tag{6.55}
$$

Now let us analyse the vanishing cosmological constant limit $\Lambda \to 0$ of all these expressions. Firstly, we obtain that

$$\lim_{\Lambda\to 0} r'_{\mathrm{A}} = \lim_{\Lambda\to 0} r'_{\mathrm{E}} = \tfrac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1) \equiv r'_0,$$

which is just the Poincaré r-matrix (6.9) of Case 0 coming from the DD of $\mathfrak{sl}(2,\mathbb{R})$ and trivial cocommutator. Secondly, we have that

$$\lim_{\Lambda\to 0} r'_{\mathrm{C}} = \lim_{\Lambda\to 0} r'_{\mathrm{F}} = \tfrac{1}{2}(P_0 \wedge K_2 + P_1 \wedge J - P_2 \wedge K_1) \propto r'_2,$$

which thus corresponds to Case 2 with the r-matrix expressed in the form (6.26). This, in turn, means that we have obtained a common DD r-matrix for the (A)dS and Poincaré Lie algebras that is just a twisted version of the 'space-like' κ-(A)dS and κ-Poincaré r-matrices studied in [111].

Finally, cases B, D and G seem to give rise to divergencies in the limit $\Lambda \to 0$. However we can rescale globally these r-matrices (the multiplication of a given r-matrix by a constant is also an r-matrix) in such a way that the limit does exist, namely

$$\lim_{\Lambda\to 0} \sqrt{\Lambda}\, r'_{\mathrm{B}} = P_2 \wedge P_1, \qquad \lim_{\Lambda\to 0} \sqrt{\Lambda}\, r'_{\mathrm{D}} = P_2 \wedge P_0, \qquad \lim_{\Lambda\to 0} \sqrt{-\Lambda}\, r'_{\mathrm{G}} = \frac{(1-\rho^2)}{4}\, P_0 \wedge P_1.$$

None of these limits provides a Poincaré r-matrix coming from a DD structure, and all of them belong to Class (V) in [166] (see (A.12)).

6.5 The extended (1+1) Poincaré algebra as a Drinfel'd double

Since the (1+1) Poincaré algebra

$$[K, P_0] = P_1, \qquad [K, P_1] = P_0, \qquad [P_1, P_0] = 0, \tag{6.56}$$

is odd-dimensional, no DD structure (3.51) can be defined within it. Nevertheless, if we consider the non-trivial central extension of the Poincaré Lie algebra $\overline{\mathfrak{iso}(1,1)} = \overline{\mathfrak{p}(1+1)}$ given by

$$[K, P_0] = P_1, \qquad [K, P_1] = P_0, \qquad [P_1, P_0] = F, \qquad [F, \cdot] = 0, \tag{6.57}$$

we will show in the sequel that the new central generator F allows the introduction of two non-equivalent DD structures. This extended algebra is also called the Nappi-Witten Lie algebra [232] and plays a relevant role in (1+1) gravity [233]. Casimir operators for the algebra (6.57) are given by

$$C_1 = P_0^2 - P_1^2 + F\,K + K\,F, \qquad\qquad C_2 = F, \tag{6.58}$$

and the first of them already suggests the existence of a non-degenerate symmetric bilinear form underlying possible DD structures.

The corresponding (1+1)-dimensional Poincaré group with a nontrivial central extension, $\overline{ISO(1,1)} = \overline{P(1+1)}$, is obtained by considering the faithful representation $\rho : \overline{\mathfrak{p}(1+1)} \to \mathrm{End}(\mathbb{R}^4)$ given by

$$\rho(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho(K) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(P_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \rho(P_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{6.59}$$

along with local coordinates on the Lie group $\{\phi, \xi, x^0, x^1\}$ associated with the generators $\{F, K, P_0, P_1\}$, respectively. Hence we obtain the group element

$$G = \exp\left(\phi\,\rho(F)\right)\exp\left(x^0\,\rho(P_0)\right)\exp\left(x^1\,\rho(P_1)\right)\exp\left(\xi\,\rho(K)\right), \tag{6.60}$$

namely,

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^0 & \cosh\xi & \sinh\xi & 0 \\ x^1 & \sinh\xi & \cosh\xi & 0 \\ x^0\,x^1 - 2\phi & -x^1\cosh\xi + x^0\sinh\xi & x^0\cosh\xi - x^1\sinh\xi & 1 \end{pmatrix}. \tag{6.61}$$

From it, left- and right-invariant vector fields on the $\overline{P(1+1)}$ group are found to be

$$\begin{aligned} &\nabla^L_F = \partial_\phi, \qquad \nabla^L_K = \partial_\xi, \\ &\nabla^L_{P_0} = \cosh\xi\left(x^1\,\partial_\phi + \partial_{x^0}\right) + \sinh\xi\,\partial_{x^1}, \\ &\nabla^L_{P_1} = \sinh\xi\left(x^1\,\partial_\phi + \partial_{x^0}\right) + \cosh\xi\,\partial_{x^1}, \end{aligned} \tag{6.62}$$

$$\begin{aligned} &\nabla^R_K = \tfrac{1}{2}\big((x^0)^2 + (x^1)^2\big)\partial_\phi + x^1\partial_{x^0} + x^0\partial_{x^1} + \partial_\xi, \\ &\nabla^R_F = \partial_\phi, \qquad \nabla^R_{P_0} = \partial_{x^0}, \qquad \nabla^R_{P_1} = x^0\partial_\phi + \partial_{x^1}. \end{aligned} \tag{6.63}$$

In this context, we define the (1+1) Minkowski spacetime M^{1+1} and its extended counterpart $\overline{M}^{1+1}$ (see [234]) as the following quotients by the Lorentz subalgebra $\mathfrak{l}$ and by the trivially extended one $\bar{\mathfrak{h}}$:

$$\begin{aligned} M^{1+1} &= \overline{P(1+1)}/\overline{L}, \quad \bar{\mathfrak{l}} = \text{Lie}(\overline{L}) = \mathfrak{so}(1,1) \oplus \mathbb{R} = \text{span}\{K, F\}, \quad \text{coordinates: } x^0, x^1. \\ \overline{M}^{1+1} &= \overline{P(1+1)}/L, \quad \mathfrak{l} = \text{Lie}(L) = \mathfrak{so}(1,1) = \text{span}\{K\}, \qquad \text{coordinates: } x^0, x^1, \phi. \end{aligned} \tag{6.64}$$

6.5.1 Two-dimensional real Lie bialgebras and their Drinfel'd double structures

The only two-dimensional non-abelian real Lie algebra is the so-caled $\mathfrak{b}_2$ algebra with bracket

$$[Y_1, Y_2] = Y_2. \tag{6.65}$$

It is also known that there exists, up to isomorphism, three real Lie bialgebra structures δ for this algebra, which are the 'trivial one' with $\delta(Y_i) = 0$ plus the two non-trivial ones given in [225]. As we will show in what follows, two of these Lie bialgebras have the centrally extended (1+1) Poincaré algebra (6.57) as its DD algebra. Explictly, with the notation used in [225] these two Lie bialgebras are:

- Case 0: $(\mathfrak{g}, \mathfrak{g}^*) = \left(\mathfrak{b}_2, \mathbb{R}^2\right)$ with $\chi = 0$, $\tilde{\delta} = 0$.
- Case 1: $(\mathfrak{g}, \mathfrak{g}^*) = (\mathfrak{b}_2, \mathfrak{b}_2)$ with $\chi = e_0 \wedge e_1$, $\tilde{\delta} = 0$.

The remaining two-dimensional real Lie bialgebra structure was shown in [235] to have as its DD Lie algebra a central extension of $\mathfrak{sl}(2, \mathbb{R})$, which is just the centrally extended (1+1) (A)dS Lie algebra. Note that the classification of nonisomorphic four-dimensional real DD structures was also given in [236].

Commutation rules for these two DD structures for $\overline{\mathfrak{p}(1+1)}$ are given in Table 6.4, and in the following we will present these two structures in the kinematical basis, together with the associated classical r-matrices and (extended) noncommutative Minkowski spacetimes. We recall that the extended noncommutative Minkowski spacetimes studied in [234] come from a (different) trivial central extension of the (1+1) Poincaré group.

6.5.2 Case 0

A Lie algebra isomorphism is given by

$$K = -Y_1, \qquad P_0 = \frac{1}{\sqrt{2}}(y^2 + Y_2), \qquad P_1 = \frac{1}{\sqrt{2}}(y^2 - Y_2), \qquad F = -y^1, \tag{6.66}$$

that from (3.48) gives the following non-zero entries for the pairing

$$\langle K, F \rangle = 1, \qquad \langle P_0, P_0 \rangle = 1, \qquad \langle P_1, P_1 \rangle = -1. \tag{6.67}$$

Table 6.4: The two non-equivalent DD Lie algebras which are isomorphic to the extended (1+1) Poincaré algebra.

	Case 0	Case 1
$[Y_1, Y_2]$	Y_2	Y_2
$[y^1, y^2]$	0	y^1
$[y^1, Y_1]$	0	$-Y_2$
$[y^1, Y_2]$	0	0
$[y^2, Y_1]$	y^2	$y^2 + Y_1$
$[y^2, Y_2]$	$-y^1$	$-y^1$

By inserting the inverse isomorphism in (3.53) we obtain the classical r-matrix

$$r_0 = \sum_{i=1}^{2} y^i \otimes Y_i = K \otimes F + \frac{1}{2} \left(P_0 \otimes P_0 - P_1 \otimes P_1 + P_0 \wedge P_1 \right), \tag{6.68}$$

and by subtracting the tensorized Casimirs C_1 (6.58) we obtain the skew-symmetric r-matrix

$$r_0' = \frac{1}{2}(K \wedge F + P_0 \wedge P_1). \tag{6.69}$$

The DD cocommutator reads

$$\begin{aligned} &\delta_D(K) = 0, \qquad \delta_D(F) = 0, \\ &\delta_D(P_0) = -\tfrac{1}{2} \left(P_0 \wedge F + P_1 \wedge F \right), \\ &\delta_D(P_1) = -\tfrac{1}{2} \left(P_0 \wedge F + P_1 \wedge F \right). \end{aligned} \tag{6.70}$$

Since $\delta_D(K) = \delta_D(F) = 0$, we have that, trivially, $\delta_D(\bar{\mathfrak{l}}) \subset \bar{\mathfrak{l}} \wedge \bar{\mathfrak{l}}$ and the associated Poisson Minkowski spacetime is a Poisson subgroup one. Obviously, $\delta_D(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{l}$ and the Poisson extended Minkowski spacetime is of Poisson subgroup type as well.

A two-parameter generalization of the r-matrix (6.69) fulfilling the modified CYBE is given by

$$r'_{0,(\alpha_0,\beta_0)} = \alpha_0 K \wedge F + \beta_0 P_0 \wedge P_1, \tag{6.71}$$

with $\alpha_0, \beta_0 \in \mathbb{R}$. The associated fundamental Poisson brackets for the coordinates $\{x^0, x^1, \phi\}$ (see (6.64)) are obtained from the Sklyanin bracket and turn out to be

$$\{x^0, x^1\} = 0, \qquad \{\phi, x^0\} = \beta_0 x^0 + \alpha_0 x^1, \qquad \{\phi, x^1\} = \alpha_0 x^0 + \beta_0 x^1, \tag{6.72}$$

which are linear and can be straightforwardly quantized. Hence, although the Minkowski coordinates $\{x^0, x^1\}$ Poisson-commute, the extended Minkowski spacetime $\{x^0, x^1, \phi\}$ defines a noncommutative structure, thus suggesting that the role of central extensions in noncommutative spacetimes deserves a deeper analysis along the lines presented in [234]. Notice also that for the DD spacetime, obtained when $\alpha_0 = \beta_0$, the Poisson algebra (6.72) is isomorphic to $\mathfrak{b}_2 \oplus \mathbb{R}$ such that $\mathfrak{b}_2 = \text{span}\{\phi, x^0 + x^1\}$ and $\mathbb{R} = \text{span}\{x^0 - x^1\}$.

6.5.3 Case 1

The Lie algebra isomorphism is now given by

$$K = -Y_1, \quad P_0 = \frac{1}{\sqrt{2}}(y^2 + Y_1 + Y_2), \quad P_1 = \frac{1}{\sqrt{2}}(y^2 + Y_1 - Y_2), \quad F = -y^1 + Y_2, \quad (6.73)$$

and the pairing is exactly (6.67).

The inverse of the above isomorphism inserted into (3.53) gives rise to the classical r-matrix

$$r_1 = F \otimes K + \frac{1}{\sqrt{2}}(K \wedge P_0 + P_1 \wedge K) + \frac{1}{2}(P_1 \wedge P_0 + P_0 \otimes P_0 - P_1 \otimes P_1), \quad (6.74)$$

and by subtracting the tensorized Casimir C_1 (6.58) we obtain the skew-symmetric r-matrix

$$r_1' = \frac{1}{2}(F \wedge K + P_1 \wedge P_0) + \frac{1}{\sqrt{2}}(P_1 \wedge K + K \wedge P_0). \quad (6.75)$$

The DD cocommutator is

$$\begin{aligned}
\delta_D(K) &= \frac{1}{\sqrt{2}}(-K \wedge P_0 + K \wedge P_1), \qquad \delta_D(F) = 0, \\
\delta_D(P_0) &= \frac{1}{\sqrt{2}}(P_0 \wedge P_1 + K \wedge F) + \frac{1}{2}(P_0 \wedge F + P_1 \wedge F), \\
\delta_D(P_1) &= \frac{1}{\sqrt{2}}(P_0 \wedge P_1 + K \wedge F) + \frac{1}{2}(P_0 \wedge F + P_1 \wedge F),
\end{aligned} \quad (6.76)$$

which is of 'true' coisotropic type for both Minkowski and extended Minkowski spacetimes (6.64).

The r-matrix (6.75) can be generalized to the following two-parameter solution of the modified CYBE

$$r'_{1,(\alpha_1,\beta_1)} = \alpha_1(K \wedge F + P_0 \wedge P_1) + \beta_1(K \wedge P_1 + P_0 \wedge K), \quad (6.77)$$

with $\alpha_1, \beta_1 \in \mathbb{R}$ (the DD case corresponds to set $\beta_1 = \sqrt{2}\alpha_1$). The associated fundamental Poisson brackets for the coordinates x^0, x^1, ϕ are given by

$$\begin{aligned}
\{x^0, x^1\} &= -\beta_1(x^0 + x^1), \\
\{\phi, x^0\} &= \alpha_1(x^0 + x^1) + \tfrac{1}{2}\beta_1\left(x^0 + x^1\right)^2, \\
\{\phi, x^1\} &= \alpha_1(x^0 + x^1) + \tfrac{1}{2}\beta_1(x^0 + x^1)(x^0 - x^1).
\end{aligned} \quad (6.78)$$

Therefore the first bracket defines the Poisson Minkowski spacetime and can be trivially quantized. We remark that this comes from the β_1-term in (6.77), which is a solution of the CYBE, that corresponds to the so-called 'null-plane' noncommutative Minkowski spacetime studied in [237] (see [238] for its (3+1) generalization). In light-cone coordinates $x^\pm = x^0 \pm x^1$ the Poisson structure takes the simpler form

$$\{x^+, x^-\} = 2\beta_1 x^+. \quad (6.79)$$

Interestingly enough, the Poisson extended Minkowski spacetime defined by the three brackets (6.78) is quadratic and cannot be straightfordwardly quantized. Notice that the α_1-term in the r-matrix (6.77) is just the one for the above Case 0 (6.69) and, consequently, (6.77) can be regarded either as a generalization of Case 0 with additional deformation parameter β_1, or as a generalized 'null-plane' Minkowski spacetime with additional parameter α_1.

6.6 Drinfel'd double Euclidean r-matrices and Poisson homogeneous spaces

Once the complete set of DD structures for the Poincaré group has been clarified, it certainly makes sense to study the DD structures for the Euclidean group in 3-dimensions, both because of its inherent interest related with the construction of Poisson Euclidean spaces and also for the possibility of comparing DD structures for these two closely related Lie algebras. So let us consider the Euclidean Lie algebra $\mathfrak{e}(3) = \mathfrak{iso}(3)$ in terms of generators of rotations J_i and translations P_i $(i = 1, 2, 3)$. The commutation rules read

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, P_j] = \epsilon_{ijk} P_k, \qquad [P_i, P_j] = 0, \qquad i, j, k = 1, 2, 3. \tag{6.80}$$

The two quadratic Casimir elements for this algebra are given by

$$C_1 = P_1^2 + P_2^2 + P_3^2, \qquad C_2 = J_1\, P_1 + J_2\, P_2 + J_3\, P_3. \tag{6.81}$$

The Euclidean space in three dimensions, E^3, can be constructed as the homogeneous space of the Euclidean isometry group $ISO(3) = E(3)$ having the subgroup $H = SO(3)$ as the isotropy subgroup of the origin, that is, $E^3 \equiv ISO(3)/SO(3)$. Hence we have that $\mathfrak{a} = \mathfrak{iso}(3) = \mathfrak{e}(3) = \operatorname{span}\{J_1, J_2, J_3, P_1, P_2, P_3\}$, and $\mathfrak{h} = \operatorname{Lie}(H) = \mathfrak{so}(3) = \operatorname{span}\{J_1, J_2, J_3\}$. If we denote $\mathfrak{t} = \operatorname{span}\{P_1, P_2, P_3\}$, we have that $\mathfrak{e}(3) = \mathfrak{h} \oplus \mathfrak{t}$ as a vector space.

According to the classification [225] there is no 'non-trivial' DD structure for $E(3)$. However, $E(3)$ has the 'trivial' DD structure induced by its semidirect product form. Notice that $E(3) = SO(3) \ltimes \mathbb{R}^3$ is the semidirect product of the rotation subgroup and the translations, inherited by its Lie algebra $\mathfrak{e}(3) = \mathfrak{so}(3) \oplus_S \mathbb{R}^3$, and this is just the DD structure arising in correspondence with the Lie bialgebra structure $(\mathfrak{g}, \delta) = (\mathfrak{so}(3), \delta \equiv 0)$. It is straightforward to check that such unique DD structure for $\mathfrak{e}(3)$ is given by the isomorphism $Y_i = J_i$ and $y^i = P_i$, and thus we have $c_{ij}^k = \epsilon_{ijk}$ and $f_j^{ik} = 0$. In this way we obtain

$$r = \sum_i P_i \otimes J_i\,, \qquad C_2 = \sum_i J_i \otimes P_i\,,$$

so the skew-symmetric component of the r-matrix reads

$$r' = r - C_2 = \sum_i P_i \wedge J_i, \tag{6.82}$$

while the induced pairing has as non-vanishing entries $\langle P_i, J_i \rangle = 1$. This DD structure is the Euclidean analogue of the Case 0 in the Poincaré clasification, and is directly related to the semidirect product structure of both Lie groups. The striking difference between the Euclidean and the Poincaré Lie groups is that while in the latter there is a plurality of DD structures (eight non-isomorphic ones), in the former only one does exist.

With the classification of r-matrices for the three-dimensional Euclidean Lie group [166] at hand, which is given in Appendix A, we can easily identify the only three-dimensional Euclidean DD r-matrix (6.82) with the one in Class (II) (A.14) in [166]. Quite interestingly, exactly as it happened for the Poincaré case, the 'trivial' DD r-matrix is the one corresponding to the only non-parametric family of coboundary PL structures on the three-dimensional Euclidean group.

Regarding different dimensions, no DD structure exists for the Euclidean group. In higher dimensions than three, this is due to the lack of existence of a non-degenerate associative symmetric bilinear form. In the two-dimensional case, the statement follows because there are only three non-isomorphic DD structures [225], two of them isomorphic to the non-trivially centrally extended Poincaré group and the other one isomorphic to the non-trivially centrally extended AdS group. Therefore, no centrally extended Euclidean group can be endowed with a DD.

6.7 Contraction of Drinfel'd double r-matrices from $\mathfrak{so}(3,1)$

The complete study of DD structures for the Lie algebra $\mathfrak{so}(3,1)$ was carried out in [110]. In what follows we analyze the contraction of such structures to the Euclidean case. This contraction procedure can be understood in geometric terms as the zero-curvature limit of the three-dimensional hyperbolic space whose isometry group is just $SO(3,1)$.

The classification of $\mathfrak{so}(3,1)$ DD r-matrices in [110] was performed in the usual Chern-Simons basis $\{J_0, J_1, J_2, P_0, P_1, P_2\}$. It turns out that there are four DDs for $\mathfrak{so}(3,1)$, called cases A, B, C and D in [110]. For the cases A and C the relationship with the geometrical basis used throughout the present paper is established by means of the isomorphism given by

$$J_0 \to -J_1, \quad J_1 \to \tfrac{1}{\eta} P_3, \quad J_2 \to \tfrac{1}{\eta} P_2, \quad P_0 \to P_1, \quad P_1 \to \eta\, J_3, \quad P_2 \to \eta\, J_2, \tag{6.83}$$

where η is a non-zero real parameter. And for the cases B and D the isomorphism reads

$$J_s \to J_{s+1}, \qquad P_s \to P_{s+1}, \qquad s = 0,1,2. \tag{6.84}$$

By applying these two isomorphisms, we find that the commutation relations for the Lie algebra $\mathfrak{so}(3,1)$ adopt the form

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \qquad [J_i, P_j] = \varepsilon_{ijk} P_k, \qquad [P_i, P_j] = -\eta^2 \varepsilon_{ijk} J_k, \qquad i,j,k = 1,2,3. \tag{6.85}$$

In this basis, the two quadratic Casimir elements for $\mathfrak{so}(3,1)$ can be written as

$$C_1 = P_1^2 + P_2^2 + P_3^2 - \eta^2 \big(J_1^2 + J_2^2 + J_3^2\big), \qquad C_2 = J_1\, P_1 + J_2\, P_2 + J_3\, P_3. \tag{6.86}$$

Now we consider the three-dimensional hyperbolic space as the homogeneous space of the isometry group $SO(3,1)$ with isotropy subgroup $H = SO(3)$, $H^3 \equiv SO(3,1)/SO(3)$, provided that $\mathfrak{a} = \mathfrak{so}(3,1) = \text{span}\{J_1, J_2, J_3, P_1, P_2, P_3\}$ and $\mathfrak{h} = \text{Lie}(H) = \mathfrak{so}(3) = \text{span}\{J_1, J_2, J_3\}$. The hyperbolic space H^3 has negative constant sectional curvature equal to $-\eta^2$, so that the parameter η is related with the radius of the space R through $\eta = 1/R$. The 'flat' contraction to the Euclidean algebra and space thus corresponds to applying the limit $\eta \to 0$ ($R \to \infty$). In this manner, the commutation rules 6.85 and Casimirs (5.27) reduce to the Euclidean ones (6.80) and (6.81), respectively.

