A FINITE ELEMENT APPROACH FOR THE TRAFFIC ASSIGNMENT PROBLEM

Víctor H. Cortínez
Centro de Investigaciones en Mecánica Teórica y Aplicada, Universidad Tecnológica Nacional,
Patricia N. Dominguez
Departamento de Ingeniería, Universidad Nacional del Sur

ABSTRACT

Traffic assignment problem (TAP) in static equilibrium conditions is based on the Wardrop´s first principle. The mathematical solution of this problem involves significant computation times for the analysis of large transportation networks, especially in the context of design. Thus, the development of efficient computational methodologies is of the greatest concern.

In this article, a reduced order model for TAP, based on a Finite Element Approach (FEA), is proposed. Such methodology involves four main ideas: a) the TAP formulation including as variables the travel times from any node of the network to the corresponding destination points, b) the solution of the governing algebraic equation system by means of an efficient iterative approach, known as “Physarum”, that takes the travel times as the main unknowns, c) the interpolation of travel times in certain urban subdomains, denominated “finite elements”, in terms of the values corresponding to some main nodes previously selected (reduced unknowns), d) the use of the Galerkin’s method to express the TAP in terms of the reduced unknowns. The model formulation is presented and a numerical example is given to show its efficiency.

1. INTRODUCTION

Estimation of users’ choice of routes and the resulting traffic flow on every arc of the transportation network, known as TAP, is a very important task in studies of urban planning. In static equilibrium conditions, this problem is based on the Wardrop´s first principle (Sheffi, 1984). One of the most used approaches for this problem corresponds to the Beckmann’s formulation that constitutes a convex optimization problem having the link flows as the main unknowns. The TAP Beckmann’s formulation may be appropriately solved by means of the Frank-Wolfe method (FW).

The computational burden to solve this problem increases with the number of network nodes. Accordingly, involved calculation times for large networks are significant, especially in the context of transportation optimal design because the TAP must be solved many times.

For this reason, remarkable research efforts have been directed to the development of efficient methodologies to solve the TAP.

Some investigations were directed to improve the efficiency of the FW, thus originating the conjugated FW and the bi-conjugated FW (Hearn et al., 1985; Mitradjieva and Lindberg, 2013). Also, efficient algorithms based on trajectories were proposed showing faster convergence with respect to arc-based methods but with higher requirements of computational memory (Florian et al., 2009). On the other hand, Xu et al. (2008) presented an improved origin-based algorithm for assignment and distribution problems.

A strategy to improve the efficiency for solving the TAP is the use of parallel computation.

For example, Lotito (2006) proposed an approach to parallelize, by origin-destination pairs, the disaggregated simplicial decomposition algorithm previously developed by Larsson and Patricksson (1992). Another interesting parallelization method for TAP was recently developed by Jafari et al. (2017) consisting in the partition of the transportation network in small sub-networks. The algorithm alternates between the equilibrium of every sub-network and the equilibrium of a simplified version of the whole network.

In the last years, a new model denominated “Physarum” for solving different optimization problems of scientific and technological interest has been proposed (Tero et al., 2007).

This biologically-inspired algorithm is based on an analogy with the foraging behaviour of the slime mould Physarum Polycephalum that consists in generating protoplasmic tubes following the shortest route to the food sources. The mathematical modelling of such behaviour has been conveniently applied to the determination of the shortest path in networks, the reduction of costs in transportation and communication networks, etc. Very recently, the Physarum approach has been extended to the TAP in user equilibrium conditions following the Beckmann´s formulation (Zang, 2018; Xu et al., 2018). As shown in these articles, the Physarum approach may be more efficient than classical methods based on the FW. On the other hand, Cortínez and Dominguez (2018) have demonstrated that Physarum model, when applied to TAP, may be considered as an iterative strategy for solving an alternative TAP formulation based on travel times (Cortínez y Dominguez, 2017).

This interpretation allows the application of the Physarum approach to the generalized traffic assignment problems (including variable demand, multiple user classes, etc.).

Despite these efforts, the computational burden is still very important in the case of optimal design of large transportation networks. Consequently, several methods have been directed to reduce the number of unknowns. Some of these ones consist in eliminating arcs with low vehicular flow, and other techniques consist in aggregating several arcs in another fictitious equivalent link (Raadsen et al., 2020).

Many of these approaches present certain arbitrariness in the selection of the reduced network topology that can lead to inaccuracies in the calculated flows.