Next, by using (6.83) and (6.84) in the results given in [110] we obtain the following four DD r-matrices for $\mathfrak{so}(3,1)$:

$$
\begin{aligned}
r'_{\mathrm{A}} &= \tfrac{1}{\eta}\, P_3 \wedge P_2 + \tfrac{1}{2}(P_1 \wedge J_1 - P_2 \wedge J_2 - P_3 \wedge J_3), \\
r'_{\mathrm{B}} &= -\eta J_2 \wedge J_3 + \tfrac{1}{2}(P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3), \\
r'_{\mathrm{C}} &= \tfrac{1}{2}\left(\tfrac{1}{\eta}\, P_3 \wedge P_1 + \eta J_1 \wedge J_3 + P_2 \wedge J_2\right), \\
r'_{\mathrm{D}} &= J_1 \wedge P_2 - J_2 \wedge P_1 + \frac{(1+\mu^2)}{2\mu}\, P_3 \wedge J_3 + \frac{(\mu^2-1)}{2\eta\mu}\left(\eta^2 J_1 \wedge J_2 - P_1 \wedge P_2\right), \qquad \mu > 0.
\end{aligned}
\tag{6.87}
$$

In principle, only the r-matrix r'_{B} has a well defined flat limit $\eta \to 0$. Nevertheless, we can scale the remaining cases in order to obtain four contracted Euclidean r-matrices; these are

$$
\begin{aligned}
&\lim_{\eta\to 0} \eta\, r'_{\mathrm{A}} = P_3 \wedge P_2, \qquad && \lim_{\eta\to 0} r'_{\mathrm{B}} = \tfrac{1}{2}(P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3), \\
&\lim_{\eta\to 0} \eta\, r'_{\mathrm{C}} = \frac{1}{2}\, P_3 \wedge P_1, \qquad && \lim_{\eta\to 0} \eta\, r'_{\mathrm{D}} = \frac{(1-\mu^2)}{2\mu}\, P_1 \wedge P_2, \quad \mu > 0.
\end{aligned}
\tag{6.88}
$$

Consequently, the cases A, C and D give rise to Euclidean r-matrices belonging to Class (III) (A.15) of [166], meanwhile the case B gives exactly the complete Class (II) (A.14). Note that this Class is the one obtained from the DD structure for the Euclidean group (6.82). We recall that previously in this Chapter (see [169]), it was found that four (A)dS (two for dS and two for AdS) DD r-matrices contract to two DD Poincaré r-matrices (Cases 0 and 2).

We remark that the initial pairing for the cases A, B and C for the $\mathfrak{so}(3,1)$ DD structures in the basis (6.87) has the non-vanishing entries $\langle P_i, J_i \rangle = 1$, which does not depend on η, so that it remains unchanged under contraction. In contrast, the pairing for case D diverges under the limit $\eta \to 0$.

6.8 Euclidean Poisson homogeneous spaces

In the previous sections we have studied the DD structures for the Euclidean group in three dimensions and we have identified to which of the coboundary PL structures they correspond. Now we present the full construction of Poisson homogeneous Euclidean spaces, based in the classification presented in Apendix A. In order to perfom that, we first need to introduce suitable coordinates on the Euclidean group and we write a generic element Q of the Lie algebra $\mathfrak{e}(3)$ as

$$Q = x^1 P_1 + x^2 P_2 + x^3 P_3 + \theta^1 J_1 + \theta^2 J_2 + \theta^3 J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \tag{6.89}$$

Next we introduce the coordinates on the Lie group as the ones associated to each Lie algebra generator through the exponential map

$$g = \exp(x^1 P_1) \exp(x^2 P_2) \exp(x^3 P_3) \exp(\theta^1 J_1) \exp(\theta^2 J_2) \exp(\theta^3 J_3),$$

and then we compute left- and right-invariant vector fields in these coordinates, thus obtaining

$$\begin{aligned}
X^L_{J_1} =& \frac{\cos\theta^3}{\cos\theta^2}\left(\partial_{\theta^1} - \sin\theta^2 \partial_{\theta^3}\right) + \sin\theta^3 \partial_{\theta^2}, \\
X^L_{J_2} =& \frac{\sin\theta^3}{\cos\theta^2}\left(-\partial_{\theta^1} + \sin\theta^2 \partial_{\theta^3}\right) + \cos\theta^3 \partial_{\theta^2}, \\
X^L_{J_3} =& \partial_{\theta^3}, \\
X^L_{P_1} =& \cos\theta^2 \cos\theta^3 \partial_{x^1} + \left(\sin\theta^1 \sin\theta^2 \cos\theta^3 + \cos\theta^1 \sin\theta^3\right) \partial_{x^2} - \\
& - \left(\cos\theta^1 \sin\theta^2 \cos\theta^3 - \sin\theta^1 \sin\theta^3\right) \partial_{x^3}, \\
X^L_{P_2} =& -\cos\theta^2 \sin\theta^3 \partial_{x^1} - \left(\sin\theta^1 \sin\theta^2 \sin\theta^3 - \cos\theta^1 \cos\theta^3\right) \partial_{x^2} + \\
& + \left(\cos\theta^1 \sin\theta^2 \sin\theta^3 + \sin\theta^1 \cos\theta^3\right) \partial_{x^3}, \\
X^L_{P_3} =& \sin\theta^2 \partial_{x^1} - \cos\theta^2 \left(\sin\theta^1 \partial_{x^2} - \cos\theta^1 \partial_{x^3}\right),
\end{aligned} \tag{6.90}$$

for the left-invariant vector fields, and

$$\begin{aligned}
X^R_{J_1} &= -x^3 \partial_{x^2} + x^2 \partial_{x^3} + \partial_{\theta^1}, \\
X^R_{J_2} &= x^3 \partial_{x^1} - x^1 \partial_{x^3} + \cos\theta^1 \partial_{\theta^2} + \frac{\sin\theta^1}{\cos\theta^2}\left(\sin\theta^2 \partial_{\theta^1} - \partial_{\theta^3}\right), \\
X^R_{J_3} &= -x^2 \partial_{x^1} + x^1 \partial_{x^2} + \sin\theta^1 \partial_{\theta^2} + \frac{\cos\theta^1}{\cos\theta^2}\left(-\sin\theta^2 \partial_{\theta^1} + \partial_{\theta^3}\right), \\
X^R_{P_1} &= \partial_{x^1}, \\
X^R_{P_2} &= \partial_{x^2}, \\
X^R_{P_3} &= \partial_{x^3},
\end{aligned} \tag{6.91}$$

for the right-invariant vector fields. Thus we have all the ingredients to study explicitly the Poisson homogeneous Euclean spaces. The final result is as follows.

Class (II). This is the only r-matrix coming from a DD structure. Its cocommutator reads

$$\delta(J_i) = 0, \qquad \delta(P_1) = 2P_2 \wedge P_3, \qquad \delta(P_2) = -2P_1 \wedge P_3, \qquad \delta(P_3) = 2P_1 \wedge P_2, \tag{6.92}$$

which shows that its associated PHS is of Poisson subgroup type in a trivial way (this is consistent with the fact that the r-matrix (A.14) is the analogue of the Poincaré Case 0 studied previously in this Chapter, see Table 6.2). The associated PHS is given by the fundamental Poisson bracket

$$\{x^i, x^j\} = 2\varepsilon_{ijk}x^k, \tag{6.93}$$

which is so isomorphic to the $\mathfrak{so}(3)$ Lie algebra, while its Poincaré counterpart was isomorphic to $\mathfrak{so}(2,1)$, as shown in Table 6.3. As a matter of fact, if we compute the full Sklyanin bracket we get that the remaining group coordinates Poisson commute $\{x^i, \theta^j\} = \{\theta^i, \theta^j\} = 0$.

Class (III). This family of r-matrices are solutions of the CYBE, and its cocommutator reads

$$\begin{aligned} \delta_a(J_1) &= -a_{13}P_1 \wedge P_2 + a_{12}P_1 \wedge P_3, \\ \delta_a(J_2) &= -a_{23}P_1 \wedge P_2 + a_{12}P_2 \wedge P_3, \qquad \delta_a(P_i) = 0, \\ \delta_a(J_3) &= -a_{23}P_1 \wedge P_3 + a_{13}P_2 \wedge P_3. \end{aligned} \tag{6.94}$$

This cocommutator is not coisotropic with respect to the isotropy subgroup of rotations (apart from the trivial case $r = a = 0$), and we shall not write down the Poisson brackets for the group coordinates as we are only interested in describing coisotropic PHS.

Class (I). This multiparametric family of r-matrices is composed by solutions of the form of Class (III) plus some new terms. They satisfy the CYBE iff $\alpha = 0$. The cocommutator reads

$$\begin{aligned} &\delta(J_1) = \alpha(-P_3 \wedge J_1 + P_1 \wedge J_3) - \rho(P_3 \wedge J_2 + P_2 \wedge J_3) + \delta_a(J_1), \\ &\delta(J_2) = \alpha(-P_3 \wedge J_2 + P_2 \wedge J_3) + \rho(P_3 \wedge J_1 + P_1 \wedge J_3) + \delta_a(J_2), \qquad \delta(J_3) = \delta_a(J_3), \\ &\delta(P_1) = \alpha P_1 \wedge P_3 + \rho P_2 \wedge P_3, \qquad \delta(P_2) = \alpha P_2 \wedge P_3 - \rho P_1 \wedge P_3, \qquad \delta(P_3) = 0, \end{aligned} \tag{6.95}$$

proving that they are coisotropic deformations if the a-terms vanish, but they are never of Poisson subgroup type. So, in the case with $a = 0$ the associated Poisson Euclidean spaces read

$$\{x^1, x^2\} = 0, \qquad \{x^1, x^3\} = \alpha x^1 - \rho x^2, \qquad \{x^2, x^3\} = \alpha x^2 + \rho x^1. \tag{6.96}$$

The distinguished behavior of the third coordinate becomes evident both in the cocommutator and in the Poisson bracket, in contrast what happens in the DD noncommutative space (6.93).

In this way we have exhausted the construction of Poisson homogeneous Euclidean spaces, and it has been shown how two of the three classes of coboundary PL structures on the Euclidean group generate coisotropic Poisson homogeneous spaces, one of them being the only DD structure.

6.9 Remarks

In this Chapter we have presented the DD structure for the Poincaré group in (2+1) dimensions, for its non-trivial central extension in (1+1) dimensions and for the Euclidean group in three dimensions. For every nonisomorphic DD structure we have worked out an explicit isomorphism between the canonical basis in the double and the Poincaré or Euclidean kinematical basis. Moreover, for the cases in which the Poisson-Lie on the group satisfy the coisotropy condition (3.79) with respect to the relevant subgroup, we have constructed the associated DD Poisson spaces. In this way, we have obtained five (2+1)-dimensional Poisson Minkowski spacetimes (see Table (6.3)), two 2-dimensional (possibly extended) Poisson Minkowski spacetimes and only one 3-dimensional Riemannian Poisson space. All these Poisson spaces have been constructed from the canonical Poisson-Lie structure on the respective group provided by the Sklyanin bracket for the canonical quasitriangular r-matrix arising from the DD structures. These results complete the chart of DD structures for the Poincaré and Euclidean groups, since they do not exist in $(N+1)$ dimensions with $N \geq 3$ (Poincaré) or in N-dimensions with $N \geq 4$ (Euclidean). Recall that DD structures on the 2 dimensional extended Euclidean group are not possible since the Poincaré and (A)dS ones exhaust the three non-isomorphic 4-dimensional DD structures, as explained above. An easy way to see that these higher dimensional Lie groups do not admit DD structures is by noticing that their Lie algebras do not admit associative scalar products (see Theorem 3.5). The existence of these scalar products was studied for higher dimensional kinematical groups in [203].

In the (2+1)-dimensional Lorentzian and in the 3-dimensional Euclidean cases, DD structures have a remarkable interest due to their connection with the Chern-Simons approach to gravity in (2+1) dimensions or its euclidean version, in which the gauge group is identified with the isometry group. Indeed, it is the existence of a DD structure that guarantees the Fock-Rosly conditions for the r-matrix defining the Poisson structure on the phase space of the theory. We have explicitly constructed the eight nonisomorphic DD structures on $\mathfrak{iso}(2,1)$, displayed in Table 6.2, meanwhile in [110] the four nonisomorphic DD structures for $\mathfrak{so}(3,1)$ and the three ones for $\mathfrak{so}(2,2)$ were deduced. We have also presented the only DD structure on $\mathfrak{iso}(3)$. Therefore, the results presented in this Chapter complete the study of DD structures on the three Lorentzian kinematical groups in (2+1) dimensions, and shows how to connect Poincaré DD structures with (A)dS ones through the contraction induced by the vanishing cosmological constant limit $\Lambda \to 0$, and similarly for the Euclidean version, in which the relevant contraction is from the DD structures on the group of isometries of the 3-dimensional hyperbolic space, when the curvature of this space tends to zero. In particular, it is found that only two of the eight Poincaré DD r-matrices (Cases 0 and 2) can be obtained through such a contraction procedure. For the Euclidean case, the only DD r-matrix can also be obtained through contraction.

As mentioned before, for each DD structure we have also investigated whether the corresponding Lie bialgebra was coisotropic with respect to the relevant subalgebra, i.e. if the condition $\delta_D(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$, and so it defines a Poisson homogeneous space. For the (2+1) Poincaré case, the result is that five out of the eight DD structures do fulfill this condition, as shown in Table 6.2, thus providing five (noncommutative) Poisson Minkowski spacetimes. Only two of them (Cases 0 and 1) fulfill the stronger requirement of being generated by a Lorentz Poisson subgroup, i.e $\delta_D(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$. Case 0 is the DD obtained from a $\mathfrak{sl}(2,\mathbb{R})$ Lie bialgebra with a trivial cocommutator map, and was previously known in the (2+1) quantum gravity literature. Interestingly enough, Case 1 gives rise to a quadratic Poisson Minkowski spacetime which, to the best of our knowledge, has not been considered previously in the literature and whose quantization deserves further study. The remaining three coisotropic DD structures (Cases 2, 6 and 7) provide noncommutative spacetimes of Lie-algebraic type, whose quantization is straightforward. For the Euclidean case, the only DD structure satisfy the coisotropy condition and defined a Poisson homogeneous space of Lie-algebraic type. The Euclidean DD (the 'rotation double' or the '$\mathfrak{su}(2)$ double' [163, 227, 228, 229]) is the analogue of the Case 0 Poincaré DD (the 'Lorentz double' [123]). These DD structures are structurally similar since both of them are canonically induced by the semidirect product structure of the group of isometries (see Theorem 3.7). Regarding the rest of the DD structures, it is worth stressing that the large plurality for the Poincaré group is lost in its Euclidean counterpart, and this is clearly due to the flexibility of the Lorentz sector in order to give rise to DDs.

In (1+1) dimensions, only three nonisomorphic DD structures do exist and two of them correspond to the centrally extended (1+1) Poincaré Lie algebra $\overline{\mathfrak{p}(1+1)}$. The remaining four-dimensional DD Lie algebra leads to $\overline{\mathfrak{so}(2,1)} \simeq \mathfrak{gl}(2,\mathbb{R})$ (see [235, 236]), which is the centrally extended Lie algebra of isometries of (1+1) (A)dS spacetimes. For the two Poincaré structures the coisotropy condition holds and (extended) Poisson Minkowski spacetimes can be explicitly constructed. Moreover, in Case 0 the noncommutative Minkowski space is generated by a commutative Poisson subgroup, while the extended space is of Lie-algebraic type. Case 1 gives rise to the 'null-plane' noncommutative Minkowski spacetime, and its extended version is again defined by a quadratic Poisson algebra. Therefore, the study of DD structures for (1+1) Lorentzian groups has been also completed.

Non-existence of DD structures for kinematical groups can be deduced by non-existence of a non-degenerate, symmetric, and 'associative' bilinear form, as mentioned previously. We stress that, in contradistinction with the rich variety of DD structures on the (2+1) Poincaré algebra, it is easy to see that in the (3+1) case no DD structures can be found, since no such bilinear form exists for $\mathfrak{iso}(3,1)$ (see [202, 201]). Furthermore, neither the static, nor the Galilean and Newton-Hooke kinematical algebras admit any DD structure in (2+1) and (3+1) dimensions, and that neither Carroll nor the Euclidean Lie algebras admit DD structures in (3+1) and 4 dimensions, respectively. Thus, we have that the Galilean limit of all the DD r-matrices for $\mathfrak{iso}(2,1)$ obtained in this paper would lead (in case that such a limit does converge) to Galilean r-matrices which would not come from a DD structure. In this respect, we also point out that DD structures for the twice extended (2+1) Galilei algebra do exist, and one of them was fully constructed in [239], thus providing a meaningful connection with previous quantum group models for Galilean (2+1) gravity [240, 53]. Indeed, it could happen that this 'exotic' Galilean DD could be obtained as the appropriate contraction of a relativistic DD structure for a twice (trivially) extended (2+1) Poincaré algebra, a problem that would also deserve some attention in the future.

Moreover, although no DD structure exists for $\mathfrak{iso}(3,1)$, the problem of the classification of DD structures on both $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$ is worth to be faced, and should be based in the classification of Lie bialgebra structures for 5-dimensional real Lie algebras. We recall that a first DD structure on $\mathfrak{so}(3,2)$ was fully worked out in the kinematical basis in [113, 114]. In fact, as proved in [203], for kinematical Lie algebras in (N+1) dimensions

($N \geq 4$), only for $\mathfrak{so}(N+1,1)$, $\mathfrak{so}(N+2)$ and $\mathfrak{so}(N,2)$ such a suitable non-degenerate bilinear form does exist. Therefore these three Lie algebras could admit DD structures in higher dimensions.

Chapter 7

Dual Poisson homogeneous spaces

The main aim of this Chapter is to study the dual notion, in the sense of Poisson-Lie groups, of the well-known theory of reductive and symmetric homogeneous spaces, that was introduced in §2.1.6 and §2.1.7 of Chapter 2, respectively. We also study the consequences of these two notions for the uncertainty relations arising noncommutative spacetimes.

More in detail, in §7.1 we introduce the motivating ideas in order to consider the notions treated in the rest of the Chapter. Section §7.2 is devoted to introduce the notion of coreductive and cosymmetric Lie bialgebras, and from them the notion of dual (reductive and symmetric) Poisson homogenous spaces $M^* = G^*/T^*$ will follow. As we will see in §7.3, these new concepts are meaningful for a novel approach to the Lie bialgebra structures for Lorentzian Lie algebras, since coisotropy and coreductivity conditions provide strong constraints on the r-matrices generating (A)dS and Poincaré Lie bialgebras in (2+1) and (3+1) dimensions. In particular, the well-known κ-deformation of the (A)dS and Poincaré Lie algebras, introduced in Chpater 4, will be analyzed from this viewpoint. The fact that all the dual spaces for the κ-deformation cannot be endowed with a G^*-invariant metric, leads to the consideration of alternative approaches in order to unveil some of their geometric properties.

With this aim, in §7.4 we discuss the geometry of dual PHS from the viewpoint of K-structures on manifolds, which allows the definition of their curvature, torsion and Ricci tensors. Finally, in Section 7.5 the connection between the coreductivity condition for a given Lie bialgebra and the properties of the uncertainty relations that would arise from the noncommutative spacetime coordinates of the associated quantum homogeneous space is discussed in terms of the representation theory of the full dual algebra $\mathfrak{g}^*$ and its restriction to the first order noncommutative space. In §7.6 some final remarks close the Chapter.

7.1 Introduction

Let us consider a connected Lie group G and one of its Lie subgroups H, that for simplicity we assume that is connected. Recall from Definition 2.15 that the smooth manifold M is called a G-homogeneous space if it is equipped with a transitive action $\alpha : G \times M \to M$

(see Definition 2.5 and (2.10)). If we set $\mathfrak{g} = \mathrm{Lie}(G)$, its subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$ and its complement $\mathfrak{t}$ then, as vector spaces,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}\,, \tag{7.1}$$

and the most generic Lie brackets for a Lie algebra $\mathfrak{g}$ with decomposition (7.1) will be of the form

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{t}] \subset \mathfrak{h} + \mathfrak{t}\,, \qquad [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{h} + \mathfrak{t}\,. \tag{7.2}$$

Recall from Definition 2.18 of Chapter 2, that a homogeneous space $M = G/H$ is said to be a reductive space if the splitting (7.1) is left invariant by the adjoint action of H, which means that (recall that we assume H to be connected)

$$[\mathfrak{h},\mathfrak{t}] \subset \mathfrak{t}\,. \tag{7.3}$$

In other words, this means that $\mathfrak{t}$ is an H-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Finally, if in addition to (7.3), the following relation holds

$$[\mathfrak{t},\mathfrak{t}] \subset \mathfrak{h}\,, \tag{7.4}$$

then $M = G/H$ is called a symmetric homogeneous space (see Definition 2.20), which ensures that the reductive splitting (7.1) is endowed with a $\mathbb{Z}_2$-grading assigning grade 0 to elements of $\mathfrak{h}$ and 1 to elements of $\mathfrak{t}$, defined by means of the involutive automorphism

$$\sigma(\mathfrak{t}) = -\mathfrak{t} \qquad \text{and} \qquad \sigma(\mathfrak{h}) = \mathfrak{h}\,. \tag{7.5}$$

(see [175, 173] for a complete treatment on the subject). Summarizing, each one of the three conditions listed above (isotropy with respect to H, reductivity and symmetry) removes one specific type of contribution to the most generic Lie algebra brackets for $\mathfrak{g}$, namely:

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} + \cancel{\mathfrak{t}}_{\text{ subgroup}}\,, \qquad [\mathfrak{h},\mathfrak{t}] \subset \cancel{\mathfrak{h}}_{\text{ reductive space}} + \mathfrak{t}\,, \qquad [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{h} + \cancel{\mathfrak{t}}_{\text{ symmetric space}}\,. \tag{7.6}$$

As explained in §3.5 of Chapter 3, in order to endow M with a PHS π, we start from a Poisson-Lie structure Π onto G and we impose that the homogeneous space action $\alpha : G \times M \to M$ is a Poisson map. Moreover, the (noncommutative) Poisson homogeneous structure π onto M can be obtained as the canonical projection of Π provided that the unique Lie bialgebra $(\mathfrak{g},\delta)$ associated to (G,Π) is coisotropic with respect to the Lie algebra $\mathfrak{h}$ of H, namely, if

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \tag{7.7}$$

As we will see, this coisotropy condition can be interpreted as the Lie subalgebra condition for the dual translation generators $\mathfrak{t}^*$ within the dual Lie algebra $\mathfrak{g}^*$ induced by the Lie bialgebra cocommutator:

$$[\mathfrak{t}^*,\mathfrak{t}^*] \subset \mathfrak{t}^*. \tag{7.8}$$

By construction, the bracket (7.8) will be just the linearization of the (in general, nonlinear) algebra defining the Poisson homogeneous space (PHS) given by (M,π) (and, after quantization, of the quantum homogeneous space given by the comodule algebra M_h).

As stated before, the aim of this Chapter is to analyse the consequences of imposing the notions of (co)reductivity and (co)symmetry for the dual Lie algebra $\mathfrak{g}^*$ associated to a given Lie bialgebra structure $(\mathfrak{g},\delta)$. Firstly, imposing the Lie bialgebra $(\mathfrak{g},\delta)$ to be coreductive (*i.e.*, imposing $\mathfrak{g}^*$ to be reductive) will be tantamount to say that

$$[\mathfrak{t}^*,\mathfrak{h}^*]\subset\mathfrak{h}^*, \tag{7.9}$$

where $\mathfrak{h}^*$ is the dual Lie algebra of the isotropy subalgebra $\mathfrak{h}$ of M. Therefore, any coisotropic and coreductive Lie bialgebra $(\mathfrak{g},\delta)$ implies the existence of a reductive PHS $M^*=G^*/T^*$ which can be considered as the dual to $M=G/H$ through δ, where G^* is the dual Poisson-Lie group and T^* is the subgroup of G^* generated by the dual translation generators $\mathfrak{t}^*$. Note that the dimension of M^* (resp. M) is the one of the isotropy subgroup of M (resp. M^*).

The dual reductive space M^* can be thus considered to be 'paired' to the initial reductive space M through the Lie bialgebra $(\mathfrak{g},\delta)$, and its geometry can be characterized -for instance- by making use of the theory of K-structures (see Example 2.2), since in general the dual spaces M^* cannot be endowed with a G^*-invariant metric, but maybe some of them could be endowed with some interesting geometry. Moreover, we will show that coreductivity is essential for the representation theory of the (linearized) quantum homogeneous space (7.8), since the fact that $\mathfrak{g}^*$ is reductive implies that the restriction of the representations of the full $\mathfrak{g}^*$ onto the noncommutative space $\mathfrak{t}^*$ provides representations of the latter. In this way, coreductivity arises both as a relevant property for noncommutative spaces and also in order to introduce new non-trivial dual geometric objects. Finally, the space M^* turns out to be a symmetric reductive space provided that $\mathfrak{g}^*$ is a symmetric reductive Lie algebra, which implies that

$$[\mathfrak{h}^*,\mathfrak{h}^*]\subset\mathfrak{t}^*. \tag{7.10}$$

This notion of cosymmetry completes the duality framework between M and M^* and, when satisfied, implies that the canonical connection on M^* is torsionless (see Theorem 2.7).

7.2 Coreductivity, cosymmetry and dual homogeneous spaces

If we consider PL structures on a group G with reductive Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{t}$, then the generic form for a given cocommutator δ will be

$$\begin{aligned}\delta(\mathfrak{h})&\subset\mathfrak{h}\wedge\mathfrak{h}+\mathfrak{h}\wedge\mathfrak{t}+\mathfrak{t}\wedge\mathfrak{t}\,,\\ \delta(\mathfrak{t})&\subset\mathfrak{h}\wedge\mathfrak{h}+\mathfrak{h}\wedge\mathfrak{t}+\mathfrak{t}\wedge\mathfrak{t}\,,\end{aligned} \tag{7.11}$$

onto which the co-Jacobi and cocycle conditions have to be imposed (conditions (B2) and (B3) from Definition 3.6 in §3.3). Note that the notion of Lie bialgebra is self-dual: for any Lie bialgebra $(\mathfrak{g},\delta)$ there exists a dual Lie bialgebra $(\mathfrak{g}^*,\eta)$ where $\mathfrak{g}^*$ is defined by ${}^t\delta$, and the dual cocommutator map η is given by dualizing the Lie algebra relations in $\mathfrak{g}$.

Essentially, the correspondence between PL groups and Lie bialgebras is based on the fact that the dual map ${}^t\delta$ coincides with the linearization (in terms of the local coordinates on G) of the PL bracket Π. This can be made explicit by introducing a basis for the dual Lie algebra $\mathfrak{g}^*$ given by $\mathfrak{h}^* = \mathrm{span}\{\hat{\xi}^i\}$ and $\mathfrak{t}^* = \mathrm{span}\{\hat{x}^j\}$ together with the following pairing with the generators of $\mathfrak{g}$, where $\mathfrak{h} = \mathrm{span}\{H_i\}$ and $\mathfrak{t} = \mathrm{span}\{T_j\}$:

$$\langle \hat{\xi}^i, H_j\rangle = \delta^i_j\,, \qquad \langle \hat{\xi}^i, T_j\rangle = 0\,, \qquad \langle \hat{x}^i, H_j\rangle = 0\,, \qquad \langle \hat{x}^i, T_j\rangle = \delta^i_j\,. \tag{7.12}$$

In such a basis, cocommutators (7.11) imply the following dual Lie bracket δ^*:

$$[\mathfrak{h}^*, \mathfrak{h}^*] \subset \mathfrak{h}^* + \mathfrak{t}^*\,, \qquad [\mathfrak{h}^*, \mathfrak{t}^*] \subset \mathfrak{h}^* + \mathfrak{t}^*\,, \qquad [\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{h}^* + \mathfrak{t}^*\,, \tag{7.13}$$

which are obviously subjected to the nonlinear equations for the structure constants that arise from Jacobi identity (or co-Jacobi if we think in terms of δ).

Remember from Definition 3.18 that a necessary condition for a PHS is the coisotropy condition (3.79), namely

$$\delta(\mathfrak{h}) \subset \mathfrak{h}\wedge\mathfrak{h} + \mathfrak{h}\wedge\mathfrak{t} + \cancel{\mathfrak{t}\wedge\mathfrak{t}}. \tag{7.14}$$

In terms of the dual Lie algebras it is tantamount to say that $\mathfrak{t}^*$ is a sub-Lie algebra, so we have that

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^*, \tag{7.15}$$

Therefore, the coisotropy condition stated as (7.15) is just a dual counterpart of the isotropy condition for the subalgebra $\mathfrak{h}$ and implies that the basis elements $\hat{x}^j$ dual to the translations T_i close a Lie subalgebra (7.15) within the dual Lie bracket $\mathfrak{g}^*$. As we will see in the sequel, coreductivity and cosymmetry can be thought of as further refinements of the notion of coisotropic Lie bialgebras that will arise when reductivity and symmetry are implemented at the level of the dual Lie algebra $\mathfrak{g}^*$.

7.2.1 Coreductive Lie bialgebras and dual reductive homogeneous spaces

By following this line of thought, the dual to the reductivity condition (7.3) for $\mathfrak{g}$, which we will call the coreductivity condition for δ, is obtained by imposing that no $\mathfrak{h}\wedge\mathfrak{t}$ term is contained in $\delta(\mathfrak{t})$, namely

$$\delta(\mathfrak{t}) \subset \mathfrak{h}\wedge\mathfrak{h} + \cancel{\mathfrak{h}\wedge\mathfrak{t}} + \mathfrak{t}\wedge\mathfrak{t}\,, \tag{7.16}$$

which admits a neat interpretation when expressed in terms of the dual Lie algebra $\mathfrak{g}^*$

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^* + \cancel{\mathfrak{h}^*}_{\text{coisotropy}}\,, \qquad [\mathfrak{t}^*, \mathfrak{h}^*] \subset \cancel{\mathfrak{t}^*}_{\text{coreductivity}} + \mathfrak{h}^*\,, \qquad [\mathfrak{h}^*, \mathfrak{h}^*] \subset \mathfrak{t}^* + \mathfrak{h}^*\,, \tag{7.17}$$

which is thus constrained to be a reductive Lie algebra. As we will see in Section 7, the coreductivity condition will give rise to a strong constraint for the uncertainty relations associated to the noncommutative coordinates on the quantum group G_q associated to δ.

As a consequence, each coisotropic and coreductive Lie bialgebra structure $(\mathfrak{g}, \delta)$ can be used to define a reductive homogeneous space which is 'dual' to $M = G/H$. Since the dual Lie algebra $\mathfrak{g}^*$ is of the form

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^*\,, \qquad [\mathfrak{t}^*, \mathfrak{h}^*] \subset \mathfrak{h}^*\,, \qquad [\mathfrak{h}^*, \mathfrak{h}^*] \subset \mathfrak{t}^* + \mathfrak{h}^*\,, \tag{7.18}$$

(plus the corresponding Jacobi identities), if we call G^* and T^* the Lie groups whose Lie algebras are given by $\mathfrak{g}^* \equiv \mathrm{span}\{\hat{x}, \hat{\xi}\}$ and $\mathfrak{t}^* \equiv \mathrm{span}\{\hat{x}\}$, respectively, we will define the dual reductive homogeneous space of M with respect to the coreductive Lie bialgebra $(\mathfrak{g}, \delta)$ as the coset space defined by $M^* = G^*/T^*$, where now T^* will play the role of the isotropy subgroup and the space M^* will be parametrized by the local coordinates associated to the dual Lie algebra generators $\hat{\xi}$, which we will denote as ξ^*. Note that the dimension of M^* is just the dimension of the vector space $\mathfrak{h}^*$, which in general does not coincide with the dimension of M.

Moreover, M^* is by construction a Poisson homogeneous space, whose Poisson bracket π^* on $\mathcal{C}^\infty(M^*)$ will be given by the canonical projection of the dual PL structure Π^* onto the ξ^* coordinates of M^*. Recall that Π^* has as its linearization the dual Lie bialgebra structure $(\mathfrak{g}^*, \eta)$ whose cocommutator map comes from the commutation rules of the reductive Lie algebra $\mathfrak{g}$ and is of the form

$$\begin{aligned} \eta(\mathfrak{t}^*) &\subset \mathfrak{t}^* \wedge \mathfrak{t}^* + \mathfrak{t}^* \wedge \mathfrak{h}^* \,, \\ \eta(\mathfrak{h}^*) &\subset \mathfrak{t}^* \wedge \mathfrak{t}^* + \mathfrak{h}^* \wedge \mathfrak{h}^* \,. \end{aligned} \tag{7.19}$$

Indeed, this Lie bialgebra is coisotropic for the subalgebra $\mathfrak{t}^*$ (which generates the isotropy subgroup of the dual space), since

$$\eta(\mathfrak{t}^*) \subset \mathfrak{t}^* \wedge \mathfrak{g}^*. \tag{7.20}$$

We stress that once the coreductivity condition (7.18) is imposed, the full construction is self-dual, and the dual PHS of M^* with respect to the Lie bialgebra $(\mathfrak{g}^*, \eta)$ will be just $M = G/H$. Thus, the coreductivity condition establishes a one-to-one correspondence (mediated by Lie bialgebras) between two different Poisson homogeneous spaces M and M^* with coordinates x and ξ^*, respectively, which have in general different dimensionality and geometric properties.

7.2.2 Cosymmetric Lie bialgebras

A further definition can be considered in a natural way, and we will say that a coreductive Lie bialgebra is cosymmetric if the dual reductive homogeneous space M^* is a symmetric space. This implies the existence of the following involutive automorphism σ^* leaving invariant the generators of the isotropy subgroup $\mathfrak{t}^*$, namely

$$\sigma^*(\mathfrak{t}^*) = \mathfrak{t}^* \,, \qquad \sigma^*(\mathfrak{h}^*) = -\mathfrak{h}^* \,. \tag{7.21}$$

Now, if the dual Lie algebra $\mathfrak{g}^*$ (7.18) has to be invariant under σ^*, then the cosymmetry condition for $(\mathfrak{g}, \delta)$ implies that the Lie brackets for $\mathfrak{g}^*$ have to be of the form

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^* \,, \qquad [\mathfrak{t}^*, \mathfrak{h}^*] \subset \mathfrak{h}^* \,, \qquad [\mathfrak{h}^*, \mathfrak{h}^*] \subset \mathfrak{t}^* \,, \tag{7.22}$$

which means that

$$\delta(\mathfrak{h}) \subset \cancel{\mathfrak{h} \wedge \mathfrak{h}} + \mathfrak{h} \wedge \mathfrak{t} \,. \tag{7.23}$$

In this way a symmetric and reductive dual Lie algebra is obtained, and in which $\mathfrak{t}^*$ (resp. $\mathfrak{h}^*$) play completely interchanged roles with respect to their duals $\mathfrak{t}$ (resp. $\mathfrak{h}$).

In the following section all these notions will be exemplified by considering coreductive and cosymmetric structures for maximally symmetric Lorentzian homogeneous spaces in (2+1) and (3+1) dimensions.

7.3 Coreductive Lorentzian Lie bialgebras

It is useful to remember here (see §2.3 for the details) the description of the three maximally symmetric (3+1)-dimensional Lorentzian spacetimes of constant curvature as coset spaces

- $\Lambda < 0$: Anti de Sitter spacetime $\mathbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.
- $\Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$.
- $\Lambda > 0$: de Sitter spacetime $\mathbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$.

The Lie brackets (see §2.2) of the Lie algebras $\mathfrak{so}(3,2)$, $\mathfrak{so}(4,1)$ and $\mathfrak{iso}(3,1)$ in terms of the cosmological constant, and written in the kinematical basis, are given by

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & [K_a, K_b] = -\epsilon_{abc} J_c, \\ [P_0, P_a] = -\Lambda\, K_a, & [P_a, P_b] = \Lambda\, \epsilon_{abc} J_c, & [P_0, J_a] = 0, \end{array} \tag{7.24}$$

Recall that we denote this family of Lie algebras by $\mathfrak{g}_\Lambda$. The decomposition of $\mathfrak{g}_\Lambda$ (as a vector space) is given by

$$\mathfrak{g}_\Lambda = \mathfrak{l} \oplus \mathfrak{t}, \qquad \mathfrak{l} = \mathrm{span}\{\mathbf{K}, \mathbf{J}\} \simeq \mathfrak{so}(3,1), \qquad \mathfrak{t} = \mathrm{span}\{P_0, \mathbf{P}\}, \tag{7.25}$$

where $\mathfrak{l}$ is the Lorentz subalgebra (2.75). Therefore, (A)dS and Minkowski spacetimes, which we will denote as M_Λ, are symmetric reductive homogeneous with $\mathfrak{t}$ being the translations subalgebra, and the commutation rules (7.24) can be schematically summarized in the form

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}\,, \qquad [\mathfrak{l}, \mathfrak{t}] \subset \mathfrak{t}\,, \qquad [\mathfrak{t}, \mathfrak{t}] \subset \Lambda\, \mathfrak{h}\,. \tag{7.26}$$

In the Minkowski ($\Lambda \to 0$) case we have $[\mathfrak{t}, \mathfrak{t}] = 0$, which means that $\mathfrak{t}$ generates a normal subgroup and we have the well-known semidirect product structure for the Poincaré algebra $\mathfrak{g}_0$.

7.3.1 Lorentzian Lie bialgebras

It seems natural to investigate how the conditions of coisotropy, coreductivity and cosymmetry define a very specific subset within the family of all possible Lie bialgebra structures for Lorentzian Lie algebras $\mathfrak{g}_\Lambda$ with commutation rules of the form (7.26) (see [230, 166, 160, 167, 162] for classification approaches to Lorentzian Lie bialgebras).

It is well-known [230] that in (2+1) and (3+1) dimensions all (A)dS and Poincaré Lie bialgebras $(\mathfrak{g}_\Lambda, \delta)$ are coboundary ones, which means that all of them can be obtained through r-matrices in the form

$$\delta(X) = \mathrm{ad}_X(r), \qquad \forall\, X \in \mathfrak{g}_\Lambda \tag{7.27}$$

with $r \in \mathfrak{g}_\Lambda \otimes \mathfrak{g}_\Lambda$ being a skew-symmetric solution of the modified Classical Yang-Baxter Equation (mCYBE). Let us consider a generic r-matrix in the schematic form

$$r \subset \alpha\, \mathfrak{l} \wedge \mathfrak{l} + \beta\, \mathfrak{l} \wedge \mathfrak{t} + \gamma\, \mathfrak{t} \wedge \mathfrak{t}, \tag{7.28}$$

with $\{\alpha, \beta, \gamma\}$ denoting generic tensor coefficients for each component of the r-matrix. Then it is straightforward to prove that the cocommutator (7.27) arising from (7.28) and the commutation rules (7.24) will be of the form

$$\delta(\mathfrak{l}) \subset \alpha\, \mathfrak{l} \wedge \mathfrak{l} + \beta\, \mathfrak{l} \wedge \mathfrak{t} + \gamma\, \mathfrak{t} \wedge \mathfrak{t}, \tag{7.29}$$

$$\delta(\mathfrak{t}) \subset \beta\, \Lambda\, \mathfrak{l} \wedge \mathfrak{l} + \alpha\, \mathfrak{l} \wedge \mathfrak{t} + \gamma\, \Lambda\, \mathfrak{l} \wedge \mathfrak{t} + \beta\, \mathfrak{t} \wedge \mathfrak{t}\,, \tag{7.30}$$

where the cosmological constant parameter that distinguishes between the (A)dS and Poincaré cases appears explicitly.

From these expressions the following conclusions can be immediately derived:

1. A Lorentzian Lie bialgebra is coisotropic ($\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{g}$) iff $\gamma = 0$.

2. For coisotropic Lie bialgebras ($\gamma = 0$), coreductivity ($\delta(\mathfrak{t}) \subset \mathfrak{h} \wedge \mathfrak{h} + \mathfrak{t} \wedge \mathfrak{t}$) is obtained iff $\alpha = 0$. Therefore, coreductive Lorentzian Lie bialgebras are given exclusively by r-matrices of the form
$$r \subset \beta\, \mathfrak{l} \wedge \mathfrak{t}, \tag{7.31}$$
and whose coefficients β are constrained by the modified CYBE. This automatically precludes Lie bialgebras such that $\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{l}$ with $\delta(\mathfrak{l}) \neq 0$ to be coreductive, and these are just Lie bialgebras associated to Poisson homogeneous spaces $M_\Lambda = G/L$ for which L is a Poisson-Lie subgroup [60].

3. Cosymmetry for coreductive Lorentzian Lie bialgebras ($\delta(\mathfrak{l})$ does not contain $\mathfrak{l} \wedge \mathfrak{l}$) is obtained iff $\alpha = 0$. Therefore, in this case if we have both coisotropy and coreductivity then cosymmetry is automatically verified.

4. Note that the answer to the question whether coreductivity implies coisotropy is Lie algebra dependent: for $\Lambda = 0$ it is negative, while for $\Lambda \neq 0$ the answer is positive.

5. The same abovementioned conclusions can be extracted for Lie bialgebras corresponding to $\mathfrak{so}(5)$ and $\mathfrak{iso}(4)$, with $\mathfrak{l}$ replaced by $\mathfrak{h} = \mathfrak{so}(4)$, since they are structurally equivalent to the Lorentzian ones with the cosmological constant being replaced by the inverse of the square of the radius of the sphere (see [143]).

We recall that classifications of r-matrices for the (2+1) Poincaré and (A)dS Lie algebras have been presented, respectively, in [166, 167, 168], while for the (3+1) Poincaré case a classification can be found in [230]. Nevertheless, we remark that most of these results are not written in the kinematical basis (7.26). In the sequel we comment on some (2+1) and (3+1) Lorentzian examples, thus making more explicit the previous definition and constructions. As have been already noted, coisotropy and coreductivity are quite restrictive properties and will select a very specific class of Lorentzian PHS.

7.3.2 Dual Poisson homogenous spaces and the κ-Lie bialgebra in (2+1) dimensions

Let us firstly illustrate the previous construction of coreductive Lie bialgebras and their dual homogeneous spaces with the analysis of the Lie bialgebra associated to the κ-deformation of the (2+1) dimensional Lorentzian algebras. This Lie bialgebra is coisotropic, coreductive and cosymmetric since it is generated by the r-matrix (see Chapter 4)

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2), \tag{7.32}$$

which is indeed of the form (7.31). We recall that in (2+1) dimensions the Lie brackets of the Lie algebra $\mathfrak{g}_\Lambda$ take the form (see §2.2)

$$\begin{aligned} &[J, P_a] = \epsilon_{ab} P_b, && [J, K_a] = \epsilon_{ab} K_b, && [J, P_0] = 0, \\ &[K_a, P_b] = \delta_{ab} P_0, && [K_a, P_0] = P_a, && [K_1, K_2] = -J, \\ &[P_0, P_a] = -\Lambda K_a, && [P_1, P_2] = \Lambda J. && \end{aligned} \tag{7.33}$$

The cocommutator map obtained from (7.32) reads

$$\begin{aligned} \delta(P_0) &= \delta(J) = 0, \\ \delta(P_1) &= \frac{1}{\kappa}(P_1 \wedge P_0 + \Lambda\, K_2 \wedge J), \\ \delta(P_2) &= \frac{1}{\kappa}(P_2 \wedge P_0 - \Lambda\, K_1 \wedge J), \\ \delta(K_1) &= \frac{1}{\kappa}(K_1 \wedge P_0 + P_2 \wedge J), \\ \delta(K_2) &= \frac{1}{\kappa}(K_2 \wedge P_0 - P_1 \wedge J), \end{aligned} \tag{7.34}$$

which is indeed coisotropic (3.79) with respect to the Lorentz Lie subalgebra

$$\mathfrak{l} = \operatorname{span}\{K_1, K_2, J\} \simeq \mathfrak{so}(2,1). \tag{7.35}$$

Therefore, if we denote the dual generators to $\{P_0, P_1, P_2, K_1, K_2, J\}$ by, respectively, $\{\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{\xi}^1, \hat{\xi}^2, \hat{\theta}\}$, the Lie brackets defining the (solvable) Lie algebra $\mathfrak{g}^*$ of the dual Poisson-Lie group G^*_Λ are straightforwardly deduced from (7.34) and read

$$\begin{aligned} &[\hat{x}^0, \hat{x}^1] = -\tfrac{1}{\kappa}\,\hat{x}^1, && [\hat{x}^0, \hat{x}^2] = -\tfrac{1}{\kappa}\,\hat{x}^2, && [\hat{x}^1, \hat{x}^2] = 0, \\ &[\hat{x}^0, \hat{\xi}^1] = -\tfrac{1}{\kappa}\,\hat{\xi}^1, && [\hat{x}^0, \hat{\xi}^2] = -\tfrac{1}{\kappa}\,\hat{\xi}^2, && [\hat{\xi}^1, \hat{\xi}^2] = 0, \\ &[\hat{\theta}, \hat{x}^2] = -\tfrac{1}{\kappa}\,\hat{\xi}^1, && [\hat{\theta}, \hat{\xi}^1] = \tfrac{1}{\kappa}\,\Lambda\,\hat{x}^2, && [\hat{\xi}^1, \hat{x}^2] = 0, \\ &[\hat{\theta}, \hat{x}^1] = \tfrac{1}{\kappa}\,\hat{\xi}^2, && [\hat{\theta}, \hat{\xi}^2] = -\tfrac{1}{\kappa}\,\Lambda\,\hat{x}^1, && [\hat{\xi}^2, \hat{x}^1] = 0, \\ &[\hat{\theta}, \hat{x}^0] = 0, && [\hat{\xi}^1, \hat{x}^1] = 0, && [\hat{\xi}^2, \hat{x}^2] = 0. \end{aligned} \tag{7.36}$$

On the other hand, the dual cocommutator map η is obtained as the dual of the Lie bracket (7.33) for the $\mathfrak{g}_\Lambda$ algebra, namely

$$\begin{aligned}
\eta(\hat{x}^0) &= \hat{\xi}^1 \wedge \hat{x}^1 + \hat{\xi}^2 \wedge \hat{x}^2, \\
\eta(\hat{x}^1) &= -\hat{\theta} \wedge \hat{x}^2 + \hat{\xi}^1 \wedge \hat{x}^0, \\
\eta(\hat{x}^2) &= \hat{\theta} \wedge \hat{x}^1 + \hat{\xi}^2 \wedge \hat{x}^0, \\
\eta(\hat{\theta}) &= \Lambda\, \hat{x}^1 \wedge \hat{x}^2 - \hat{\xi}^1 \wedge \hat{\xi}^2, \\
\eta(\hat{\xi}^1) &= -\Lambda\, \hat{\theta} \wedge \hat{\xi}^2, \\
\eta(\hat{\xi}^2) &= \Lambda\, \hat{\theta} \wedge \hat{\xi}^1.
\end{aligned} \tag{7.37}$$

Note that Λ plays now the role of a deformation parameter for the dual cocommutator, although in this case the $\Lambda \to 0$ limit does not lead to the zero cocommutator.