Another kind of models conceives the traffic network as an equivalent two-dimensional continuum medium. To solve these models, discretization methodologies, such as finite differences or finite elements, could be applied with a number of unknowns lesser than that of the original network (Sasaki et al., 1990; Ho et al., 2006, Dominguez, 2013, Cortínez and Dominguez, 2013, 2017).

In the present paper, in order to reduce the number of necessary variables to define the TAP, a finite element approach is proposed. This methodology involves four main ideas: a) the TAP formulation including, as variables, the travel times from any node of the network to the corresponding destination points, b) the solution of the governing algebraic equation system by means of an efficient iterative approach known as "Physarum" that takes the travel times as the main unknowns, c) the interpolation of travel times in certain urban subdomains, denominated "finite elements", in terms of the values corresponding to some main nodes previously selected (reduced unknowns), d) the use of the Galerkin's method to express the TAP in terms of the reduced unknowns.

The model formulation is presented and a numerical example is given to show its efficiency.

2. TRAFFIC ASSIGNMENT PROBLEM IN TERMS OF TRAVEL TIMES

A traffic urban network during the rush hour, when users travel from their homes distributed over the city to certain destination points d (d=1,2,...Nd), is considered.

The network topology, assumed as a set of nodes i (i=1,2,…Nn) connected by directed arcs a (a=1,2,…Na), and the properties of every arc (capacity, length, maximum speed, etc.) are known. Moreover, trip generation rates q_i^d (veh/h), at every node i of the network to the different destinations d, are known. TAP consists in obtaining the link flows $\overline{g}_a$ (veh/h) and the travel times u_i^d (h), from each node i to the corresponding destinations d, according to the users route choices in equilibrium condition.
In the following, the TAP is formulated incorporating the travel times u_i^d as variables (Dominguez, 2013; Cortínez y Dominguez, 2017).

2.1 Congestion function

Urban traffic congestion may be defined as the increment of the link travel time t_a caused by the increment of the link flow $\bar{g}_a$ (veh/h).

From a mathematical point of view, congestion may be defined by means of the following analytical expression:

$$t_a = t_a(\bar{g}_a) \tag{1}$$

This strictly increasing function (Figure 1a) is known in the literature as cost function. Several empirical functions have been developed for expressing (1) explicitly, such as the well-known BPR formula (Sheffi, 1984).

The total flow $\bar{g}_a$ on a link *a* may be expressed as the sum of the flow directed to a certain particular destination *d*, g_a^d, and the flows going to the rest of destinations, g_{aR}^d. This last one may be denominated residual flow with respect to the destination *d*:

$$g_{aR}^d = \sum_{dd} g_a^{dd} \qquad \forall dd \neq d \tag{2}$$

From (1) and (2), the flow on the link *a* going to *d* may be expressed as:

$$g_a^d = \bar{g}_a\left(t_a\right) - g_{aR}^d \tag{3}$$

where $\bar{g}_a(t_a)$ is the inverse function of (1).

2.2 Wardrop's first principle

Wardrop's first principle (Sheffi, 1984) for user's equilibrium (UE) states that the used paths travel times, between every origin-destination pair, are less than, or equal to, the travel times of the other paths. According to this sentence, the travel time between a given point of the network to the destination point is unique (and the minimum) for all the paths really used.

Then, such principle may also be formulated by postulating that the travel time between a given point (*x*,*y*) of the transportation network until the destination *d* is only a function of (*x*,*y*) and not of the employed trajectory (it is a potential function):

$$u^d = u^d(x, y) \tag{4}$$

being by definition $u^d(x_d, y_d) = 0$, where x_d and y_d correspond to the coordinates of the destination *d*. Then, travel time from node *i* to destination *d* may be written as:

$$u_i^d = u^d(x_i, y_i) \tag{5}$$

Using this potential function, the link travel time can be expressed as:

$$t_a = t_a^d = \left| u_i^d - u_j^d \right| \tag{6}$$

It is important to observe that, in equilibrium conditions, the link travel time t_a is the same for every user circulating on the link independently of its destination.