We recall that the Poisson homogeneous Lorentzian spacetimes (M_Λ, π) associated to the κ-PL structure defined by (7.32) are explicitly given by the Poisson structure (see [144] for details)

$$\{x^0, x^1\}_\pi = -\frac{1}{\kappa}\, \frac{\tan \sqrt{\Lambda}\, x^1}{\sqrt{\Lambda}\, \cos^2(\sqrt{\Lambda}\, x^2)}, \qquad \{x^0, x^2\}_\pi = -\frac{1}{\kappa}\, \frac{\tan(\sqrt{\Lambda}\, x_2)}{\sqrt{\Lambda}}, \qquad \{x^1, x^2\}_\pi = 0, \tag{7.38}$$

which in the limit $\Lambda \to 0$ gives rise to the Poisson version of the well-known (2+1) κ-Minkowski noncommutative spacetime

$$\{x^0, x^1\}_\pi = -\frac{1}{\kappa}\, x^1, \qquad \{x^0, x^2\}_\pi = -\frac{1}{\kappa}\, x^2, \qquad \{x^1, x^2\}_\pi = 0. \tag{7.39}$$

Note that (7.38) is just the projection to (2+1) dimensions of (4.52) and (4.53). Moreover, since the (A)dS and Minkowski spacetimes are obtained as cosets by the Lorentz isotropy subgroup generated by $\mathfrak{l} = \mathrm{span}\{J, K_1, K_2\}$, we have that $\mathfrak{t}^* = \mathrm{span}\{\hat{x}^0, \hat{x}^1, \hat{x}^2\}$ and the generators of $\mathfrak{l}^*$ are $\{\hat{\xi}^1, \hat{\xi}^2, \hat{\theta}\}$. Therefore, the commutation rules for $\mathfrak{g}^*$ (7.36) are of the form

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^*\,, \qquad [\mathfrak{t}^*, \mathfrak{l}^*] \subset \mathfrak{l}^*\,, \qquad [\mathfrak{l}^*, \mathfrak{h}^*] \subset \Lambda\, \mathfrak{t}^*\,. \tag{7.40}$$

As a consequence, the reductive homogeneous spaces M^*_Λ which are dual to the Lorentzian spacetimes M_Λ through the Lie bialgebra δ, would be defined as $M^*_\Lambda = G^*_\Lambda / T^*$ where T^* is the subgroup of G^* generated by the dual translations $\{\hat{x}^0, \hat{x}^1, \hat{x}^2\}$. This is the so-called κ-Minkowski subgroup [63, 64, 65] and whose Lie algebra coincides with (7.39) (note that this algebra does not depend on Λ). In the limit $\Lambda \to 0$ (the Minkowski case) the $\hat{\xi}$ generators form an abelian subalgebra of infinitesimal translations on the dual space M^*_0.

Also, we stress that G^*_Λ with Lie algebra $\mathfrak{g}^*$ (7.36) is by no means a semisimple Lie group, and the dual isotropy subgroup T^* is a solvable one. Therefore, the associated Killing-Cartan form for the dual group G^* is degenerate, and no guarantee of the existence of a G^*-invariant (indefinite) metric onto M^*_Λ is expected. In fact, it is easy to prove by direct computation that for the particular case of the (2+1)-dimensional κ-Lie bialgebra, the dual space M^*_Λ cannot be endowed with such a metric, just by noticing that G-invariant metrics on reductive spaces are in one-to-one correspondence (see Proposition 2.4) with Ad_H-invariant non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle$ on $\mathfrak{l}^* = \mathrm{Lie}(L^*)$, i.e.

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle, \qquad \forall X, Y \in \mathfrak{l}^*, Z \in \mathfrak{t}^* \tag{7.41}$$

where $\mathfrak{t}^* = \mathrm{Lie}(T^*)$. This implies that, in general, the geometric features of the three-dimensional dual spaces M^*_Λ will be quite different from their Lorentzian counterparts M_Λ.

Nevertheless, we stress that M^* are Poisson homogeneous spaces whose Poisson structure π^* can be also obtained as the canonical projection of the dual PL bracket Π^* with associated Lie bialgebra is $(\mathfrak{g}^*, \eta)$ given by (7.19). The latter is, by construction, coisotropic with respect to the dual isotropy subalgebra generated by the generators of $\mathfrak{t}^*$. Recall also that π^* is defined on $\mathcal{C}^\infty(M^*) \times \mathcal{C}^\infty(M^*)$, and the coordinates on M^* are the local coordinates associated to the $\hat{\xi}$ generators.

In the particular case of the κ-deformation, the dual Lie bialgebra $(\mathfrak{g}^*, \eta)$ given by (7.36) and (7.37) is not a coboundary one since there is no r-matrix defined within $\mathfrak{g}^* \otimes \mathfrak{g}^*$ that could generate η. Despite of this fact (which implies that no Sklyanin bracket is available for the construction of Π^*) its full dual PL bracket Π^* can be computed through the method based on a Poisson version of the quantum duality principle [19, 208, 147, 59] which was introduced in [205] (see expressions (11) in [145]). The canonical projection of this bracket onto the $\{K_1^*, K_2^*, J^*\}$ coordinates, which are just the local coordinates associated to the $\hat{\xi} = \{\hat{\xi}^1, \hat{\xi}^2, \hat{\theta}\}$ generators, respectively, gives the π^* bracket for the dual PHS W_Λ, which reads

$$\{J^*, K_1^*\}_{\pi^*} = K_2^*, \qquad \{J^*, K_2^*\}_{\pi^*} = -K_1^*, \qquad \{K_1^*, K_2^*\}_{\pi^*} = -\frac{\sin(2\sqrt{\Lambda}J^*/\kappa)}{2\sqrt{\Lambda}/\kappa}. \tag{7.42}$$

Here it is worth remarking that, due to the non-coboundary nature of $(\mathfrak{g}^*, \eta)$, the construction of the PL bracket Π^* on G^* from which (7.42) is obtained as a projection, was performed in [145] by imposing its Poisson map compatibility with the coalgebra structure provided by the group multiplication, and its computation is by no means a trivial one. For that reason, the coordinates employed to describe G^* were fixed in such a way that they are well-defined functions on the coset space $M^*_\Lambda = G^*_\Lambda / T^*$, since the commutation rules of the dual generators of translations and boosts guarantee an appropriate ordering in the exponentiation of the dual group G^* (see (5.57)). We also stress that getting a common description in terms of 'dual' coordinates of two different coset spaces and of their corresponding non-coboundary PHS is, in general, a difficult problem.

The dual Poisson homogeneous space $M^*_\Lambda = G^*_\Lambda / T^*$ endowed with (7.42) deserves several comments:

- As expected, the linearization of (7.42) coincides with the dual of the cocommutator η (7.37), or equivalently with the Poisson version of the commutation rules for the Lorentz Lie algebra sector in (7.33).
- When $\frac{1}{\kappa} \neq 0$, the bracket (7.42) is a cosmological constant deformation of the $\mathfrak{so}(2,1)$ algebra, that is recovered in the limit $\Lambda \to 0$. This is similar to what occurs in (7.38), which in the case $\frac{1}{\kappa} \neq 0$ is just a Λ-deformation of the κ-Minkowski Lie algebra (see also [144]).
- Therefore, we can say that the Poisson algebra (M_Λ, π) (7.38) is a cosmological constant deformation of the Lie algebra (7.39), while the dual Poisson homogeneous

space (M^*_Λ, π^*) (7.42) is a Λ-deformation of the Lie algebra generated by the dual coordinates to the translations in M^*, which is a Poisson analogue of the Lorentz subalgebra (7.33).

Summarizing, we realize that given any (2+1) dimensional Lorentzian Lie bialgebra, its dual homogeneous space M^*_Λ will have as its Poisson bracket π^* either the Lorentz Lie algebra or a deformation of it. As we will see in the sequel, all these are structural properties imposed by the coreductivity constraint, and will also appear in (3+1) dimensions.

7.3.3 The (3+1) dimensional case

In (3+1) dimensions the r-matrix for the κ-deformation of Lorentzian Lie algebras is given by (see Chapter 4)

$$r = \frac{1}{\kappa}\left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{-\Lambda}\, J_1 \wedge J_2\right), \tag{7.43}$$

which is always coisotropic. However, due to the presence of the $\sqrt{-\Lambda}\, J_1 \wedge J_2$ term, only for $\Lambda = 0$ this r-matrix gives rise to a coreductive (and thus cosymmetric) Lie bialgebra structure, as seen directly from the cocommutator (4.36) with $\eta = \sqrt{-\Lambda}$. This means that reductive duals of (A)dS spaces are excluded in (3+1) dimensions. On the contrary, the dual (M^*_0, π^*) of the κ-Minkowski Poisson homogeneous space (M_0, π) can be constructed as a reductive space. This M^*_0 space will be 6-dimensional, since G^* is 10-dimensional and the isotropy subgroup for W_0 will be generated by the $\{\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3\}$ generators dual to the P_α translations.

In this case the κ-Minkowski [63, 64, 65] PHS (M_0, π) obtained by projecting the Sklyanin bracket for the r-matrix (7.43) onto the spacetime coordinates can be proven to be given (see Chapter 4) by the (3+1)-dimensional generalization of (7.44), namely

$$\{x^0, x^1\}_\pi = -\frac{1}{\kappa}\, x^1, \quad \{x^0, x^2\}_\pi = -\frac{1}{\kappa}\, x^2, \quad \{x^0, x^3\}_\pi = -\frac{1}{\kappa}\, x^3, \quad \{x^a, x^b\}_\pi = 0. \tag{7.44}$$

On the other hand, the Poisson homogeneous structure (M^*_0, π^*) has to be obtained as the canonical projection onto the ξ^* sector of the corresponding PL structure on G^*, which was explicitly constructed in [114] by following the method introduced in [205]. It is straightforward to check that this projection gives rise to a Poisson structure π^* which is a (undeformed) Poisson version of the Lorentz Lie algebra, namely

$$\{J^*_a, J^*_b\}_{\pi^*} = \epsilon_{abc} J^*_c, \qquad \{J^*_a, K^*_b\}_{\pi^*} = \epsilon_{abc} K^*_c, \qquad \{K^*_a, K^*_b\}_{\pi^*} = -\epsilon_{abc} J^*_c. \tag{7.45}$$

which again generalizes the $\Lambda \to 0$ case of the (2+1)-dimensional one (7.42). As it was mentioned in the previous section, this Poisson homogeneous structure (M^*_0, π^*) is well defined because $\{J^*_a, K^*_a\}$ are, by construction, suitable coordinates on the coset space M^*_0. This statement can be explicitly checked from in (5.57), since the vanishing commutation relations among dual generators of boosts and translations ensures that the ordering in the exponentiation is the suitable one for the description of the coset space in terms of local coordinates. As a consequence, the noncommutative spacetime arising from quantizing π^* will be just isomorphic to the Lorentz Lie algebra $\mathfrak{so}(3,1)$, whose representation theory is well-known [241].

7.4 On the geometry of dual Poisson homogeneous spaces

In the previous Section we have dealt with the Poisson geometry of the space M^* since, by construction, these spaces are naturally endowed with a Poisson structure compatible with the left action of G^*. The general method for constructing such Poisson homogeneous structure on M^* has been given, and some interesting examples have been worked out in detail. However, while on the spacetimes M_Λ the pseudo-riemannian structure coexists with the Poisson structure in a natural way, the first one describing the classical geometry (general relativity) and the second one describing semi-classical quantum corrections, we have seen that, in general, the dual PHS M^*_Λ do not admit a G-invariant pseudo-riemannian metric. Therefore, alternative approaches for the characterization of the geometric properties of the dual PHS are needed.

A natural approach is to consider the general setting of K-structures on manifolds, by following [181] and [173] (see Example 2.2). Let M^* be the PHS dual to a coreductive Lie bialgebra $(\mathfrak{g}, \delta)$, and let w be the dimension of M^*. Consider the frame bundle $F(M^*)$ viewed as a principal bundle over M^* with structure group $GL(w, \mathbb{R})$. With this notation, a K-structure is a reduction of $F(M^*)$ to the subgroup K of $GL(w, \mathbb{R})$. A connection in the principal bundle defined by the K-structure on M^* induces a linear connection on the tangent bundle of the manifold M^* which is said to be adapted to the K-structure. Associated to each connection we have its torsion and curvature, which indeed give information not only about the geometry but also the topology of the manifold, and this will be the route we propose in order to extract some explicit geometric information about the space M^*.

Recall the notions of the torsion and curvature tensors T and R of a given connection, defined in §2.1.5. It should be stressed that such connections are far from being unique (so having different associated torsion and curvature forms), but in the particular case of reductive spaces the so-called canonical connection having a particularly simple form can be defined (see §2.1.6).

Let us consider the dual PHS corresponding to the coreductive Lie bialgebra $(\mathfrak{g}, \delta)$ defined as the coset space $M^* = G^*/T^*$. We know that $\mathfrak{g}^* = \mathrm{Lie}(G^*)$ admits a reductive decomposition of the form $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{t}^*$ where $\mathfrak{t}^* = \mathrm{Lie}(T^*)$, so we can identify $T_w M^* \simeq \mathfrak{h}^*$. We define the Lie bracket projection onto the subspaces associated to this decomposition as

$$[X, Y] = [X, Y]_{\mathfrak{h}^*} + [X, Y]_{\mathfrak{t}^*} , \tag{7.46}$$

for all $X, Y \in \mathfrak{g}^*$, where $[\cdot, \cdot]_{\mathfrak{h}^*}$ stands for the projection to the subspace $\mathfrak{h}^*$ of the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}^*$. With this notation, the so-called canonical connection for the dual PHS corresponding to a coreductive Lie bialgebra fulfills the following relations (see Theroem 2.5)

$$\begin{aligned}
&T(X, Y)_{eT^*} = -\, [X, Y]_{\mathfrak{h}^*} , \\
&(R(X, Y)Z)_{eT^*} = -\, [[X, Y]_{\mathfrak{t}^*} , Z] , \\
&\nabla T = 0, \\
&\nabla R = 0.
\end{aligned} \tag{7.47}$$

The last two identities are a direct consequence of the fact that every G^*-invariant tensor field is parallel transported by the canonical connection. Here should be noticed that the canonical connection just defined is complete for every dual PHS corresponding to a coreductive Lie bialgebra.

In the case of M^* being the dual PHS corresponding to a cosymmetric Lie bialgebra $(\mathfrak{g}, \delta)$, further simplifications arise for the torsion and curvature tensors by taking into account that $[\mathfrak{l}^*, \mathfrak{l}^*] \subset \mathfrak{t}^*$ (see Theorem 2.7), and therefore

$$\begin{aligned} &T(X,Y)_{eT^*} = 0, \\ &(R(X,Y)Z)_{eT^*} = -\left[[X,Y],Z\right], \\ &\nabla T = 0, \\ &\nabla R = 0, \end{aligned} \tag{7.48}$$

for all $X, Y, Z \in \mathfrak{h}^*$. In general, the fact that the torsion tensor vanishes identically if $(\mathfrak{g}, \delta)$ is cosymmetric means that the canonical connection on M^* coincides with the Levi-Civita connection associated to a G^*-invariant Riemannian metric (provided it exists).

As an example, let $(\mathfrak{g}, \delta)$ be the κ-Lie bialgebra in (2+1) dimensions given by (7.34). A straightforward computation gives that the only non vanishing components of the curvature tensor are

$$\left(R(\hat{\xi}^a, \hat{\theta}), \hat{\theta}\right)_0 = \frac{\Lambda}{\kappa^2}\, \hat{\xi}^a, \tag{7.49}$$

while the Ricci tensor has the only non-vanishing component given by

$$\left(R(\hat{\theta}, \hat{\theta})\right)_0 = 2\frac{\Lambda}{\kappa^2}. \tag{7.50}$$

Therefore, the dual space M^* associated to the κ-Lie bialgebra in (2+1) dimensions turns out to be Ricci flat iff the corresponding model spacetime M_Λ is flat, i.e. only in the Minkowski case where $\Lambda = 0$ (the limit $\kappa \to \infty$ corresponds to the trivial PHS structure with abelian dual group G^*). The scalar curvature of M^*_Λ cannot be defined at this stage because we have not endowed M^*_Λ with a metric. Note that although we have previously proved that M^*_Λ does not admit a G^*-invariant metric, such G^*-invariance condition -which in fact is quite restrictive- could perhaps be relaxed.

For the $(3+1)$ dimensional κ-Poincaré deformation, whose dual Lie algebra was presented in [146], it is straightforward to check that $\mathfrak{l}^*$ is a commutative Lie subalgebra, so the Riemann tensor for the dual space M^*_0, with Poisson structure given in (7.45), vanishes identically.

7.5 Coreductivity and uncertainty relations

The main physical motivation for the introduction of noncommutative spacetimes is based on the widely shared idea that some quantum gravity effects could be described (in an effective or dynamical way) by the introduction of a ‘quantum’ geometry in which spacetime coordinates are replaced by noncommutative operators (see for instance [20, 21, 26, 22, 242, 30, 234] and references therein). In particular, quantum homogeneous spaces M_q are noncommutative spacetimes covariant under quantum groups G_q (co)actions. We have seen that the dual Lie algebra $\mathfrak{g}^*$ provides the first order of the noncommutative algebra defining G_q, while the first order of the noncommutative spacetime's commutation relations M_q is given by the $\mathfrak{t}^*$ subalgebra of dual translations. In this framework, the noncommutativity of the algebra of coordinates of spacetime events implies the existence of Heisenberg-type uncertainty relations in the case of simultaneous measurements of different components of the noncommutative coordinates $\hat{x}$ (and their functions).

We have seen that the coisotropy condition (3.79) for a given Lie bialgebra guarantees that the 'quantum' spacetime coordinates $\hat{x}$ close a subalgebra within the dual Lie algebra $\mathfrak{g}^*$ of quantum group coordinates, and that $\mathfrak{t}^*$ generates the isotropy subgroup of the dual homogeneous space M^*. Moreover, by definition, the coreductivity constraint (7.16) imposes onto $\mathfrak{g}^*$ the condition of being a reductive Lie algebra, and this fact will be reflected in the representation theory of $\mathfrak{g}^*$, a fact which, as we will show now, has far-reaching consequences from a physical viewpoint.

Let us firstly assume that the dual Lie algebra $\mathfrak{g}^*$ can be endowed with a C^*-algebra structure, and let us consider a unitary irreducible representation of this algebra on a Hilbert space of physical states denoted by $|\psi\rangle$. Then, if coreductivity does not hold and we allow for elements of $\hat{x}$ to appear on the right-hand-side of the commutation rules $[\hat{x},\hat{\xi}]$ in the dual Lie algebra, namely,

$$[\hat{x},\hat{x}] \subset \hat{x}\,, \qquad [\hat{x},\hat{\xi}] \subset \hat{\xi}+\hat{x}\,, \qquad [\hat{\xi},\hat{\xi}] \subset \hat{\xi}+\hat{x}, \tag{7.51}$$

then there will exist at least one uncertainty relation of the form

$$\Delta\hat{x}\,\Delta\hat{\xi} \geq \frac{1}{2}\langle\hat{\xi}\rangle + \frac{1}{2}\langle\hat{x}\rangle\,, \qquad\qquad \text{where} \qquad \Delta\hat{y} = \sqrt{\langle\hat{y}^2\rangle - \langle\hat{y}\rangle^2}\,. \tag{7.52}$$

Now, let us consider the subset of states such that $\hat{\xi}|\psi\rangle = 0$. Since by definition such states have vanishing uncertainty and $\Delta\hat{\xi} = \langle\psi|\hat{\xi}^2|\psi\rangle - \langle\hat{\xi}\rangle^2 = 0$, then relations (7.52) impose singular constraints onto the expectation values of the momenta of $\hat{x}$. In particular, (7.52) implies that either $\langle\psi|\hat{x}|\psi\rangle = 0$ or $\Delta\hat{x} \to \infty$. If we consider the representation space for the Lie subalgebra $[\hat{x},\hat{x}] \subset \hat{x}$ alone, there could be some states such that $\langle\hat{x}\rangle = 0$, but these ones most certainly do not exhaust, in general, the set of all possible states. Similarly, there could be sequences of states for the subalgebra generated by $\hat{x}$ whose uncertainty is divergent, but, again, they will not be generic ones.

On the contrary, if the coreductivity condition holds this implies that we have commutation rules of the type

$$[\hat{x},\hat{\xi}] \subset \hat{\xi}, \tag{7.53}$$

which give rise to uncertainty relations of the form

$$\Delta\hat{x}\,\Delta\hat{\xi} \geq \frac{1}{2}\langle\hat{\xi}\rangle\,. \tag{7.54}$$

Now, if we consider the set of eigenstates of $\hat{\xi}$ with vanishing eigenvalue, $\hat{\xi}|\psi\rangle = 0$, then on these states the previous crossed uncertainty relations do not constrain in any way the momenta of the spacetime observables $\hat{x}$. This argument can be illustrated with a well-known example: the theory of unitary irreducible representations (UIR) of the Poincaré Lie algebra. Note that the UIR of null-vector type are the ones with zero eigenvalues for the generators of the subalgebra of translations (the $\hat{\xi}$ operators in (7.53)), and in this case the representation theory for the isotropy subgroup (the $\hat{x}$ operators) completely decouples with the one for the translation generators (see [241] for details).

In summary, if there are $\hat{x}$ contributions on the right-hand-side of $[\hat{x}, \hat{\xi}]$, the subset of states for the $\mathfrak{g}^*$ algebra such that $\hat{\xi}|\psi\rangle = 0$ cannot provide with the full set of representation states for the noncommutative spacetime subalgebra $\hat{x}$. Therefore, the coreductivity condition allows us to get a physical insight of the subalgebra $\hat{x}$ on its own, and consider it as the keystone for the construction of a noncommutative algebra of functions on a quantum homogeneous space M_q.

A more specific illustration of this argument can be extracted from the recent work [96], where the representation theory for the κ-Minkowski spacetime has been thouroughly studied. In the simpler (1+1)-dimensional case, the commutation relations between the noncommutative coordinates over the full quantum (1+1) Poincaré group are (see also [206])

$$[\hat{a}^0, \hat{a}^1] = i\lambda\, \hat{a}^1\,, \qquad [\hat{\xi}, \hat{a}^0] = -i\lambda \sinh \hat{\xi}\,, \qquad [\hat{\xi}, \hat{a}^1] = i\lambda \left(1 - \cosh \hat{\xi}\right)\,, \tag{7.55}$$

Note that the linearization of these relations leads to

$$[\hat{a}^0, \hat{a}^1] = i\lambda\, \hat{a}^1\,, \qquad [\hat{\xi}, \hat{a}^0] = -i\lambda\, \hat{\xi}\,, \qquad [\hat{\xi}, \hat{a}^1] = 0\,, \tag{7.56}$$

which exactly coincides with the (1+1)-dimensional version of (7.36) provided that $z = -i\lambda$ and we identify the quantum group translation coordinates as $\hat{x}^0 = \hat{a}^0$ and $\hat{x}^1 = \hat{a}^1$. Again, (7.56) illustrates the fact that the dual Lie algebra $\mathfrak{g}^*$ (7.36) provides just the linearization of the full quantum group relations.

As it is shown in [96], finite translation ($\hat{a}^\mu$) and Lorentz rapidity ($\hat{\xi}$) operators can be represented as differential operators on a Hilbert space of functions on the Cartesian product between the Lorentz group and $\mathbb{R}$ in the following way:

$$\begin{aligned} \hat{a}^0 &= i\lambda \left(\frac{1}{2} + q\frac{\partial}{\partial q}\right) + i\lambda \left(\frac{1}{2}\cosh \xi + \sinh \xi\, \frac{\partial}{\partial \xi}\right)\,, \\ \hat{a}^1 &= q + i\lambda \left(\frac{1}{2}\sinh \xi + (\cosh \xi - 1)\, \frac{\partial}{\partial \xi}\right)\,, \end{aligned} \tag{7.57}$$

while ξ is the coordinate associated with the eigenvalues of the multiplicative operator $\hat{\xi}$. The meaning of our uncertainty-relation argument is made clear in [96], where it is shown that there exists a sequence of well-normalized product wavefunctions $Q(\xi)$, such that for any function of q, $f(q)$, all the expectation values

$$\langle Q(\xi) f(q) | (\hat{a}^1)^n (\hat{a}^0)^m | Q(\xi) f(q) \rangle\,, \tag{7.58}$$

tend to the following:

$$\langle f(q) | (\hat{y}^1)^n (\hat{y}^0)^m | f(q) \rangle\,, \tag{7.59}$$

where now $\hat{y}^\mu$ provide a faithful representation of the commutation relations of the κ-Minkowski quantum homogeneous space:

$$\begin{aligned} \hat{y}^0 &= i\lambda\left(\frac{1}{2}+q\frac{\partial}{\partial q}\right), \\ \hat{y}^1 &= q\,. \end{aligned} \tag{7.60}$$

In this way we see that, by choosing a product state between this sequence of functions $Q(\xi)$ (which tend, in an appropriate way, to a function localized at $\xi = 0$) and an arbitrary wavefunction f, we can reproduce the expectation values of any polynomial in $\hat{y}^0$ and $\hat{y}^1$, and so we can define the whole wealth of possible states on the κ-Minkowski algebra as a limit of states on the κ-Poincaré group, in which the rapidity (ξ) contribution is sent to zero in a controlled way.

7.6 Remarks

Summarizing, given a Poisson homogeneous space (M, π), where $M = G/H$ and the Poisson-Lie structure Π on G is characterized by the Lie bialgebra $(\mathfrak{g}, \delta)$, the coisotropy, coreductivity and cosymmetry conditions for δ are given as the following constraints

$$\begin{aligned} \delta(\mathfrak{h}) \subset \cancel{\mathfrak{h}\wedge\mathfrak{h}}^{\,\text{cosymmetry}} + \mathfrak{h}\wedge\mathfrak{t} + \cancel{\mathfrak{t}\wedge\mathfrak{t}}^{\,\text{coisotropy}}, \\ \delta(\mathfrak{t}) \subset \mathfrak{h}\wedge\mathfrak{h} + \cancel{\mathfrak{h}\wedge\mathfrak{t}}^{\,\text{coreductivity}} + \mathfrak{t}\wedge\mathfrak{t}, \end{aligned} \tag{7.61}$$

thus leading to a dual Lie algebra $\mathfrak{g}^*$ which is reductive and symmetric

$$[\mathfrak{t}^*, \mathfrak{t}^*] \subset \mathfrak{t}^* + \cancel{\mathfrak{h}^*}_{\text{coisotropy}}, \qquad [\mathfrak{t}^*, \mathfrak{h}^*] \subset \cancel{\mathfrak{t}^*}_{\text{coreductivity}} + \mathfrak{h}^*, \qquad [\mathfrak{h}^*, \mathfrak{h}^*] \subset \mathfrak{t}^* + \cancel{\mathfrak{h}^*}_{\text{cosymmetry}}, \tag{7.62}$$

As we have seen, these conditions allows the construction of a reductive and symmetric dual Poisson homogeneous space (M^*, π^*) where $M^* = G^*/T^*$ and π^* is the canonical projection onto M^* of the Π^* bracket on $\mathcal{C}^\infty(G^*)$ with associated Lie bialgebra given by $(\mathfrak{g}^*, \eta)$.

As we have seen through explicit examples, coreductivity and cosymmetry provide a novel insight into the classification problem of Lie algebra structures (and therefore, of PL groups and PHS, as well as their quantum analogues) which -to the best of our knowledge- had not been considered yet. As far as quantum deformations of Lorentzian Lie algebras are concerned, we have shown that the coreductivity condition impose strong constraints: for instance, non-trivial Poisson-subgroup homogeneous spacetimes are precluded in general, and the (A)dS cases with non-vanishing cosmological constant also face strong obstructions in the (3+1) dimensional case. In particular, the κ-Poincaré Lie bialgebra is coreductive in any dimension, while the κ-(A)dS Lie bialgebra is only coreductive in (2+1) dimensions.

We would also like to stress that the notion of dual reductive PHS completes the set of structures that can be defined onto a group G, a homogeneous space M and their duals G^* and M^*. In order to summarize this global and self-dual picture, we can consider the well-known example of the Poincaré group G, its associated Minkowski spacetime $M = G/L$, the coisotropic and coreductive PL structure on G provided by the κ-deformation, together with their duals G^* and M^*. In this way we have four Poisson structures:

1. Π: The PL structure on the Poincaré group associated to the Lie bialgebra $(\mathfrak{g}, \delta)$ given by the r-matrix which corresponds to the κ-deformation.

2. π: The Poisson homogeneous structure on the Minkowski spacetime $M = G/L$ (the Poisson κ-Minkowski spacetime), whose bracket is obtained through canonical projection from Π, since δ is coisotropic with respect to $\mathfrak{l} = \mathrm{Lie}(L)$.

3. Π^*: The PL structure on the dual Poincaré group G^*, whose associated Lie bialgebra is $(\mathfrak{g}^*, \eta)$. As we have shown, the Killing-Cartan form for G^* is degenerate (G^* is a solvable Lie algebra), and no G^*-invariant metric does exist.

4. π^*: Since the Lie bialgebra $(\mathfrak{g}, \delta)$ is coreductive, the dual reductive homogeneous space $M^* = G^*/T^*$, whose isotropy subgroup is generated by the dual of the Poincaré translations T^*, can be defined. Since the dual Lie bialgebra $(\mathfrak{g}^*, \eta)$ is coisotropic, the bracket π^* can be thus obtained through canonical projection form Π^*, and provides a Poisson κ-Lorentz space. Despite that such dual PHS cannot be endowed with a G^*-invariant metric, its geometry can be analysed from the viewpoint of K-structures and turns out to be torsionless and with vanishing curvature tensor.

Finally, we recall that the corresponding quantum homogeneous spacetimes will be just the quantizations of the Poisson spaces (M, π) and (M^*, π^*). As we have seen, it turns out that the coreductivity condition for δ guarantees that the representation theory of the algebra obtained from (M, π) after quantization will not depend on the rest of the quantum group G_q, and the same would happen with the quantum analogue of (M^*, π^*).

Chapter 8

Conclusions and open problems

Finally, we would like to summarize the most relevant original results presented in this Thesis, together with some of the future research lines that arise as natural continuations of this work.

8.1 Conclusions

The main results presented within this Thesis are the following:

1. We have generalized the well-known κ-Poincaré quantum deformation to the case of a non-vanishing cosmological constant, proving that this generalization, called the κ-(A)dS deformation, is unique under the assumption that the time generator is primitive, which implies that the deformation parameter has mass units, a necessary requirement from a physical point of view.

2. We have explicitly constructed the Poisson-Lie group associated to the κ-(A)dS deformation, together with its covariant Poisson homogeneous space, called κ-(A)dS spacetime. The first order in the cosmological constant parameter $\eta = \sqrt{-\Lambda}$ of this Poisson homogeneous space is quadratic and we have presented its quantization, namely

$$\begin{aligned}
&[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\,\hat{x}^a, \\
&[\hat{x}^1, \hat{x}^2] = -\frac{\eta}{\kappa}\,(\hat{x}^3)^2, \qquad [\hat{x}^1, \hat{x}^3] = \frac{\eta}{\kappa}\,\hat{x}^3\,\hat{x}^2, \qquad [\hat{x}^2, \hat{x}^3] = -\frac{\eta}{\kappa}\,\hat{x}^1\,\hat{x}^3.
\end{aligned} \tag{8.1}$$

   From these expressions we directly see that, in contradistinction to the κ-Minkowski case, space coordinates $\hat{x}^a$ do not commute among themselves but close a homogeneous quadratic algebra which, defines a quantum sphere related to the quantum $SU(2) \simeq SO(3)$ subalgebra of the (3+1)-dimensional κ-(A)dS quantum group. In general, we have shown that the κ-(A)dS spacetime is a smooth deformation of the κ-Minkowski one in terms of the cosmological constant parameter η.

3. The quantization to all orders in η of the κ-(A)dS spacetime has been also obtained. In order to achieve this result we have introduced ambient space coordinates s^α in terms of which the commutation relations are again homogeneous quadratic. We have also shown that the Casimir operator $\hat{\Sigma}_{\eta,\kappa}$ is the quantum analogue of the pseudosphere defining the (A)dS spacetime.

4. We have introduced a novel construction to study the consequences of quantum group symmetries for the homogeneous space of worldlines, which can be identified with inertial observers. This construction can be performed for every spacetime whose space of time-like geodesics is a homogeneous space, and for every quantum deformation which is coisotropic with respect to the stabilizer subgroup of a time-like geodesic.

5. As an application of this framework, we have explicitly studied the Poisson homogeneous space of worldlines associated to the κ-Minkowski spacetime, finding that the Poisson structure is almost a symplectic one, which therefore implies that its quantization is straightforward. Therefore, we have constructed a quantum homogeneous space of worldlines which is covariant under the κ-Poincaré quantum group, and which is the noncommutative version of the space of worldlines of the Minkowski spacetime.

6. The momentum space associated to the κ-(A)dS quantum group has explicitly been constructed, and we have shown that in order to extend the method applied so far to the κ-Poincaré quantum group, 'momentum' coordinates associated to Lorentz boosts and spatial rotations have to be included. Moreover, we have shown that the geometry of the momentum space for this quantum deformation is dimension dependent, since for lower dimensions it is (half of) a higher dimensional dS space, while in (3+1) dimensions it depends on the sign of the cosmological constant Λ: it maintains this geometry for $\Lambda \geq 0$, but for $\Lambda < 0$ the momentum manifold has an $SO(4,4)$ invariance group.

7. The technique employed to construct the momentum space associated to a quantum deformation makes use of the Poisson version of the 'quantum duality principle' and therefore, as a subsidiary result, we have constructed the full dual κ-(A)dS Poisson-Lie group.

8. We have studied, based on its previous classification and by using a kinematical basis in which the physical meaning of the results are apparent, all the possible Drinfel'd double structure of the Poincaré group in (2+1) dimensions. We have constructed the eight non-isomorphic Minkowski Poisson homogeneous spacetimes canonically defined by these Drinfel'd double structures, finding that only one of these Poisson structures is quadratic, while the remaining ones are linear.

9. The only Drinfel'd double structure of the Euclidean group in three dimensions have also been analyzed. This Drinfel'd double is the 'rotation double', the analogous structure to the Lorentz double for the Poincaré case. Although these two structures are induced by the semidirect product structure of these groups, in the Poincaré case there exists a plurality of Drinfel'd double structures which is absent in the Euclidean

case. These results have been compared with the ones for the (A)dS groups, and we have found that not every (A)dS Drinfel'd double structure survives the contraction limit $\Lambda \to 0$.

10. For the nontrivially centrally extended (1+1)-Poincaré group, the two existing non-isomorphic Drinfel'd doubles were also analyzed and their canonical Poisson homogeneous spaces constructed. In higher dimensions, we have shown that most of the kinematical groups cannot have any Drinfel'd double structure, being the (A)dS groups the most important exception.

11. Based on the coisotropy condition for a Lie bialgebra, which guarantees that the associated homogeneous space is indeed a Poisson homogeneous space, we have introduced two new types of Lie bialgebras called coreductive and cosymmetric ones. The intuition behind them is that one can see the coisotropy condition as the condition for the existence of a dual homogeneous space, and so coreductive and cosymmetric Lie bialgebras are those for which these dual homogeneous spaces are indeed reductive and symmetric, respectively.

12. We have considered, as an explicit example, the κ-Poincaré and κ-(A)dS Lie bialgebras, and we have constructed the associated dual homogeneous spaces. We have explicitly proven that, in contradistinction to their associated spacetimes, they do not admit invariant metrics. For this reason, in order to study their geometry we have made use of the theory of K-structures, and thus we have analyzed the canonical connections on these spaces, finding that they are flat (in the sense that the Riemann tensor vanishes identically) if and only if the associated spacetimes are flat.

13. Finally, we have found that these dual homogeneous spaces are relevant from a physical point of view, since they are related to the uncertainty relations of the associated noncommutative spacetime operators. In this sense, the notion of coreductive and cosymmetric quantum deformations provide a way to classify noncommutative spacetimes covariant under quantum group symmetries in terms of their type of associated uncertainty relations.

8.2 Open problems

Some interesting open problems suggested by the results presented in this Thesis are the following:

1. This Thesis has been focused in the three maximally symmetric Lorentzian spacetimes of constant curvature, Minkowski and (anti-)de Sitter. It certainly would be interesting to consider their non-relativistic limit and thus construct noncommutative spaces for the Galilean and Newton-Hooke groups, together with their noncommutative spaces of worldlines and curved momentum spaces.

2. The introduction of noncommutative spaces of worldlines opens an interesting window from a phenomenological point of view. Since worldlines of free massive particles

can be identified with inertial observers, these spaces can be a first step in the direction of proposing an explicit rigorous mathematical model of quantum observers with quantum group symmetry. The consequences of noncommuting positions and momenta for quantum observers, together with its associated uncertainty relations, could imply physical limits (different, although necessarily related, from those induced by a noncommutative spacetime) in our ability to probe the physical world, which would be induced by the Planck scale deformation parameter.

3. The well-known connection between Drinfel'd doubles and the non-abelian version of Poisson-Lie T-dual σ-models (see [236, 243, 244, 245, 246, 247, 248] and references therein) indicate that the study of the 'eightfold' Poisson-Lie T-plurality for the (2+1) Poincaré group certainly deserves some attention, as well as the comparison with the case with Euclidean signature.

4. Finally, a natural generalization of Poisson-Lie groups are the so-called dynamical Poisson-Lie groups [249, 250, 251, 252], which have been already employed in the context of gauge fixing in (2+1) gravity [253, 254]. The analysis of their usefulness in constructing 'dynamical' noncommutative spacetimes certainly deserves further study.

Appendix A

Poisson-Lie structures on $(2+1)$D Poincaré and 3D Euclidean groups

The classification of equivalence classes (under automorphisms) of Poisson-Lie structures on the $(2+1)$-dimensional Poincaré and 3-dimensional Euclidean groups is due to P. Stachura, who presented it in [166]. In this Appendix we summarize these results and write them in the notation employed throughout the Thesis.

Since for both the $(2+1)$-dimensional Poincaré and 3-dimensional Euclidean groups all Poisson-Lie structures are coboundary ones, the classification of Poisson-Lie structures on them is equivalent to the classification of skew-symmetric solutions of the mCYBE (see Proposition 3.3). Using the fact that both the $(2+1)$-dimensional Poincaré and 3-dimensional Euclidean Lie algebras are symmetric Lie algebras (see Definition 2.21), one can decompose any element $r \in \mathfrak{g} \wedge \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{p}(2+1)$ or $\mathfrak{g} = \mathfrak{e}(3)$, as

$$r = a + b + c, \qquad a \subset \mathfrak{t} \wedge \mathfrak{t}, \qquad b \subset \mathfrak{t} \wedge \mathfrak{h}, \qquad c \subset \mathfrak{h} \wedge \mathfrak{h}, \tag{A.1}$$

where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ is the Ad_H-invariant splitting of Proposition 2.5, and call $\mathfrak{t} = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{h} = \operatorname{span}\{k_1, k_2, k_3\}$. With this notation one can simplify the mCYBE (3.36) given by

$$\mathrm{ad}_X[[r,r]] = 0 \tag{A.2}$$

for all $X \in \mathfrak{g}$, and obtain the set of equations

$$[[a,b]] = p\,\tilde{\eta}, \quad 2[[a,c]] + [[b,b]] = \mu\,\Omega, \quad [[b,c]] = [[c,c]] = 0, \quad \tilde{\eta} \in \bigwedge^3 \mathfrak{t}, \quad \Omega \in \bigwedge^2 \mathfrak{t} \otimes \mathfrak{h}, \tag{A.3}$$

depending of two arbitrary real parameters $\mu, p \in \mathbb{R}$. The cases with $p = 0$ correspond to solutions leading to coisotropic Lie bialgebras with respect to $\mathfrak{h}$, and so to coisotropic Poisson homogeneous spaces $M = G/H$.

A.1 Poisson-Lie structures on the (2+1) Poincaré group

In this case $\mathfrak{g} = \mathfrak{p}(2+1) \simeq \mathfrak{so}(2,1) \ltimes \mathbb{R}^3$. In order to translate this classification in terms of the kinematical basis we have used throughout the Thesis (and in particular in (6.1)), the appropriate isomorphism is given by the map

$$e_1 = P_0, \qquad e_2 = P_1, \qquad e_3 = P_2, \qquad k_1 = -J, \qquad k_2 = -K_2, \qquad k_3 = K_1, \tag{A.4}$$

with $\{e_1, e_2, e_3, k_1, k_2, k_3\}$ being the basis used in [166]. Then $\mathfrak{t} = \mathrm{span}\{P_0, P_1, P_2\} \simeq \mathbb{R}^3$ and $\mathfrak{h} = \mathrm{span}\{J, K_1, K_2\} \simeq \mathfrak{so}(2,1)$. The cases with $p = 0$ correspond to solutions giving rise to coisotropic Lie bialgebras with respect to $\mathfrak{h} \simeq \mathfrak{so}(2,1)$, and so they give rise to 2-dimensional Poisson-Minkowski spacetimes $M = P(2+1)/SO(2+1)$.

There exist eight equivalence classes of Poisson-Lie structures on the (2+1)-dimensional Poincaré group, defined by:

Class (I):

$$c = \frac{1}{\sqrt{2}} K_1 \wedge (J + K_2), \qquad b = \alpha(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1), \qquad a = 0. \tag{A.5}$$

Here $\alpha = \{0, 1\}$, $\mu = 2\alpha^2$, $p = 0$. This is the only case in the classification with $c \neq 0$, so hereafter $c = 0$.

Class (IIa):

$$b = \rho P_2 \wedge K_1 - \alpha(P_1 \wedge J + P_0 \wedge K_2), \qquad a = a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \tag{A.6}$$

with $\alpha = \{0, 1\}$, $\rho \geq 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = -2\alpha^2$, $p \in \mathbb{R}$, and where from now on $\{a_{01}, a_{02}, a_{12}\}$ denote free real parameters. Note that the automorphism (6.32) transforms $\rho \geq 0$ into $\rho \leq 0$.

Class (IIb):

$$b = -\rho P_0 \wedge J - \alpha(P_1 \wedge K_1 + P_2 \wedge K_2), \qquad a = a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \tag{A.7}$$

with $\alpha = \{0, 1\}$, $\rho \geq 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = 2\alpha^2$, $p \in \mathbb{R}$.

Class (IIc):

$$\begin{aligned} b &= \frac{\alpha}{\sqrt{2}}\big(-P_2 \wedge (J + K_2) + (P_0 - P_1) \wedge K_1\big) - \rho(P_0 - P_1) \wedge (J + K_2), \\ a &= a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \end{aligned} \tag{A.8}$$

with $\alpha = \{0, 1\}$, $\rho \geq 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = 0$, $p \in \mathbb{R}$.

Class (IIIa):

$$b = \frac{1}{\sqrt{2}}(P_0 - P_1) \wedge K_1, \qquad a = a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \tag{A.9}$$

with $\mu = 0$, $p \in \mathbb{R}$.

Class (IIIb):

$$\begin{aligned} b &= -P_0 \wedge J - (\rho - 1) P_1 \wedge J - (\rho + 1) P_0 \wedge K_2 + P_1 \wedge K_2 + \rho P_2 \wedge K_1, \\ a &= a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \end{aligned} \tag{A.10}$$

with $\rho \in \mathbb{R}^*$, $\mu = 2\rho^2$, $p \in \mathbb{R}$.

Class (IV):

$$b = -P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1, \qquad a = 0, \tag{A.11}$$

with $\mu = 2$, $p = 0$.

Class (V):

$$b = 0, \qquad a = a_{01} P_0 \wedge P_1 + a_{02} P_0 \wedge P_2 + a_{12} P_1 \wedge P_2, \tag{A.12}$$

with $\mu = 0$, $p = 0$.