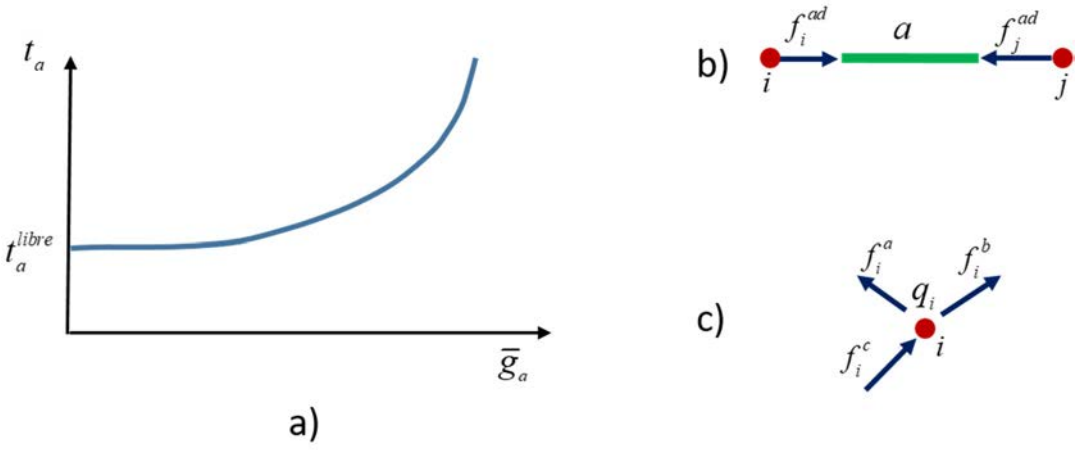

Fig. 1 – Basic definition of UE: a) Cost function, b) flow on link *a*, c) nodal flows at *i*.

2.3 Conservation of vehicles

To formulate the TAP, the conservation of vehicles law at a node must be considered (the number of vehicles entering and exiting a node is a conservative quantity). In order to express such a law, it is convenient to write, making use of (6), the following identity:

$$g_a^d = \frac{g_a^d \left| u_i^d - u_j^d \right|}{t_a} \tag{7}$$

Then, the nodal flows, corresponding to a link *a* (Figure 1b), are defined in the following form:

$$f_i^{ad} = \frac{g_a^d \left(u_i^d - u_j^d \right)}{t_a} = -f_j^{ad} \tag{8}$$

In (8) such flows have been defined positive when they enter the node and negative when they exit. In fact, in (8) $\left(u_i^d - u_j^d\right)$ is positive when the flow is directed from *i* to *j* (in the direction of decreasing *u*). Expression (8) is then useful for non-directed arcs, that is to say those links in which flows can be directed from *i* to *j* or from *j* to *i* according to the sign of $\left(u_i^d - u_j^d\right)$. However, TAP involves directed arcs (one-way links). To define the correct direction of a link *a*, the indicator λ_{ij}^a is introduced:

$$\lambda_{ij}^a = \begin{cases} 1 & i \to j \\ -1 & j \to i \end{cases} \qquad \lambda_{ij}^a = -\lambda_{ij}^a \tag{9}$$

Therefore, the expression for nodal flows corresponding to directed arcs should be re-written, generalizing (8), in the following form:

$$f_i^{ad} = \frac{g_a^d}{t_a} \xi_e^d \left(u_i^d - u_j^d\right) \tag{10}$$

where:

$$\xi_a^d = \begin{cases} 1 & \lambda_{ij}^a \left(u_i^d - u_j^d\right) \geq 0 \\ 0 & \lambda_{ij}^a \left(u_i^d - u_j^d\right) < 0 \end{cases} \tag{11}$$

As can be observed, ξ_a^d has a unit value when the travel time decreases in the allowed link direction and is null in the other case (vehicles cannot flow in a not allowed direction). One can observe that:

$$g_a^d = \left| f_i^{ad} \right| \tag{12}$$

According to expressions (8) and (10), the nodal flows on link *a* going to *d*, may be re-written as:

$$\begin{cases} f_i^{ad} = k_{ii}^{ad} u_i^d + k_{ij}^{ad} u_j^d \\ f_j^{ad} = k_{ij}^{ad} u_i^d + k_{jj}^{ad} u_j^d \end{cases} \tag{13}$$

where the link conductivity matrix has been introduced:

$$k_{ii}^{ad} = k_{jj}^{ad} = \frac{g_a^d \xi_a^d}{t_a^d}; \quad k_{ij}^{ad} = k_{ji}^{ad} = -\frac{g_a^d \xi_a^d}{t_a^d} \tag{14}$$

This matrix depends on $g_a^d(t_a)$ and t_a, and then, according to (6), depends on the potential functions u_i. Expressions (13) involve the travel time corresponding to the considered link *a*. However, it is possible to generalize expressions (13) involving all the nodes of the network by defining in an enlarged form the conductivity matrix. This definition can be performed by adding zeros in those elements connecting two nodes *m*, *n* not belonging to the considered link *a*:

$$k_{mn}^{ad} = 0, \quad m,n \neq i,j \tag{15}$$

In this way, nodal flow to destination *d*, at node *i* of link *a*, can be written as:

$$f_i^{ad} = \sum_{j=1}^{NN} k_{ij}^{ad} u_j^d \tag{16}$$

Therefore, the conservation of vehicles, at generic node *i*, may be expressed by the following equation (Figure 1c):

$$q_i^d - \sum_a f_i^{ad} = 0 \tag{17}$$

In (17), the sum involves all the arcs, although, obviously, the nodal flows are null for the links not connected with node *i* (in agreement with expression 15):

$$f_i^{ad} = 0 \qquad i \notin a \tag{18}$$

Substituting (16) into (17), the vehicle conservation equation may be written in the following form:

$$\sum_j \left(\sum_a k_{ij}^{ad} \right) u_j^d = q_i^d \tag{19}$$

Or, using a matrix notation:

$$\mathbf{K}^d \mathbf{u}^d = \mathbf{q}^d \qquad d = 1,2,...,Nd \tag{20}$$

In (20), the elements of the global conductivity matrix for vehicles going to *d* have been defined in the form:

$$K_{ij}^{d} = \sum_{a} k_{ij}^{ad} \tag{21}$$

Such system must be complemented with the expressions (corresponding to the definition of travel times):

$$u_{d}^{d} = 0 \qquad d = 1, 2, ..., Nd \tag{22}$$

As can be seen in (14), conductivity matrices $\mathbf{K}^{d}$ depend on t_a and g_a^d and, accordingly, expressions (1), (2), (3), (6), (20) and (22), constitute a non-linear algebraic equation system whose unknowns are given by u_i^d, g_a^d, $\overline{g}_a$ and t_a. Cortínez and Dominguez (2017) have shown that this system can be iteratively solved by means a sequence of problems whose unknowns are given by u_i^d. It can be demonstrated that the system (1), (2), (3), (6), (20) and (22) is equivalent to Beckmann's variational formulation (Dominguez, 2013; Cortínez y Dominguez, 2017).

3. NUMERICAL SOLUTION BY USING PHYSARUM APPROACH

The previous system can be iteratively solved by means of a Newton-Raphson technique (Cortínez and Dominguez, 2017). One of the difficulties found with this methodology is due the fact that if some links are decongested, g_a^d tends to zero and then, according to (14), also the conductivity matrix tends to zero.

Therefore, the system becomes indeterminate. This methodology makes use of the Physarum iterative approach (Zhang and Mahadevan, 2018; Xu et al., 2018; Cortínez and Dominguez, 2018), based on an analogy with a biological process, that works appropriately even in presence of decongested links.

The methodology starts with the approximation of the numerator and the denominator of the conductivity matrix elements (see expressions 14 and 15):

$$\begin{aligned} g_a^d \xi_a^d &\rightarrow D_a^d \\ t_a &\rightarrow L_a \end{aligned} \tag{23}$$

Consequently, expression (10) is approximated as:

$$f_i^{*ad} = \frac{D_a^d}{L_a}\left(u_i^d - u_j^d\right) \tag{24}$$

where f_i^{*ad} is an approximation to f_i^{ad}. Therefore, following (14), matrix k_{ij}^{ad} may be approximated by k_{ij}^{*ad} defined as:

$$k_{ij}^{*ad} = \begin{cases} D_a^d / L_a & i = j \\ -D_a^d / L_a & i \neq j \in a \\ 0 & i, j \notin a \end{cases} \tag{25}$$

Thus, considering (21), system (20) is approximated by the following linear system with unknowns $\mathbf{u}^d$:

$$\mathbf{K}^{*d}\mathbf{u}^d = \mathbf{q}^d \tag{26}$$

Before solving (26), it is necessary to take into account conditions (22). This may be performed easily modifying $\mathbf{K}^{*d}$ by means of a penalization approach (summing very large values to the diagonal elements corresponding to the unknowns u_d^d, Chandrupatla and Belegundu, 2012). One can observe that (26) (modified by 22) constitutes a decoupled system of linear equations for every destination. Formally, the solution may be expressed as:

$$\mathbf{u}^d = \left(\mathbf{K}^{*d}\right)^{-1}\mathbf{q}^d \tag{27}$$

Once determined $\mathbf{u}^d$, better approximations for link flows g_a^{*d} are obtained by means of (12) and (24):

$$g_a^{*d} = \frac{D_a^d}{L_a}\left|u_i^d - u_j^d\right| \tag{28}$$

Then, new approximations for ξ_a^{*d} are calculated using (11). Now, it is possible to obtain updated values for coefficients *D*:

$$D_a^{d(new)} = \frac{D_a^d + g_a^{*d}\xi_a^{*d}}{2} \tag{29}$$

The new approximation for total flow on the link *a* is obtained as:

$$\bar{g}_a^* = \sum_d g_a^{*d} \xi_a^{*d} \tag{30}$$

Finally, using expression (1), a new approximation for the link travel time t_a^* is achieved and then, the updated coefficient *L* is expressed as:

$$L_a^{(new)} = \frac{L_a + t_a^*\left(\bar{g}_a^*\right)}{2} \tag{31}$$

With the updated values, matrix conductivity $\mathbf{K}^{*d(new)}$ is re-calculated and the procedure is iterated. When convergence is achieved, D_a^d and L_a converge to $g_a^d \xi_a^d$ and t_a, respectively. The fact that, in each iteration, the systems (26) should be solved separately for every destination *d*, allows the application of this methodology in a context of parallel computing.

4. REDUCED MODEL: FINITE ELEMENT METHOD

System (26) for every *d* is often very large for urban or regional networks and, accordingly, very demanding from the computational point of view, because it should be solved many times up to convergence. This is especially true in the context of optimal design. For this reason, it is of interest to develop a model with lesser unknowns being approximately equivalent to system (26). In this section, an approach for reducing unknowns is developed by taking ideas from the Finite Element Method (FEM) commonly used in the field of continuum mechanics (Chandrupatla y Belegundu, 2012).

The methodology starts with the subdivision of the transportation network in subdomains, denominated finite elements (FE), each one containing a part of the network, as shown in Figure 2. The geometrical shapes of such elements may be relatively simple: rectangles, triangles, irregular quadrilaterals, etc.

Over these elements certain nodes are identified corresponding to the main unknowns U_K^d that represent the travel times from each point *K* to the corresponding destination *d*. Inside each FE, travel time from a generic point, with (*x*,*y*) coordinates, to destination *d*, is approximated by means of an interpolation in terms of travel times corresponding to the main nodes U_K^d:

$$u^d(x,y) = \sum_K N_K^e(x,y) U_K^d \tag{32}$$

For a consistency reason, the interpolation functions $N_K^e(x,y)$, must adopt the values $N_K^e = 1$ if $(x,y) = (x_K, y_K)$ and $N_K^e = 0$ if (x,y) correspond to any of the other main element nodes. Moreover, such functions are null outside the considered *e* element: $N_K^e = 0$ if $(x,y) \notin e$. There are many ways to select the interpolation functions (Chandrupatla y Belegundu, 2012). One of them will be presented in the numerical example given below. According to (32), travel time from *i*-node to destination *d* may be approximated as:

$$u_i^d = \sum_K \sum_e N_{iK}^e U_K^d \tag{33}$$

where $N_{iK}^e = N_K^e(x_i, y_i)$. On the other hand, if one defines:

$$N_{iK} = \sum_e N_{iK}^e \tag{34}$$

expression (33) could be written as:

$$u_i^d = \sum_K N_{iK} U_K^d \tag{35}$$

Or, in matrix form:

$$\mathbf{u}^d = \mathbf{N}\mathbf{U}^d \tag{36}$$

That is to say, the travel time u_i^d may be expressed in terms of travel times corresponding to the main nodes U_K^d. It should be observed that, using (36), the number of unknowns are significantly reduced with respect to the original network.

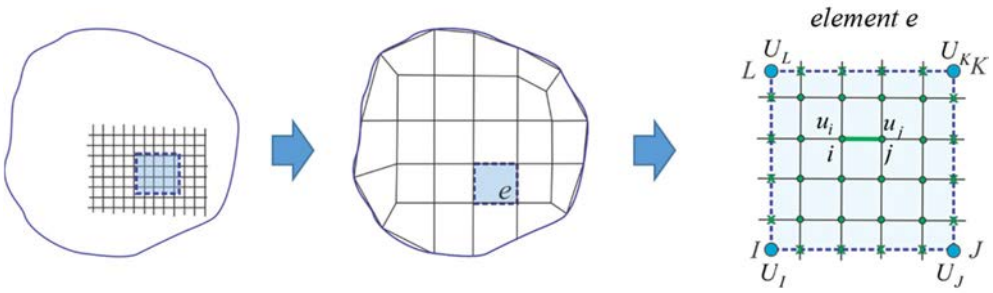

Fig. 2 – Scheme of the finite element reduced model

To obtain the reduced equation system with unknowns $\mathbf{U}^d$, the Galerkin's method is employed (Chandrupatla y Belegundu, 2012; Cortínez y Dominguez, 2017). Accordingly, a virtual vector $\delta\mathbf{u}$ is defined, with arbitrary values at all nodes excepting at those corresponding to destinations *d* where takes zero value. Now, pre-multiplying expression (26) by the transposed of $\delta\mathbf{u}$ one arrives at:

$$\delta\mathbf{u}^T\left(\mathbf{K}^{*d}\mathbf{u}^d-\mathbf{q}^d\right)=0 \tag{37}$$