As it is shown in Table 6.2, we have proven that only four of the above classes contain DD structures. Among them, only Class (IV) is by itself a DD, while Classes (I), (IIa) and (IIIb) contain DD structures for some specific values of the parameters.

A.2 Poisson-Lie structures on the 3D Euclidean group

In this case $\mathfrak{g} = \mathfrak{e}(3) \simeq \mathfrak{so}(3) \ltimes \mathbb{R}^3$. In order to translate this classification in terms of the kinematical basis we have used in particular in §6.6, the appropriate isomorphism is given by the map

$$e_i = P_i, \qquad k_i = J_i, \qquad\qquad i = 1, 2, 3 \tag{A.13}$$

with $\{e_1, e_2, e_3, k_1, k_2, k_3\}$ being the basis used in [166]. Then $\mathfrak{t} = \mathrm{span}\{P_1, P_2, P_3\} \simeq \mathbb{R}^3$ and $\mathfrak{h} = \mathrm{span}\{J_1, J_2, J_3\} \simeq \mathfrak{so}(3)$. The cases with $p = 0$ correspond to solutions giving rise to coisotropic Lie bialgebras with respect to $\mathfrak{h} \simeq \mathfrak{so}(3)$, and so they give rise to 2-dimensional Poisson-Riemannian spacetimes $M = E(3)/SO(3)$.

There exist there equivalence classes of Poisson-Lie structures on the 3-dimensional Euclidean group, all of them with $c = 0$,defined by:

Class (I)

$$b = \alpha(P_1 \wedge J_2 - P_2 \wedge J_1) + \rho P_3 \wedge J_3, \qquad a = a_{12} P_1 \wedge P_2 + a_{13} P_1 \wedge P_3 + a_{23} P_2 \wedge P_3,$$

with $\alpha = \{0, 1\}$, $\rho \geq 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = -2\alpha^2$, $p \in \mathbb{R}$, and where from now on $\{a_{12}, a_{13}, a_{23}\}$ denote three free real parameters.

Class (II)

$$b = P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3, \qquad a = 0, \tag{A.14}$$

with $\mu = 2$ and $p = 0$.

Class (III)

$$b = 0, \qquad a = a_{12} P_1 \wedge P_2 + a_{13} P_1 \wedge P_3 + a_{23} P_2 \wedge P_3, \tag{A.15}$$

with $\mu = 0$ and $p = 0$.

Bibliography

[1] S. L. Glashow. Partial-symmetries of weak interactions. *Nucl. Phys.*, 22(4):579–588, 1961. doi:10.1016/0029-5582(61)90469-2. (Cited in page 1).

[2] S. Weinberg. A model of leptons. *Phys. Rev. Lett.*, 19(21):1264–1266, nov 1967. doi:10.1103/PhysRevLett.19.1264. (Cited in page 1).

[3] A. Salam. Weak and electromagnetic interactions. In *8th Nobel Symp. Lerum, Sweden, May 19-25, 1968*, volume C680519, pages 367–377, 1968. (Cited in page 1).

[4] G. 't Hooft and M. Veltman. Regularization and renormalization of gauge fields. *Nucl. Phys. B*, 44(1):189–213, 1972. arXiv:gr-qc/0602086, doi:10.1016/0550-3213(72)90279-9. (Cited in page 1).

[5] F. Englert and R. Brout. Broken symmetry and the mass of gauge vector mesons. *Phys. Rev. Lett.*, 13(9):321–323, aug 1964. doi:10.1103/PhysRevLett.13.321. (Cited in page 1).

[6] P. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13(16):508–509, oct 1964. doi:10.1103/PhysRevLett.13.508. (Cited in page 1).

[7] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Global conservation laws and massless particles. *Phys. Rev. Lett.*, 13(20):585–587, nov 1964. doi:10.1103/PhysRevLett.13.585. (Cited in page 1).

[8] CMS Collaboration. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys. Lett. B*, 716(1):30–61, 2012. doi:10.1016/j.physletb.2012.08.021. (Cited in page 1).

[9] ATLAS Collaboration. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys. Lett. B*, 716(1):1–29, 2012. doi:10.1016/j.physletb.2012.08.020. (Cited in page 1).

[10] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman and Company, 1973. (Cited in page 1).

[11] R. M. Wald. *General Relativity*. University of Chicago Press, 1984. doi:10.7208/chicago/9780226870373.001.0001. (Cited in page 1).

[12] LIGO Scientific Collaboration and Virgo Collaboration. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116(6):1–16, 2016. doi:10.1103/PhysRevLett.116.061102. (Cited in page 1).

[13] The Event Horizon Telescope Collaboration. First M87 Event Horizon Telescope results. I. The shadow of the supermassive black hole. *Astrophys. J.*, 875(1):L1, 2019. doi:10.3847/2041-8213/ab0ec7. (Cited in page 1).

[14] A. Ashtekar, T. Pawlowski, and P. Singh. Quantum nature of the big bang. *Phys. Rev. Lett.*, 96(14):2–5, 2006. doi:10.1103/PhysRevLett.96.141301. (Cited in page 1).

[15] M. Bronstein. Quantentheorie schwacher Gravitationsfelder. *Phys. Zeitschrift der Sowjetunion*, 9:140–157, 1936. (Cited in page 2).

[16] M. Planck. Ueber irreversible Strahlungsvorgänge. *Ann. Phys.*, 306(1):69–122, 1900. doi:10.1002/andp.19003060105. (Cited in page 2).

[17] A. Connes. *Noncommutative Geometry*. Academic Press, 1995. http://www.alainconnes.org/docs/book94bigpdf.pdf. (Cited in page 3).

[18] A. H. Chamseddine, A. Connes, and V. Mukhanov. Quanta of Geometry: Noncommutative Aspects. *Phys. Rev. Lett.*, 114(9):091302, mar 2015. arXiv:1409.2471, doi:10.1103/PhysRevLett.114.091302. (Cited in page 3).

[19] V. Drinfeld. Quantum Groups. In *Proc. Int. Congr. Math. (Berkeley 1986)*, pages 798–820, Providence, 1987. American Mathematical Society. (Cited in pages 3, 5, 5, 6, 8, 60, 103, 112, 123, 126, 182).

[20] H. Snyder. Quantized space-time. *Phys. Rev.*, 71(1):38–41, 1947. doi:10.1103/PhysRev.71.38. (Cited in pages 3, 3, 185).

[21] P. Podleś and S. L. Woronowicz. Quantum deformation of Lorentz group. *Commun. Math. Phys.*, 130(2):381–431, 1990. doi:10.1007/BF02473358. (Cited in pages 3, 12, 185).

[22] S. Doplicher, K. Fredenhagen, and J. E. Roberts. The quantum structure of spacetime at the Planck scale and quantum fields. *Commun. Math. Phys.*, 172(1):187–220, 1995. arXiv:hep-th/0303037, doi:10.1007/BF02104515. (Cited in pages 3, 3, 3, 186).

[23] J. J. Heckman and H. Verlinde. Covariant non-commutative space-time. *Nucl. Phys. B*, 894:58–74, 2015. doi:10.1016/j.nuclphysb.2015.02.018. (Cited in page 3).

[24] C. N. Yang. On Quantized Space-Time. *Phys. Rev.*, 72(9):874–874, 1947. doi:10.1103/PhysRev.72.874. (Cited in page 3).

[25] M. Maggiore. A generalized uncertainty principle in quantum gravity. *Phys. Lett. B*, 304(1-2):65–69, 1993. arXiv:hep-th/9301067, doi:10.1016/0370-2693(93)91401-8. (Cited in page 3).

[26] M. Maggiore. The algebraic structure of the generalized uncertainty principle. *Phys. Lett. B*, 319(1-3):83–86, 1993. arXiv:hep-th/9309034, doi:10.1016/0370-2693(93)90785-G. (Cited in pages 3, 185).

[27] S. Doplicher, K. Fredenhagen, and J. E. Roberts. Spacetime quantization induced by classical gravity. *Phys. Lett. B*, 331(1-2):39–44, 1994. doi:10.1016/0370-2693(94)90940-7. (Cited in page 3).

[28] L. J. Garay. Quantum gravity and minimum length. *Int. J. Mod. Phys. A*, 10(2):145–165, 1995. arXiv:gr-qc/9403008, doi:10.1142/S0217751X95000085. (Cited in page 3).

[29] S. Hossenfelder. Minimal length scale scenarios for quantum gravity. *Living Rev. Relativ.*, 16(2):1–90, 2013. arXiv:1203.6191, doi:10.12942/lrr-2013-2. (Cited in page 3).

[30] R. J. Szabo. Quantum field theory on noncommutative spaces. *Phys. Rep.*, 378(4):207–299, 2003. arXiv:hep-th/0109162, doi:10.1016/S0370-1573(03)00059-0. (Cited in pages 3, 186).

[31] M. Jimbo. A q-difference analogue of U(g) and the Yang-Baxter equation. *Lett. Math. Phys.*, 10(1):63–69, 1985. doi:10.1007/BF00704588. (Cited in pages 5, 8, 112).

[32] L. A. Takhtadzhyan. Introduction to quantum group. In M.-L. Ge and B.-H. Zao, editors, *Introd. to quantum Gr. Integr. massive Model. quantum F. theory*, page 69, Singapore, 1990. World Scientific. doi:10.1142/1094. (Cited in page 5).

[33] N. Y. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev. Quantization of Lie groups and Lie algebras. *Leningr. Math. J.*, 1(1):193–225, 1990. (Cited in pages 5, 5, 112).

[34] S. Majid. *Foundations of quantum group theory*. Cambridge University Press, Cambridge, 1995. doi:10.1017/CBO9780511613104. (Cited in pages 5, 54, 125).

[35] V. Chari and A. Pressley. *A guide to Quantum Groups*. Cambridge University Press, Cambridge, 1994. (Cited in pages 5, 53, 54, 57, 62, 64, 64, 75, 79, 119, 125).

[36] J. Lukierski and A. Nowicki. Doubly special relativity versus κ-deformation of relativistic kinematics. *Int. J. Mod. Phys. A*, 18(1):7–18, 2003. arXiv:hep-th/0203065v3, doi:10.1142/S0217751X03013600. (Cited in pages 5, 8).

[37] L. Freidel, J. Kowalski-Glikman, and L. Smolin. 2+1 gravity and doubly special relativity. *Phys. Rev. D*, 69(4):044001, 2004. arXiv:hep-th/0307085, doi:10.1103/PhysRevD.69.044001. (Cited in pages 5, 10).

[38] G. Amelino-Camelia. Doubly-special relativity: Facts, myths and some key open issues. *Symmetry*, 2(1):230–271, 2010. arXiv:1003.3942, doi:10.3390/sym2010230. (Cited in page 5, 5).

[39] V. Drinfeld. On Poisson homogeneous spaces of Poisson-Lie groups. *Theor. Math. Phys.*, 95(2):524–525, 1993. doi:10.1007/BF01017137. (Cited in pages 5, 6, 7, 67).

[40] M. S. Dijkhuizen and T. H. Koornwinder. Quantum homogeneous spaces, duality and quantum 2-spheres. *Geom. Dedicata*, 52(3):291–315, 1994. doi:10.1007/BF01278478. (Cited in page 5).

[41] S. Zakrzewski. Poisson homogeneous spaces. In J. Lukierski, Z. Popowicz, and J. Sobczyk, editors, *Quantum groups, formalism Appl. Proc. XXX Winter Sch. Theor. Phys. 1426February 1994*, pages 629–639, Karpacz, 1995. Polish Scientific Publishers PWN. (Cited in page 5).

[42] P. Podleś. Mathematical Physics Spaces of Quantum SU(2) and SO(3) Groups. *Commun. Math. Phys.*, 170:1–20, 1995. doi:10.1007/BF02099436. (Cited in page 5).

[43] A. G. Reyman. Poisson Structures Related to Quantum Groups. In L. Castellani and J. Wess, editors, *Quantum Groups Their Appl. Physics, Proc. Int. Sch. Phys. "Enrico Fermi"*, volume 127, pages 407–443, Amsterdam, 1996. IOS Press. doi:10.3254/978-1-61499-213-4-407. (Cited in pages 5, 7).

[44] N. Ciccoli. Quantum planes and quantum cylinders from Poisson homogeneous spaces. *J. Phys. A: Math. Gen.*, 29(7):1487–1495, 1996. arXiv:q-alg/9504023, doi:10.1088/0305-4470/29/7/018. (Cited in pages 5, 7).

[45] F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace, and M. Tarlini. Free q-Schrödinger equation from homogeneous spaces of the 2-dim euclidean quantum group. *Commun. Math. Phys.*, 175(1):161–176, 1996. doi:10.1007/BF02101628. (Cited in page 5).

[46] T. Brzeziński. Quantum homogeneous spaces as quantum quotient spaces. *J. Math. Phys.*, 37(5):2388–2399, 1996. arXiv:q-alg/9509015, doi:10.1063/1.531517. (Cited in page 5).

[47] A. Y. Alekseev and A. Z. Malkin. Symplectic structure of the moduli space of flat connection on a Riemann surface. *Commun. Math. Phys.*, 169(1):99–119, 1995. arXiv:hep-th/9312004, doi:10.1007/BF02101598. (Cited in pages 5, 12).

[48] V. V. Fock and A. A. Rosly. Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix. *Am. Math. Soc. Transl.*, 191:67–86, 1999. arXiv:math/9802054v2. (Cited in pages 5, 12, 12, 145).

[49] A. Achúcarro and P. K. Townsend. A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories. *Phys. Lett. B*, 180(1-2):89–92, 1986. doi:10.1016/0370-2693(86)90140-1. (Cited in pages 5, 12).

[50] E. Witten. 2+1 dimensional gravity as an exactly soluble system. *Nucl. Phys. B*, 311(1):46–78, 1988. doi:10.1016/0550-3213(88)90143-5. (Cited in pages 5, 12).

[51] C. Meusburger and B. J. Schroers. Quaternionic and Poisson-Lie structures in three-dimensional gravity: the cosmological constant as deformation parameter. *J. Math. Phys.*, 49(8):083510, 2008. arXiv:0708.1507, doi:10.1063/1.2973040. (Cited in pages 5, 12).

[52] C. Meusburger and B. J. Schroers. Generalised Chern-Simons actions for 3d gravity and κ-Poincaré symmetry. *Nucl. Phys. B*, 806(3):462–488, 2009. arXiv:0805.3318v2, doi:10.1016/j.nuclphysb.2008.06.023. (Cited in pages 5, 12, 12).

[53] G. Papageorgiou and B. J. Schroers. Galilean quantum gravity with cosmological constant and the extended q-Heisenberg algebra. *J. High Energy Phys.*, 11:020, 2010. arXiv:1008.0279v1, doi:10.1007/JHEP11(2010)020. (Cited in pages 5, 170).

[54] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar. Tests of quantum gravity from observations of γ-ray bursts. *Nature*, 393(6687):763–765, 1998. arXiv:astro-ph/9712103, doi:10.1038/31647. (Cited in pages 5, 11).

[55] D. Mattingly. Modern tests of Lorentz invariance. *Living Rev. Relativ.*, 8(1):5, dec 2005. arXiv:gr-qc/0502097, doi:10.12942/lrr-2005-5. (Cited in page 5).

[56] S. Zakrzewski. Poisson space-time symmetry and corresponding elementary systems. In H. D. Doebner and V. K. Dobrev, editors, *'Quantum Symmetries', Procs. II Int. Wigner Symp.*, page 111, Goslar, 1991. (Cited in page 7).

[57] S. Zakrzewski. Poisson homogeneous spaces. *Proc. XXX Winter Sch. Theor. Phys.*, page 10, 1994. arXiv:hep-th/9412101. (Cited in page 7).

[58] P. Etingof and D. Kazhdan. Quantization of Poisson algebraic groups and Poisson homogeneous spaces. In A. Connes, K. Gawedki, and J. Zinn-Justin, editors, *Symmétries quantiques, (Les Houches 1995)*, page 935, Amsterdam, 1998. North-Holland. arXiv:q-alg/9510020. (Cited in page 7).

[59] N. Ciccoli and F. Gavarini. A quantum duality principle for coisotropic subgroups and Poisson quotients. *Adv. Math. (N. Y).*, 199(1):104–135, 2006. arXiv:math/0603376, doi:10.1016/j.aim.2005.01.009. (Cited in pages 7, 11, 103, 123, 126, 182).

[60] A. Ballesteros, C. Meusburger, and P. Naranjo. AdS Poisson homogeneous spaces and Drinfel'd doubles. *J. Phys. A: Math. Theor.*, 50(39):395202, 2017. arXiv:1701.04902v3, doi:10.1088/1751-8121/aa858c. (Cited in pages 7, 9, 67, 67, 70, 116, 179).

[61] A. Ballesteros, I. Gutierrez-Sagredo, and F. J. Herranz. Noncommutative spaces of worldlines. *Phys. Lett. B*, 792:175–181, 2019. arXiv:1902.09132, doi:10.1016/j.physletb.2019.03.029. (Cited in pages 7, 10, 91, 116).

[62] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoy. q-deformation of Poincaré algebra. *Phys. Lett. B*, 264(3-4):331–338, 1991. doi:10.1016/0370-2693(91)90358-W. (Cited in pages 7, 8, 8, 11, 94, 97, 108, 136).

[63] P. Maslanka. The n-dimensional κ-Poincaré algebra and group. *J. Phys. A: Math. Gen.*, 26(24):L1251–L1253, 1993. doi:10.1088/0305-4470/26/24/001. (Cited in pages 7, 8, 94, 97, 125, 130, 181, 183).

[64] S. Majid and H. Ruegg. Bicrossproduct structure of κ-Poincaré group and non-commutative geometry. *Phys. Lett. B*, 334(3-4):348–354, 1994. doi:10.1016/0370-2693(94)90699-8. (Cited in pages 7, 8, 11, 92, 94, 97, 124, 125, 128, 130, 181, 183).

[65] S. Zakrzewski. Quantum Poincaré group related to the κ-Poincaré algebra. *J. Phys. A: Math. Gen.*, 27(6):2075–2082, 1994. doi:10.1088/0305-4470/27/6/030. (Cited in pages 7, 8, 8, 97, 125, 181, 183).

[66] A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander. A new "null-plane" quantum Poincaré algebra. *Phys. Lett. B*, 351(1-3):137–145, 1995. arXiv:q-alg/9502019, doi:10.1016/0370-2693(95)00386-Y. (Cited in page 8).

[67] A. Ballesteros, N. R. Bruno, and F. J. Herranz. A non-commutative Minkowskian spacetime from a quantum AdS algebra. *Phys. Lett. B*, 574(3-4):276–282, 2003. arXiv:hep-th/0306089, doi:10.1016/j.physletb.2003.09.014. (Cited in page 8).

[68] G. Amelino-Camelia, L. Smolin, and A. Starodubtsev. Quantum symmetry, the cosmological constant and Planck-scale phenomenology. *Class. Quantum Gravity*, 21(13):3095–3110, 2004. arXiv:hep-th/0306134, doi:10.1088/0264-9381/21/13/002. (Cited in pages 8, 11, 12, 129).

[69] A. Marcianò, G. Amelino-Camelia, N. R. Bruno, G. Gubitosi, G. Mandanici, and A. Melchiorri. Interplay between curvature and Planck-scale effects in astrophysics and cosmology. *J. Cosmol. Astropart. Phys.*, 06:030, 2010. arXiv:1004.1110, doi:10.1088/1475-7516/2010/06/030. (Cited in page 8, 8).

[70] G. Amelino-camelia, L. Barcaroli, G. D'Amico, N. Loret, and G. Rosati. IceCube and GRB neutrinos propagating in quantum spacetime. *Phys. Lett. B*, 761:318–325, 2016. arXiv:1605.00496, doi:10.1016/j.physletb.2016.07.075. (Cited in pages 8, 8, 11).

[71] G. Amelino-Camelia, G. Gubitosi, and G. Palmisano. Pathways to relativistic curved momentum spaces: de Sitter case study. *Int. J. Mod. Phys. D*, 25(02):1650027, 2016. doi:10.1142/s0218271816500279. (Cited in page 8).

[72] L. Barcaroli and G. Gubitosi. Kinematics of particles with quantum-de Sitter-inspired symmetries. *Phys. Rev. D*, 93(12):1–11, 2016. arXiv:1512.03462, doi:10.1103/PhysRevD.93.124063. (Cited in page 8).

[73] S. Majid. Hopf algebras for physics at the Planck scale. *Class. Quantum Gravity*, 5(12):1587–1606, 1988. doi:10.1088/0264-9381/5/12/010. (Cited in page 8).

[74] J. Lukierski, A. Nowicki, and H. Ruegg. Real forms of complex quantum Anti de sitter algebra Uq (Sp(4;C) and their contraction schemes. *Phys. Lett. B*, 271(3-4):321–328, 1991. arXiv:hep-th/9108018, doi:10.1016/0370-2693(91)90094-7. (Cited in pages 8, 11, 108, 136).

[75] S. Giller, P. Kosinski, M. Majewski, P. Maslanka, and J. Kunz. More about the q-deformed Poincaré algebra. *Phys. Lett. B*, 286(1-2):57–62, 1992. doi:10.1016/0370-2693(92)90158-Z. (Cited in pages 8, 108, 136).

[76] J. Lukierski, A. Nowicki, and H. Ruegg. New quantum Poincaré algebra and κ-deformed field theory. *Phys. Lett. B*, 293(3-4):344–352, 1992. doi:10.1016/0370-2693(92)90894-A. (Cited in pages 8, 11, 108, 136).

[77] J. Lukierski, H. Ruegg, and W. J. Zakrzewski. Classical and quantum mechanics of free κ-relativistic systems. *Ann. Phys. (N. Y).*, 243(1):90–116, 1995. doi:10.1006/aphy.1995.1092. (Cited in page 8).

[78] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini. The quantum Heisenberg group H(1)q. *J. Math. Phys.*, 32:1155–1158, 1991. doi:10.1063/1.529311. (Cited in page 8).

[79] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini. Contractions of quantum groups. *Lect. Notes Math.*, 1510:221–244, 1992. doi:10.1007/BFb0101174. (Cited in page 8).

[80] A. Ballesteros, N. A. Gromov, F. J. Herranz, M. A. del Olmo, and M. Santander. Lie bialgebra contractions and quantum deformations of quasi-orthogonal algebras. *J. Math. Phys.*, 36(10):5916–5937, 1995. arXiv:hep-th/9412083v3, doi:10.1063/1.531368. (Cited in pages 8, 13, 100, 102, 153).

[81] G. Amelino-Camelia and S. Majid. Waves on noncommutative space-time and gamma-ray bursts. *Int. J. Mod. Phys. A*, 15(27):4301–4323, oct 2000. arXiv:hep-th/9907110, doi:10.1142/S0217751X00002779. (Cited in pages 8, 10).