Interpolating $\delta\mathbf{u}$ in a form similar to (36), that is to say:

$$\delta\mathbf{u}^d=\mathbf{N}^T\delta\mathbf{U}^d \tag{38}$$

expression (37) can be re-written in the form:

$$\delta\mathbf{U}^T\mathbf{N}^T\left(\mathbf{K}^{*d}\mathbf{N}\mathbf{U}^d-\mathbf{q}^d\right)=0 \tag{39}$$

Then, taking into account that $\delta\mathbf{U}^d$ is an arbitrary vector, equation (39) is true if:

$$\left(\mathbf{N}^T\mathbf{K}^{*d}\mathbf{N}\right)\mathbf{U}^d=\mathbf{N}^T\mathbf{q}^d \tag{40}$$

This expression may be conveniently re-written as:

$$\mathbf{\Psi}^d\mathbf{U}^d=\mathbf{Q}^d \tag{41}$$

considering the following definitions:

$$\mathbf{\Psi}^d=\mathbf{N}^T\mathbf{K}^{*d}\mathbf{N},\quad \mathbf{Q}^d=\mathbf{N}^T\mathbf{q}^d \tag{42 a, b}$$

$\mathbf{\Psi}^d$ is a reduced conductivity matrix and $\mathbf{Q}^d$ is a reduced trip demand vector. One should observe that for representing appropriately the network in a reduced form, the destinations *d* must be included among the main nodes. On the other hand, the rest of the main nodes do not need to match with the real nodes. Before solving (41) matrix $\mathbf{\Psi}^d$ should be modified using the conditions (22) by means of the penalization approach described in the previous section.

5. COMPUTATIONAL PROCEDURE

According to the methodology explained in the above sections, the present algorithm to solve TAP is the following:

I) Definition of FE model (reduced model).
   a. Choice of main nodes K.
   b. Definition of nodes and links in each element.

II) Initialization of the iterative process.
   a. Adoption of initial values for g_a^d, ξ_a^d and t_a $\forall(d,a)$.
   b. Determination of the initial values for D_a^d and L_a $\forall(d,a)$ → Expression (23)

III) Iterative calculation up to convergence
   a. Determination of the conductivity matrix for each link k_{ij}^{*ad} $\forall(d,a,i,j)$ → Expression (25).
   b. Determination of the global conductivity matrices K_{ij}^{*d} $\forall(d,i,j)$ → Expression (21).
   c. Determination of the reduced FR conductivity matrices ψ_{IJ}^{*d} $\forall(d,I,J)$ → Expression (42a).
   d. Determination of the reduced EF trip demand vector Q_I^{*d} $\forall(d,I)$ → Expression (42b).
   e. Determination of travel time from main nodes to destinations *d*: U_I^{*d} $\forall(d,I)$. → Solution of system (41) for every destination *d*.
   f. Recovery travel times for the transportation network nodes u_i^{*d} → Expression (35).
   g. Updating of g_a^{*d}, ξ_a^{*d} $\forall(d,a)$ → Expressions (28) and (11).
   h. Updating of $\bar{g}_a^*$ and t_a^* $\forall a$ → Expressions (30) and (1).
   i. Updating of D_a^d and L_a $\forall(d,a)$ → Expressions (29) and (31).
   j. Verification of convergence. If $\max\left|u_i^{*d} - \left(u_i^{*d}\right)_{previous}\right| < Tolerance$ → end of iterative calculation.

6. NUMERICAL EXAMPLE

A fictitious transportation network of 2.9x2.9 km^2 consisting in 1740 one-way links of 100m length and 900 nodes (Figure 3) is analysed. The links, normal one each other, have a BPR cost function given by $t_a = 0.1\left(1+0.15\left(g_a/600\right)^2\right)$, corresponding to a link capacity of 600 vehicles/h and a maximum allowable speed of 60 km/h. Trip demand is assumed to be 6308 trips/h uniformly distributed among all the nodes with an only one destination.

In Figure 3, three paths from different origins to the destination, with arrows showing the circulation direction, are depicted. The nodes and links identified with numbers and letters, respectively, correspond to those used in the following figures to show the numerical results.

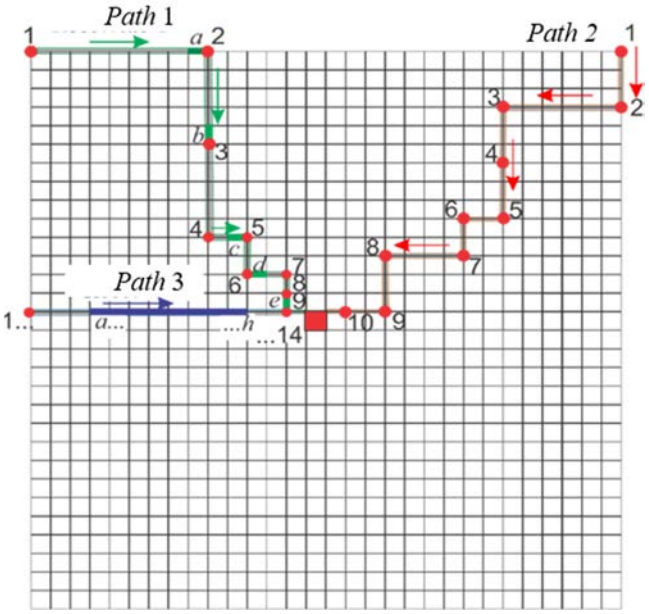

Fig. 3 – Scheme of the transportation network under analysis.

To solve the TAP corresponding to the described network, the explained methodology is applied using two different meshes, one of 16 elements (shown in Figure 4) and the other of 64 elements.

Each element has been assumed to be rectangular, with four main nodes (one per vertex). Bi-linear interpolation functions have been adopted. That is to say, they have the generic form $a_K(b_K - x)(c_K - y)$, where constants a_K, b_K, c_K are determined in such a way the following conditions are verified: $N_K^e = 1$ if $(x,y) = (x_K, y_K)$, and $N_K^e = 0$ if (x,y) correspond to any other main node.

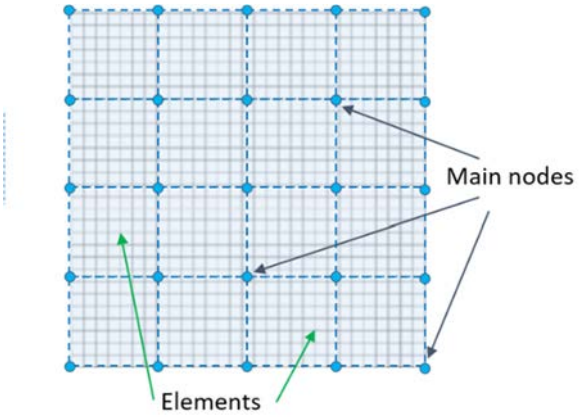

Fig. 4 – Scheme of the reduced model (considering 16 finite elements).

In order to perform a numerical study on the accuracy of the present methodology, TAP has also been solved: a) using the classical Beckmann's formulation along with Frank-Wolfe method (FW) and b) with the iterative Physarum method (Physarum) for the full network. The results of these two methods are considered to be exact.

Observe that Beckmann's formulation involves 1740 link flows as basic unknowns, Physarum solution to the full network involves 900 travel times (one for each node) while the FE reduced models of 16 and 64 finite elements involve only 25 and 81 main unknowns, respectively.