[82] A. Borowiec and A. Pachol. κ-Minkowski spacetime as the result of Jordanian twist deformation. *Phys. Rev. D*, 79(4):04501, 2009. arXiv:0812.0576, doi:10.1103/PhysRevD.79.045012. (Cited in page 8).

[83] A. Borowiec, K. S. Gupta, S. Meljanac, and A. Pachol. Constraints on the quantum gravity scale from κ-Minkowski spacetime. *EPL (Europhysics Lett.*, 92(2):20006, 2010. arXiv:0912.3299, doi:10.1209/0295-5075/92/20006. (Cited in page 8).

[84] M. Khodadi and K. Nozari. Some features of scattering problem in a κ-Deformed minkowski spacetime. *Ann. Phys.*, 528(11-12):785–795, 2016. arXiv:1607.02869, doi:10.1002/andp.201600072. (Cited in page 8).

[85] P. Aschieri, A. Borowiec, and A. Pachol. Observables and dispersion relations in κ-Minkowski spacetime. *J. High Energy Phys.*, 2017(10):152, 2017. arXiv:1703.08726, doi:10.1007/JHEP10(2017)152. (Cited in page 8).

[86] G. Amelino-Camelia, N. Loret, and G. Rosati. Speed of particles and a relativity of locality in κ-Minkowski quantum spacetime. *Phys. Lett. B*, 700(2):150–156, 2011. arXiv:1102.4637, doi:10.1016/j.physletb.2011.04.054. (Cited in page 8).

[87] G. Gubitosi and F. Mercati. Relative locality in κ-Poincaré. *Class. Quantum Gravity*, 30(14):145002, 2013. arXiv:1106.5710, doi:10.1088/0264-9381/30/14/145002. (Cited in pages 8, 10, 11, 11, 128, 131).

[88] J. Lukierski, Z. Škoda, and M. Woronowicz. κ-deformed covariant quantum phase spaces as Hopf algebroids. *Phys. Lett. B*, 750:401–406, 2015. arXiv:1507.02612, doi:10.1016/j.physletb.2015.09.042. (Cited in page 8).

[89] A. Sitarz. Noncommutative differential calculus on the κ-Minkowski space. *Phys. Lett. B*, 349(1-2):42–48, 1995. arXiv:hep-th/9409014, doi:10.1016/0370-2693(95)00223-8. (Cited in page 8).

[90] T. Juric, S. Meljanac, D. Pikutić, and R. Štrajn. Toward the classification of differential calculi on κ-Minkowski space and related field theories. *J. High Energy Phys.*, 2015(7):055, 2015. arXiv:1502.02972, doi:10.1007/JHEP07(2015)055. (Cited in page 8).

[91] B. Durhuus and A. Sitarz. Star product realizations of κ-Minkowski space. *J. Noncommutative Geom.*, 7(3):605–645, 2013. arXiv:1104.0206, doi:10.4171/JNCG/129. (Cited in page 8).

[92] M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, and M. Wohlgenannt. Deformed field theory on κ-spacetime. *Eur. Phys. J. C*, 31(1):129–138, 2003. arXiv:hep-th/0307149, doi:10.1140/epjc/s2003-01309-y. (Cited in page 8).

[93] L. Freidel, J. Kowalski-Glikman, and S. Nowak. From noncommutative κ-Minkowski to Minkowski space-time. *Phys. Lett. B*, 648(1):70–75, 2007. arXiv:hep-th/0612170, doi:10.1016/j.physletb.2007.02.056. (Cited in page 8).

[94] M. Dimitrijević, L. Jonke, and A. Pachol. Gauge theory on twisted κ-Minkowski: old problems and possible solutions. *Symmetry, Integr. Geom. Methods Appl.*, 10:063, 2014. arXiv:1403.1857, doi:10.3842/SIGMA.2014.063. (Cited in page 8).

[95] A. Agostini. κ-Minkowski representations on Hilbert spaces. *J. Math. Phys.*, 48(5):052305, 2007. arXiv:hep-th/0512114, doi:10.1063/1.2738360. (Cited in pages 8, 112).

[96] F. Lizzi, M. Manfredonia, F. Mercati, and T. Poulain. Localization and reference frames in κ-Minkowski spacetime. *Phys. Rev. D*, 99(8):085003, 2019. arXiv:1811.08409, doi:10.1103/PhysRevD.99.085003. (Cited in pages 8, 112, 187, 187, 187).

[97] F. Mercati and M. Sergola. Light cone in a quantum spacetime. *Phys. Lett. B*, 787:105–110, 2018. arXiv:1810.08134, doi:10.1016/j.physletb.2018.10.031. (Cited in page 8).

[98] G. Amelino-Camelia. Testable scenario for Relativity with minimum-length. *Phys. Lett. B*, 510(1-4):255–263, 2001. arXiv:hep-th/0012238, doi:10.1016/S0370-2693(01)00506-8. (Cited in pages 8, 10).

[99] G. Amelino-Camelia. Doubly-special relativity: first results and key open problems. *Int. J. Mod. Phys. D*, 11(10):1643–1669, 2002. arXiv:gr-qc/0210063, doi:10.1142/S021827180200302X. (Cited in page 8).

[100] G. Amelino-Camelia. Relativity in spacetimes with short-distance structure governed by an observer-independent (Planckian) length scale. *Int. J. Mod. Phys. D*, 11(1):35–59, 2002. arXiv:gr-qc/0012051, doi:10.1142/S0218271802001330. (Cited in pages 8, 10).

[101] J. Magueijo and L. Smolin. Lorentz Invariance with an Invariant Energy Scale. *Phys. Rev. Lett.*, 88(19):190403, 2002. arXiv:hep-th/0112090, doi:10.1103/PhysRevLett.88.190403. (Cited in page 8).

[102] J. Kowalski-Glikman and S. Nowak. Doubly special relativity theories as different bases of κ-poincaré algebra. *Phys. Lett. B*, 539(1-2):126–132, 2002. arXiv:hep-th/0203040, doi:10.1016/S0370-2693(02)02063-4. (Cited in pages 8, 94).

[103] J. Kowalski-Glikman. De Sitter space as an arena for doubly special relativity. *Phys. Lett. B*, 547(3-4):291–296, 2002. arXiv:hep-th/0207279, doi:10.1016/S0370-2693(02)02762-4. (Cited in pages 8, 10, 12, 124).

[104] J. Kowalski-Glikman and S. Nowak. Doubly special relativity and de Sitter space. *Class. Quantum Gravity*, 20(22):4799–1816, 2003. arXiv:hep-th/0304101, doi:10.1088/0264-9381/20/22/006. (Cited in pages 8, 10).

[105] A. Ballesteros, N. R. Bruno, and F. J. Herranz. A new doubly special relativity' theory from a quantum Weyl - Poincaré algebra. *J. Phys. A: Math. Gen.*, 36:10493–10503, 2003. arXiv:hep-th/0305033, doi:10.1088/0305-4470/36/42/006. (Cited in page 8).

[106] A. Borowiec and A. Pachol. κ-Minkowski spacetimes and DSR algebras: fresh look and old problems. *Symmetry, Integr. Geom. Methods Appl.*, 6(086):31 pages, 2010. arXiv:1005.4429, doi:10.3842/SIGMA.2010.086. (Cited in page 8).

[107] G. Amelino-Camelia, A. Marcianò, M. Matassa, and G. Rosati. Deformed Lorentz symmetry and relative locality in a curved/expanding spacetime. *Phys. Rev. D*, 86(12):124035, 2012. arXiv:1206.5315, doi:10.1103/PhysRevD.86.124035. (Cited in pages 8, 11).

[108] G. Rosati, G. Amelino-Camelia, A. Marcianò, and M. Matassa. Planck-scale-modified dispersion relations in FRW spacetime. *Phys. Rev. D*, 92(12):124042, 2015. arXiv:1507.02056, doi:10.1103/PhysRevD.92.124042. (Cited in page 8).

[109] A. Ballesteros, F. J. Herranz, and C. Meusburger. Three-dimensional gravity and Drinfel'd doubles: Spacetimes and symmetries from quantum deformations. *Phys. Lett. B*, 687(4-5):375–381, 2010. arXiv:1001.4228, doi:10.1016/j.physletb.2010.03.043. (Cited in pages 9, 13, 145).

[110] A. Ballesteros, F. J. Herranz, and C. Meusburger. Drinfel'd doubles for (2+1)-gravity. *Class. Quantum Gravity*, 30(15):155012, 2013. arXiv:1303.3080, doi:10.1088/0264-9381/30/15/155012. (Cited in pages 9, 12, 13, 13, 145, 145, 145, 157, 158, 158, 159, 165, 165, 165, 166, 169).

[111] A. Ballesteros, F. J. Herranz, C. Meusburger, and P. Naranjo. Twisted (2+1) κ-AdS algebra, Drinfel'd doubles and non-commutative spacetimes. *Symmetry, Integr. Geom. Methods Appl.*, 10:052, 26 pages, 2014. arXiv:1403.4773, doi:10.3842/SIGMA.2014.052. (Cited in pages 9, 13, 105, 110, 113, 145, 159).

[112] A. Ballesteros, F. J. Herranz, and C. Meusburger. A $(2+1)$ non-commutative Drinfel'd double spacetime with cosmological constant. *Phys. Lett. B*, 732:201–209, 2014. arXiv:1402.2884, doi:10.1016/j.physletb.2014.03.036. (Cited in pages 9, 13, 145, 146, 149).

[113] A. Ballesteros, F. J. Herranz, and P. Naranjo. Towards (3+1) gravity through Drinfel'd doubles with cosmological constant. *Phys. Lett. B*, 746:37–43, 2015. arXiv:1502.07518, doi:10.1016/j.physletb.2015.04.041. (Cited in pages 9, 102, 170).

[114] A. Ballesteros, F. J. Herranz, F. Musso, and P. Naranjo. The κ-(A)dS quantum algebra in $(3+1)$ dimensions. *Phys. Lett. B*, 766:205–211, 2017. arXiv:1612.03169, doi:10.1016/j.physletb.2017.01.020. (Cited in pages 9, 9, 11, 91, 98, 99, 102, 102, 103, 106, 107, 110, 137, 170, 183).

[115] A. Ballesteros, I. Gutierrez-Sagredo, and F. J. Herranz. The κ-(A)dS noncommutative spacetime. *Phys. Lett. B*, 796:93–101, 2019. arXiv:1905.12358, doi:10.1016/j.physletb.2019.07.038. (Cited in pages 9, 91, 99).

[116] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin. Relative locality: A deepening of the relativity principle. *Gen. Relativ. Gravit.*, 43(10):2547–2553, 2011. arXiv:1106.0313, doi:10.1007/s10714-011-1212-8. (Cited in pages 9, 10).

[117] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin. Principle of relative locality. *Phys. Rev. D*, 84(8):084010, 2011. arXiv:1101.0931, doi:10.1103/PhysRevD.84.084010. (Cited in pages 9, 10).

[118] M. Born. A suggestion for unifying quantum theory and relativity. *Proc. R. Soc. London. Ser. A. Math. Phys. Sci.*, 165(921):291–303, apr 1938. doi:10.1098/rspa.1938.0060. (Cited in page 10).

[119] D. Rätzel, S. Rivera, and F. P. Schuller. Geometry of physical dispersion relations. *Phys. Rev. D*, 83(4):044047, feb 2011. doi:10.1103/PhysRevD.83.044047. (Cited in page 10).

[120] S. Mignemi and A. Samsarov. Relative-locality effects in Snyder spacetime. *Phys. Lett. Sect. A Gen. At. Solid State Phys.*, 381(19):1655–1658, 2017. arXiv:1610.09692, doi:10.1016/j.physleta.2017.03.033. (Cited in page 10).

[121] J. Kowalski-Glikman. Living in curved momentum space. *Int. J. Mod. Phys. A*, 28(12):1330014, 2013. arXiv:1303.0195, doi:10.1142/S0217751X13300147. (Cited in pages 10, 94, 124, 126, 130, 130, 133).

[122] M. Arzano, G. Gubitosi, J. Magueijo, and G. Amelino-Camelia. Anti-de Sitter momentum space. *Phys. Rev. D*, 92(2):024028, 2015. arXiv:1412.2054, doi:10.1103/PhysRevD.92.024028. (Cited in page 10).

[123] H.-J. Matschull and M. Welling. Quantum Mechanics of a Point Particle in (2+1)-dimensional gravity. *Class. Quantum Gravity*, 15:2981–3030, 1998. arXiv:gr-qc/9708054v2, doi:10.1088/0264-9381/15/10/008. (Cited in pages 10, 146, 149, 170).

[124] C. Meusburger and B. J. Schroers. Poisson structure and symmetry in the Chern-Simons formulation of (2 + 1)-dimensional gravity. *Class. Quantum Gravity*, 20(11):2193–2233, 2003. arXiv:gr-qc/0301108, doi:10.1088/0264-9381/20/11/318. (Cited in page 10).

[125] L. Freidel and E. R. Livine. 3D quantum gravity and effective noncommutative quantum field theory. *Phys. Rev. Lett.*, 96(22):221301, 2006. arXiv:hep-th/0512113, doi:10.1103/PhysRevLett.96.221301. (Cited in page 10).

[126] S. Majid. *Towards quantum gravity*, volume 541 of *Lecture Notes in Physics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000. doi:10.1007/3-540-46634-7. (Cited in page 10).

[127] J. Kowalski-Glikman and S. Nowak. Non-commutative space-time of doubly special relativity theories. *Int. J. Mod. Phys. D*, 12(2):299–315, 2003. arXiv:hep-th/0204245, doi:10.1142/S0218271803003050. (Cited in page 10).

[128] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, and N. Loret. Dual redshift on Planck-scale-curved momentum spaces. *Class. Quantum Gravity*, 30(23):235002, 2013. arXiv:1305.5062, doi:10.1088/0264-9381/30/23/235002. (Cited in page 11).

[129] G. Amelino-Camelia, L. Barcaroli, and N. Loret. Modeling Transverse Relative Locality. *Int. J. Theor. Phys.*, 51(11):3359–3375, 2012. arXiv:1107.3334, doi:10.1007/s10773-012-1216-5. (Cited in page 11).

[130] G. Amelino-Camelia. Quantum-spacetime phenomenology. *Living Rev. Relativ.*, 16(1):5, 2013. arXiv:0806.0339, doi:10.12942/lrr-2013-5. (Cited in page 11).

[131] Fermi LAT/GBM Collaborations. A limit on the variation of the speed of light arising from quantum gravity effects. *Nature*, 462(7271):331–334, 2009. doi:10.1038/nature08574. (Cited in page 11).

[132] G. Amelino-Camelia and L. Smolin. Prospects for constraining quantum gravity dispersion with near term observations. *Phys. Rev. D*, 80(8):1–14, 2009. doi:10.1103/PhysRevD.80.084017. (Cited in page 11).

[133] G. Amelino-Camelia, G. D'Amico, G. Rosati, and N. Loret. In vacuo dispersion features for gamma-ray-burst neutrinos and photons. *Nat. Astron.*, 1(June):1–6, 2017. doi:10.1038/s41550-017-0139. (Cited in page 11).

[134] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer. Hamilton geometry: Phase space geometry from modified dispersion relations. *Phys. Rev. D*, 92(8):1–18, 2015. arXiv:1507.00922, doi:10.1103/PhysRevD.92.084053. (Cited in page 11).

[135] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer. Planck-scale-modified dispersion relations in homogeneous and isotropic spacetimes. *Phys. Rev. D*, 95(2):024036(15), 2017. arXiv:gr-qc/1612.01390, doi:10.1103/PhysRevD.95.024036. (Cited in page 11).

[136] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer. Curved spacetimes with local κ -Poincaré dispersion relation. *Phys. Rev. D*, 96(8):084010, 2017. arXiv:1703.02058, doi:10.1103/PhysRevD.96.084010. (Cited in page 11).

[137] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, S. Liberati, and N. Loret. Realization of doubly special relativistic symmetries in Finsler geometries. *Phys. Rev. D*, 90(12):125030, 2014. arXiv:1407.8143, doi:10.1103/PhysRevD.90.125030. (Cited in page 11).

[138] F. Cianfrani, J. Kowalski-Glikman, and G. Rosati. Generally covariant formulation of relative locality in curved spacetime. *Phys. Rev. D*, 89(4):044039, 2014. arXiv:1401.2057, doi:10.1103/PhysRevD.89.044039. (Cited in page 11).

[139] J. Lukierski and H. Ruegg. Quantum κ-Poincaré in any dimension. *Phys. Lett. B*, 329(2-3):189–194, 1994. arXiv:arXiv:hep-th/9310117v1, doi:10.1016/0370-2693(94)90759-5. (Cited in pages 11, 125).

[140] J. Kowalski-Glikman and S. Nowak. Quantum κ-Poincaré algebra from de Sitter space of momenta. nov 2004. arXiv:hep-th/0411154. (Cited in page 11).

[141] G. Amelino-Camelia, M. Arzano, J. Kowalski-Glikman, G. Rosati, and G. Trevisan. Relative-locality distant observers and the phenomenology of momentum-space geometry. *Class. Quantum Gravity*, 29(7):075007, 2012. arXiv:1107.1724v1, doi:10.1088/0264-9381/29/7/075007. (Cited in page 11).

[142] A. Ballesteros, F. J. Herranz, M. A. Del Olmo, and M. Santander. Quantum structure of the motion groups of the two-dimensional Cayley-Klein geometries. *J. Phys. A: Math. Gen.*, 26(21):5801–5823, 1993. doi:10.1088/0305-4470/26/21/019. (Cited in pages 11, 128).

[143] A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander. Quantum (2+1) kinematical algebras: a global approach. *J. Phys. A: Math. Gen.*, 27(4):1283–1297, 1994. doi:10.1088/0305-4470/27/4/021. (Cited in pages 11, 129, 153, 179).

[144] A. Ballesteros, N. R. Bruno, and F. J. Herranz. Non-commutative relativistic spacetimes and worldlines from 2+ 1 quantum (anti) de Sitter groups. *Adv. High Energy Phys.*, 2017:7876942 (19 pages), 2017. arXiv:hep-th/0401244, doi:10.1155/2017/7876942. (Cited in pages 11, 105, 114, 129, 181, 182).

[145] A. Ballesteros, G. Gubitosi, I. Gutierrez-Sagredo, and F. J. Herranz. Curved momentum spaces from quantum groups with cosmological constant. *Phys. Lett. B*, 773:47–53, 2017. arXiv:1707.09600, doi:10.1016/j.physletb.2017.08.008. (Cited in pages 11, 12, 94, 102, 136, 182, 182).

[146] A. Ballesteros, G. Gubitosi, I. Gutierrez-Sagredo, and F. J. Herranz. Curved momentum spaces from quantum (anti-)de Sitter groups in (3+1) dimensions. *Phys. Rev. D*, 97(10):106024, 2018. arXiv:1711.05050, doi:10.1103/PhysRevD.97.106024. (Cited in pages 11, 12, 94, 102, 103, 185).

[147] F. Gavarini. The quantum duality principle. *Ann. l'institut Fourier*, 52(3):809–834, 2002. arXiv:math/9909071, doi:10.5802/aif.1902. (Cited in pages 11, 182).

[148] I. Gutierrez-Sagredo, A. Ballesteros, G. Gubitosi, and F. J. Herranz. Quantum groups, non-commutative Lorentzian spacetimes and curved momentum spaces. pages 1–21. arXiv:1907.07979. (Cited in pages 12, 47, 50).

[149] V. G. Turaev and O. Y. Viro. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31(4):865–902, 1992. doi:10.1016/0040-9383(92)90015-A. (Cited in page 12).

[150] A. Y. Alekseev, H. Grosse, and V. Schomerus. Combinatorial quantization of the Hamiltonian Chern-Simons theory I. *Commun. Math. Phys.*, 172(2):317–358, 1995. arXiv:hep-th/9403066, doi:10.1007/BF02099431. (Cited in page 12).

[151] A. Y. Alekseev, H. Grosse, and V. Schomerus. Combinatorial quantization of the Hamiltonian Chern-Simons theory II. *Commun. Math. Phys.*, 174(3):561–604, 1996. arXiv:hep-th/9408097, doi:10.1007/BF02101528. (Cited in page 12).

[152] A. Y. Alekseev and V. Schomerus. Representation theory of Chern-Simons observables. *Duke Math. J.*, 85(2):447–510, 1996. arXiv:q-alg/9503016, doi:10.1215/S0012-7094-96-08519-1. (Cited in page 12).

[153] E. Buffenoir and P. Roche. Two dimensional lattice gauge theory based on a quantum group. *Commun. Math. Phys.*, 170(3):669–698, 1995. arXiv:hep-th/9405126, doi:10.1007/BF02099153. (Cited in page 12).

[154] C. Meusburger and B. J. Schroers. The quantisation of Poisson structures arising in Chern-Simons theory with gauge group G g. *Adv. Theor. Math. Phys.*, 7(6):1003–1043, 2003. arXiv:hep-th/0310218. (Cited in page 12, 12).

[155] K. Noui and P. Roche. Cosmological deformation of Lorentzian spin foam models. *Class. Quantum Gravity*, 20(14):3175–3213, 2003. arXiv:gr-qc/0211109, doi:10.1088/0264-9381/20/14/318. (Cited in page 12).

[156] W. J. Fairbairn and C. Meusburger. Quantum deformation of two four-dimensional spin foam models. *J. Math. Phys.*, 53(2):022501, 2012. arXiv:1012.4784, doi:10.1063/1.3675898. (Cited in page 12).

[157] S. Carlip. *Quantum gravity in 2+ 1 dimensions*. Cambridge University Press, 1998. doi:10.1017/CBO9780511564192. (Cited in page 12, 12, 12).

[158] S. Deser, R. Jackiw, and G. 't Hooft. Three-dimensional Einstein gravity: dynamics of flat space. *Ann. Phys. (N. Y).*, 152(1):220–235, 1984. doi:10.1016/0003-4916(84)90085-X. (Cited in page 12).

[159] S. Majid and B. J. Schroers. q -deformation and semidualization in 3D quantum gravity. *J. Phys. A: Math. Theor.*, 42(42):425402, 2009. arXiv:0806.2587, doi:10.1088/1751-8113/42/42/425402. (Cited in pages 12, 12, 13, 105, 143, 146, 149).

[160] A. Ballesteros, F. J. Herranz, and F. Musso. On quantum deformations of (anti-)de Sitter algebras in (2+1) dimensions. *J. Phys. Conf. Ser.*, 532(1):012002, 2014. arXiv:1302.0684v2, doi:10.1088/1742-6596/532/1/012002. (Cited in pages 12, 102, 152, 153, 178).

[161] A. Ballesteros, I. Gutierrez-Sagredo, F. J. Herranz, C. Meusburger, and P. Naranjo. Quantum groups and noncommutative spacetimes with cosmological constant. *J. Phys. Conf. Ser.*, 880(1):012023, 2017. arXiv:1702.04704, doi:10.1088/1742-6596/880/1/012023. (Cited in page 12).

[162] F. Mercati and M. Sergola. Physical constraints on quantum deformations of spacetime symmetries. *Nucl. Phys. B*, 933:320–339, 2018. arXiv:1802.09483, doi:10.1016/j.nuclphysb.2018.06.014. (Cited in pages 12, 178).

[163] F. A. Bais and N. M. Muller. Topological field theory and the quantum double of SU(2). *Nucl. Phys. B*, 530(1-2):349–400, 1998. arXiv:hep-th/9804130, doi:10.1016/S0550-3213(98)00572-0. (Cited in pages 12, 170).

[164] E. Buffenoir, K. Noui, and P. Roche. Hamiltonian quantization of Chern Simons theory with SL(2,R) group. *Class. Quantum Gravity*, 19(19):4953–5015, 2002. arXiv:hep-th/0202121, doi:10.1088/0264-9381/19/19/313. (Cited in page 12).

[165] C. Meusburger and K. Noui. The Hilbert space of 3d gravity: quantum group symmetries and observables. *Adv. Theor. Math. Phys.*, 14(6):1651–1715, 2010. arXiv:0809.2875. (Cited in page 12).

[166] P. Stachura. Poisson-Lie structures on Poincaré and Euclidean groups in three dimensions. *J. Phys. A: Math. Gen.*, 31(19):4555–4564, 1998. doi:10.1088/0305-4470/31/19/018. (Cited in pages 13, 147, 148, 152, 153, 154, 155, 155, 157, 159, 165, 165, 166, 178, 179, 195, 196, 197).

[167] A. Borowiec, J. Lukierski, and V. N. Tolstoy. Quantum deformations of D=4 Euclidean, Lorentz, Kleinian and quaternionic o*(4) symmetries in unified o(4;C) setting. *Phys. Lett. B*, 754:176–181, 2016. arXiv:1511.03653, doi:10.1016/j.physletb.2016.01.016. (Cited in pages 13, 157, 178, 179).

[168] A. Borowiec, J. Lukierski, and V. N. Tolstoy. Addendum to Quantum deformations of D=4 Euclidean, Lorentz, Kleinian and quaternionic o*(4) symmetries in unified o(4;C) setting [Phys. Lett. B 754 (2016) 176181]. *Phys. Lett. B*, 770:426–430, 2017. arXiv:1704.06852, doi:10.1016/j.physletb.2017.04.070. (Cited in pages 13, 157, 179).

[169] A. Ballesteros, I. Gutierrez-Sagredo, and F. J. Herranz. The Poincaré group as a Drinfel'd double. *Class. Quantum Gravity*, 36(2):025003, 2019. arXiv:1809.09207, doi:10.1088/1361-6382/aaf3c2. (Cited in pages 13, 146, 158, 166).

[170] A. Ballesteros, I. Gutierrez-Sagredo, and F. J. Herranz. Drinfel'd double structures for Poincaré and Euclidean groups. *J. Phys. Conf. Ser.*, 1194(1):012041, 2019. arXiv:1812.02075, doi:10.1088/1742-6596/1194/1/012041. (Cited in pages 13, 146).

[171] A. Ballesteros, I. Gutierrez-Sagredo, and F. Mercati. Coreductive Lie bialgebras and dual homogeneous spaces. 2019. arXiv:1909.01000. (Cited in page 14).

[172] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Volume I.* Interscience Publishers, 1963. (Cited in pages 16, 26, 29).

[173] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Volume II.* Interscience Publishers, 1969. (Cited in pages 16, 31, 31, 32, 32, 33, 33, 33, 34, 174, 184).

[174] R. Gilmore. *Lie groups, Lie algebras and some of their applications.* John Wiley & Sons, Inc., 1974. (Cited in page 16).

[175] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34. Academic Press, Inc., New York London, 1978. doi:10.1090/gsm/034. (Cited in pages 16, 174).

[176] B. O'Neill. *Semi-Riemannian manifolds with applications to relativity.* 1983. (Cited in pages 16, 26).

[177] V. V. Gorbatsevich, E. B. Vinberg, and A. L. Onishchik. *Foundations of Lie Theory and Lie Transformation Groups.* Springer Berlin Heidelberg, Berlin, 2nd printi edition, 1997. (Cited in pages 16, 19, 27).

[178] J. Fuchs and C. Schweigert. *Symmetries, Lie algebras and representations.* Cambridge University Press, Cambridge, 1997. (Cited in page 16).

[179] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer New York, New York, NY, 2012. doi:10.1007/978-1-4419-9982-5. (Cited in pages 16, 17).

[180] C. Chevalley. *Theory of Lie groups.* Princeton University Press, 1946. (Cited in page 26).

[181] S. S. Chern. Pseudo-groupes continus infinis. In *Géométrie différentielle*, volume 52, pages 119–136, Strasbourg, 1953. (Cited in pages 27, 184).

[182] D. Bernard. Sur la géométrie différentielle des G-structures. *Ann. l'institut Fourier*, 10:151–270, 1960. doi:10.5802/aif.99. (Cited in page 27).

[183] A. Fujimoto. On the structure tensor of G-structure. *Mem. Coll. Sci. Univ. Kyoto. Ser. A Math.*, 33(1):157–169, 1960. doi:10.1215/kjm/1250776063. (Cited in page 27).

[184] S. S. Chern. The geometry of G-structures. *Bull. Am. Math. Soc.*, 72(2):167–219, 1966. (Cited in page 27).

[185] H. Weyl. *Raum · Zeit · Materie*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1923. doi:10.1007/978-3-642-98950-6. (Cited in page 28).

[186] H. Cartan. La transgression dans un groupe de Lie et dans un espace fibré principal. In *Colloq. Topol. (espaces fibrés)*, pages 57–71, 1950. (Cited in page 28).

[187] F. J. Herranz and M. Santander. Casimir invariants for the complete family of quasisimple orthogonal algebras. *J. Phys. A: Math. Gen.*, 30(15):5411–5426, 1997. arXiv:physics/9702032, doi:10.1088/0305-4470/30/15/026. (Cited in pages 35, 43).

[188] J. Maldacena. The large N Limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2(2):231–252, 1998. arXiv:hep-th/9711200, doi:10.4310/ATMP.1998.v2.n2.a1. (Cited in page 39).

[189] M. Santander and F. J. Herranz. Geometries of orthogonal groups and their contractions: a unified classical deformation viewpoint. *Int. J. Mod. Phys. A*, 12(01):99–107, jan 1997. arXiv:hep-th/9610184, doi:10.1142/S0217751X97000128. (Cited in pages 41, 43).

[190] F. J. Herranz and M. Santander. Conformal symmetries of spacetimes. *J. Phys. A: Math. Gen.*, 35(31):6601–6618, aug 2002. doi:10.1088/0305-4470/35/31/306. (Cited in page 43).

[191] F. J. Herranz. *Grupos de Cayley Klein clásicos y cuánticos*. PhD thesis, Universidad de Valladolid, 1996. (Cited in page 44).

[192] I. Vaisman. *Lectures on the geometry of Poisson manifolds*. Springer Basel AG, 1994. doi:10.1007/978-3-0348-8495-2. (Cited in page 53).

[193] N. Ciccoli. From Poisson to Quantum Geometry, 2006. (Cited in pages 53, 58, 60).

[194] C. Meusburger. Introduction to Poisson-Lie Groups, 2018. (Cited in pages 53, 62, 64, 65).

[195] T. Tjin. An introduction to quantized Lie groups and algebras. *Int. J. Mod. Phys. A*, 7(25):6175–6213, 1992. arXiv:hep-th/9111043, doi:10.1142/S0217751X92002805. (Cited in pages 54, 65, 77).

[196] C. Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer New York, New York, 1995. arXiv:arXiv:1011.1669v3, doi:10.1007/978-1-4612-0783-2. (Cited in page 54).

[197] C. Meusburger. Hopf Algebras and Representation Theory of Hopf Algebras, 2017. (Cited in page 54).

[198] G. Darboux. Sur le problème de Pfaff. *Bull. des Sci. Mathématiques Astron.*, 6(1):14–36, 1882. (Cited in page 57).

[199] A. Weinstein. The local structure of Poisson manifolds. *J. Differ. Geom.*, 18(3):523–557, 1983. doi:10.4310/jdg/1214437787. (Cited in page 57, 57).

[200] P. Etingof and D. Kazhdan. *Quantization of Lie bialgebras, I*, volume 2. 1996. arXiv:q-alg/9506005, doi:10.1007/BF01587938. (Cited in pages 61, 61, 101).

[201] T. Andrzejewski and J. M. Figueroa-O'Farrill. Kinematical lie algebras in 2 + 1 dimensions. *J. Math. Phys.*, 59(6):061703, 2018. arXiv:1802.04048v2, doi:10.1063/1.5025785. (Cited in pages 63, 170).

[202] J. M. Figueroa-O'Farrill. Kinematical Lie algebras via deformation theory. *J. Math. Phys.*, 59(6):061701, 2018. arXiv:1711.06111v2, doi:10.1063/1.5016288. (Cited in pages 63, 170).

[203] J. M. Figueroa-O'Farrill. Higher-dimensional kinematical Lie algebras via deformation theory. *J. Math. Phys.*, 59(6):061702, 2018. arXiv:1711.07363v2, doi:10.1063/1.5016616. (Cited in pages 63, 169, 170).

[204] S. Evens and J. Lu. On the variety of Lagrangian subalgebras, I. *Ann. Sci. l'École Norm. Supérieure*, 34:631–668, oct 2001. arXiv:math/9909005, doi:10.1016/S0012-9593(01)01072-2. (Cited in page 69).

[205] A. Ballesteros and F. Musso. Quantum algebras as quantizations of dual Poisson-Lie groups. *J. Phys. A: Math. Theor.*, 46(19):195203, 2013. arXiv:1212.3809, doi:10.1088/1751-8113/46/19/195203. (Cited in pages 87, 88, 88, 103, 104, 106, 123, 125, 126, 132, 132, 182, 183).

[206] A. Ballesteros, F. J. Herranz, M. A. Del Olmo, and M. Santander. Classical deformations, Poisson-Lie contractions, and quantization of dual Lie bialgebras. *J. Math. Phys.*, 36(2):631–640, 1995. doi:10.1063/1.531331. (Cited in pages 100, 187).

[207] M. Daszkiewicz. Canonical and Lie-algebraic twist deformations of κ-Poincaré and contractions to κ-Galilei algebras. *Int. J. Mod. Phys. A*, 23(26):4387–4400, 2008. arXiv:0802.1974, doi:10.1142/S0217751X08042262. (Cited in page 102).

[208] M. A. Semenov-Tian-Shanskii. Poisson-Lie groups. The quantum duality principle and the twisted quantum double. *Theor. Math. Phys.*, 93(2):1292–1307, 1992. arXiv:hep-th/9304042, doi:10.1007/BF01083527. (Cited in pages 103, 123, 126, 182).

[209] P. K. Osei and B. J. Schroers. Classical r-matrices for the generalised Chern-Simons formulation of 3d gravity. *Class. Quantum Gravity*, 35(7):075006, 2018. arXiv:1708.07650, doi:10.1088/1361-6382/aaaa5e. (Cited in page 105).

[210] V. Drinfeld. Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations. *Sov. Math. Dokl.*, 27:68–71, 1983. (Cited in page 106).

[211] J. Grabowski. Quantum SU(2) group of Woronowicz and Poisson structures. In J. Janyska and D. Krupka, editors, *Differ. Geom. its Appl. Proc. Conf. Brno 1989*, pages 313–322, Brno, Czechoslovakia, 1990. World Scientific. (Cited in page 110).

[212] J. Grabowski. Poisson Lie groups and their relations to quantum groups. *Panoramas Math. Banach Cent. Publ.*, 34:55–64, 1995. doi:10.4064/-34-1-55-64. (Cited in page 110).

[213] S. L. Woronowicz. Twisted SU (2) group. An example of a non-commutative differential calculus. *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, 23:117–181, 1987. doi:10.2977/prims/1195176848. (Cited in page 110).

[214] S. L. Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111(4):613–665, 1987. doi:10.1007/BF01219077. (Cited in page 110).

[215] L. L. Vaksman and Y. S. Soibel'man. Algebra of functions on the quantum group SU(2). *Funct. Anal. Its Appl.*, 22(3):170–181, 1988. doi:10.1007/BF01077623. (Cited in pages 110, 112).

[216] M. Chaichian and A. Demichev. *Introduction to quantum groups*. World Scientific, Boston, 1996. doi:10.1142/3065. (Cited in pages 110, 112).

[217] A. A. Fokas and I. M. Gel'fand. Quadratic Poisson algebras and their infinite-dimensional extensions. *J. Math. Phys.*, 35(6):3117–3131, 1994. doi:10.1063/1.530457. (Cited in page 110).

[218] R. J. Low. The geometry of the space of null geodesics. *J. Math. Phys.*, 30(4):809–811, 1989. doi:10.1063/1.528401. (Cited in page 114).

[219] J. K. Beem and P. E. Parker. The space of geodesics. *Geom. Dedicata*, 38(1):87–99, apr 1991. doi:10.1007/BF00147737. (Cited in page 114).

[220] A. L. Besse. *Manifolds all of whose geodesics are closed*, volume 93. Springer Berlin Heidelberg, 1978. (Cited in page 114).

[221] D. V. Alekseevsky, B. Guilfoyle, and W. Klingenberg. On the Geometry of Spaces of Oriented Geodesics. *Ann. Glob. Anal. Geom.*, 40:389–409, 2011. arXiv:0911.2602v1, doi:10.1007/s10455-011-9261-5. (Cited in page 114).

[222] F. J. Herranz and M. Santander. Homogeneous phase spaces: the Cayley-Klein framework. In J. F. Cariñena, E. Martinez, and M. F. Rañada, editors, *Geom. y Fis. Memorias la Real Acad. Ciencias*, volume XXXII, pages 59–84, Madrid, 1998. arXiv:physics/9702030. (Cited in pages 114, 114, 116).

[223] T. D. Newton and E. P. Wigner. Localized states for elementary systems. *Rev. Mod. Phys.*, 21(3):400–406, jul 1949. doi:10.1103/RevModPhys.21.400. (Cited in page 120).

[224] P. K. Osei and B. J. Schroers. On the semiduals of local isometry groups in three-dimensional gravity. *J. Math. Phys.*, 53(7), 2012. arXiv:1109.4086, doi:10.1063/1.4731229. (Cited in page 143).

[225] X. Gomez. Classification of three-dimensional Lie bialgebras. *J. Math. Phys.*, 41(7):4939–4956, 2000. doi:10.1063/1.533385. (Cited in pages 145, 149, 149, 149, 150, 150, 150, 150, 150, 150, 150, 150, 150, 155, 157, 161, 161, 164, 165).

[226] L. Snobl and L. Hlavaty. Classification of 6-dimensional real Drinfeld doubles. *Int. J. Mod. Phys. A*, 17(28):4043–4067, 2002. arXiv:math/0202210v2, doi:10.1142/S0217751X02010571. (Cited in pages 145, 149, 149, 150, 150, 150, 150, 150, 150, 150).

[227] E. Batista and S. Majid. Noncommutative geometry of angular momentum space U(su(2)). *J. Math. Phys.*, 44(1):107–137, 2003. arXiv:hep-th/0205128, doi:10.1063/1.1517395. (Cited in pages 146, 149, 170).

[228] S. Majid. Noncommutative model with spontaneous time generation and Planckian bound. *J. Math. Phys.*, 46:103520, 2005. arXiv:hep-th/0507271, doi:10.1063/1.2084748. (Cited in pages 146, 149, 170).

[229] E. Joung, J. Mourad, and K. Noui. Three dimensional quantum geometry and deformed Poincaré symmetry. *J. Math. Phys.*, 50(5):052503, 2009. arXiv:0806.4121, doi:10.1063/1.3131682. (Cited in pages 146, 149, 170).

[230] S. Zakrzewski. Poisson Structures on the Poincaré Group. *Commun. Math. Phys.*, 185(2):285–311, 1997. arXiv:q-alg/9602001, doi:10.1007/s002200050091. (Cited in pages 147, 147, 178, 178, 179).

[231] J. Lukierski and M. Woronowicz. New Lie-algebraic and quadratic deformations of Minkowski space from twisted Poincaré symmetries. *Phys. Lett. B*, 633(1):116–124, 2006. arXiv:hep-th/0508083v3, doi:10.1016/j.physletb.2005.11.052. (Cited in page 152).

[232] C. R. Nappi and E. Witten. Wess-Zumino-Witten model based on a nonsemisimple group. *Phys. Rev. Lett.*, 71(23):3751–3753, 1993. arXiv:hep-th/9310112, doi:10.1103/PhysRevLett.71.3751. (Cited in page 160).

[233] D. Cangemi and R. Jackiw. Poincaré Gauge Theory for Gravitational Forces in (1+1) Dimensions. *Ann. Phys. (N. Y).*, 225:229–263, 1993. arXiv:hep-th/9302026, doi:10.1006/aphy.1993.1058. (Cited in page 160).

[234] A. Ballesteros and F. Mercati. Extended noncommutative Minkowski spacetimes and hybrid gauge symmetries. *Eur. Phys. J. C*, 78:615, 2018. arXiv:1805.07099, doi:10.1140/epjc/s10052-018-6097-1. (Cited in pages 161, 161, 162, 186).

[235] A. Ballesteros, E. Celeghini, and M. A. Del Olmo. Quantization of Drinfel'd doubles. *J. Phys. A: Math. Gen.*, 38:3909–3922, 2005. arXiv:math/0411389, doi:10.1088/0305-4470/38/18/003. (Cited in pages 161, 170).

[236] L. Hlavaty and L. Snobl. Classification of Poisson-Lie T-dual models with two-dimensional targets. *Mod. Phys. Lett. A*, 17(07):429–434, 2002. arXiv:hep-th/0110139, doi:10.1142/S0217732302006515. (Cited in pages 161, 170, 194).

[237] A. Ballesteros, F. J. Herranz, M. A. del Olmo, C. M. Pereña, and M. Santander. Non-standard quantum (1+1) Poincare group: a T-matrix approach. *J. Phys. A: Math. Gen.*, 28:7113–7125, 1995. arXiv:q-alg/9501029v1, doi:10.1088/0305-4470/28/24/012. (Cited in page 163).

[238] A. Ballesteros, F. J. Herranz, and C. M. Pereña. Null-plane quantum universal R-matrix. *Phys. Lett. B*, 391(1-2):71–77, 1997. doi:10.1016/S0370-2693(96)01435-9. (Cited in page 163).

[239] A. Ballesteros, F. J. Herranz, and P. Naranjo. From Lorentzian to Galilean (2+1) gravity: Drinfeld doubles, quantization and noncommutative spacetimes. *Class. Quantum Gravity*, 31:245013, 2014. arXiv:1408.3689, doi:10.1088/0264-9381/31/24/245013. (Cited in page 170).

[240] G. Papageorgiou and B. J. Schroers. A Chern-Simons approach to Galilean quantum gravity in 2+1 dimensions. *J. High Energy Phys.*, 11:009, 2009. arXiv:0907.2880, doi:10.1088/1126-6708/2009/11/009. (Cited in page 170).

[241] W.-K. Tung. *Group Theory in Physics*. World Scientific, Singapore, 1985. doi:10.1142/0097. (Cited in pages 183, 187).

[242] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu. On a Lorentz-invariant interpretation of noncommutative space-time and its implications on non-commutative QFT. *Phys. Lett. B*, 604(1-2):98–102, 2004. arXiv:hep-th/0408069, doi:10.1016/j.physletb.2004.10.045. (Cited in page 186).

[243] C. Klimčík and P. Ševera. Dual non-Abelian duality and the Drinfeld double. *Phys. Lett. B*, 351(4):455–462, 1995. arXiv:hep-th/9502122, doi:10.1016/0370-2693(95)00451-P. (Cited in page 194).

[244] K. Sfetsos. Poisson-Lie T-duality beyond the classical level and the renormalization group. *Phys. Lett. B*, 432(3-4):365–375, 1998. arXiv:hep-th/9803019, doi:10.1016/S0370-2693(98)00666-2. (Cited in page 194).

[245] M. A. Lledó and V. S. Varadarajan. SU(2) Poisson-Lie T duality. *Lett. Math. Phys.*, 45:247–257, 1998. doi:10.1023/A:1007498803198. (Cited in page 194).

[246] M. A. Jafarizadeh and A. Rezaei-Aghdam. Poisson-Lie T-duality and Bianchi type algebras. *Phys. Lett. B*, 458(4):477–490, 1999. arXiv:hep-th/9903152, doi:10.1016/S0370-2693(99)00571-7. (Cited in page 194).

[247] E. J. Beggs and S. Majid. Poisson-Lie T-duality for quasitriangular Lie bialgebras. *Commun. Math. Phys.*, 220(3):455–488, 2001. arXiv:math/9906040, doi:10.1007/s002200100463. (Cited in page 194).

[248] L. Hlavaty and L. Snobl. Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models. *J. High Energy Phys.*, 2004(05):010, 2004. arXiv:hep-th/0408126, doi:10.1088/1126-6708/2004/05/010. (Cited in page 194).

[249] P. Etingof and A. Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Commun. Math. Phys.*, 196(3):591–640, 1998. arXiv:q-alg/9708015, doi:10.1007/s002200050437. (Cited in page 194).

[250] P. Etingof and A. Varchenko. Geometry and classification of solutions of the classical dynamical Yang-Baxter equation. *Commun. Math. Phys.*, 192(1):77–120, 1998. arXiv:q-alg/9703040, doi:10.1007/s002200050292. (Cited in page 194).

[251] E. Karolinsky and A. Stolin. Classical dynamical r-matrices, Poisson homogeneous spaces, and Lagrangian subalgebras. *Lett. Math. Phys.*, 60(3):257–274, 2002. arXiv:math/0110319v2, doi:10.1023/A:1016231526203. (Cited in page 194).

[252] J. Lu. Classical dynamical r-matrices and homogeneous Poisson structures on G/H and K/T. *Commun. Math. Phys.*, 212(2):337–370, 2000. arXiv:math/9909004, doi:10.1007/s002200000209. (Cited in page 194).

[253] C. Meusburger and T. Schönfeld. Gauge fixing in (2+1)-gravity: Dirac bracket and spacetime geometry. *Class. Quantum Gravity*, 28(12), 2011. arXiv:1012.1835v1, doi:10.1088/0264-9381/28/12/125008. (Cited in page 194).

[254] C. Meusburger and T. Schönfeld. Gauge Fixing and Classical Dynamical r-Matrices in ISO(2, 1)-Chern-Simons Theory. *Commun. Math. Phys.*, 327(2):443–479, 2014. arXiv:1203.5609v2, doi:10.1007/s00220-014-1938-8. (Cited in page 194).