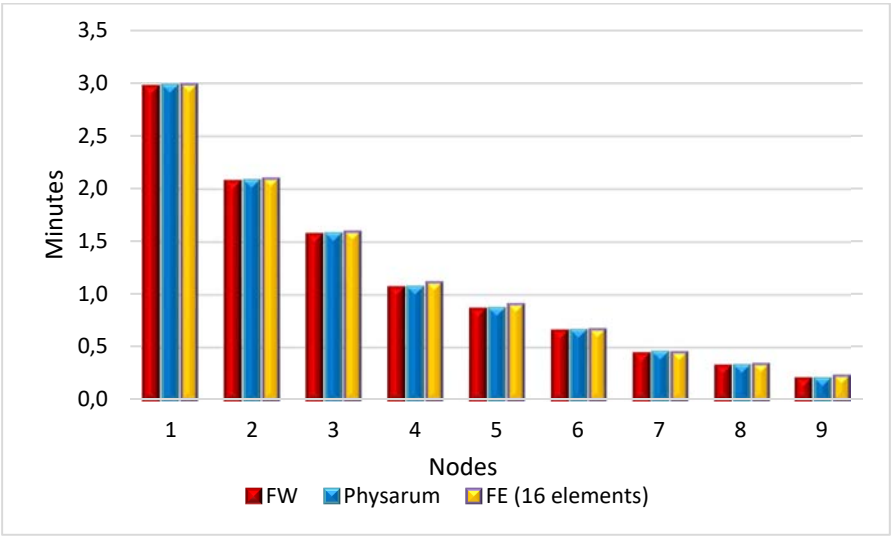

Fig. 5 – Travel times u_i from nodes corresponding to Path 1.

In Figure 5 a comparison of travel times obtained with FW, Physarum and FE (16 elements) for the nodes of Path 1 is shown. A very good agreement between the values obtained with all the models is observed. The same observation is valid for the nodes corresponding to Path 2 (Figure 6).

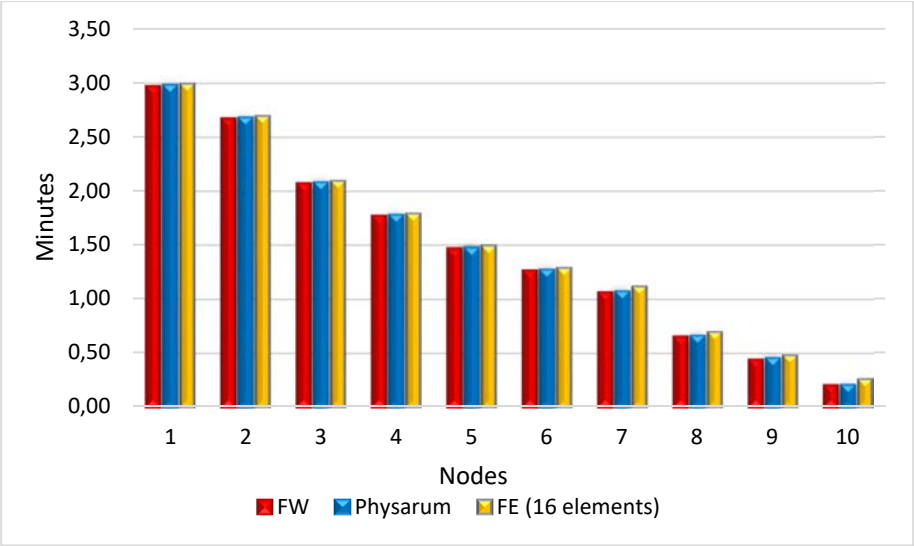

Fig. 6 – Travel times u_i from nodes corresponding to Path 2.

In Figure 7, a comparison for link travel times (expressed as the ratio between link travel time and free link travel time) determined by Physarum and the reduced FE (16 and 64 finite elements) for different links of Path 1, is shown. As observed, the results are almost identical except for link *e*, where the FE reduced model with 16 elements presents an error of approximately 10%, and the FE reduced model, with 64 elements, an error less than 5%.

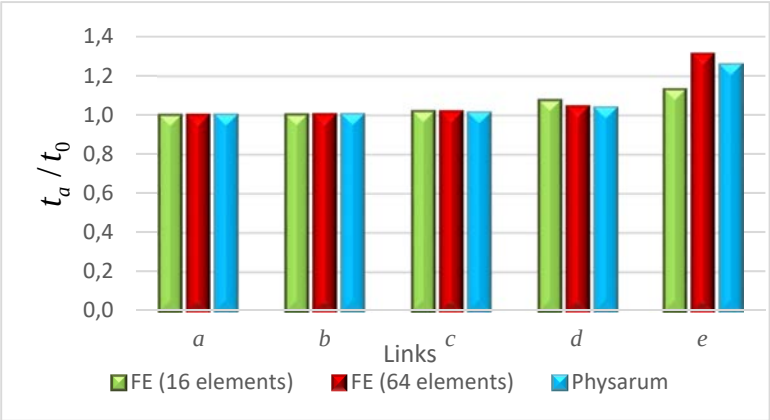

Fig. 7– Ratio between link travel time and free link travel time t_a / t_0 on links of Path 1

On the other hand, in Figure 8, a comparison for the link flows corresponding to Path 1, obtained by means of Physarum and the FE reduced model (64 elements), is shown. The maximum error of the reduced model is of 8% approximately.

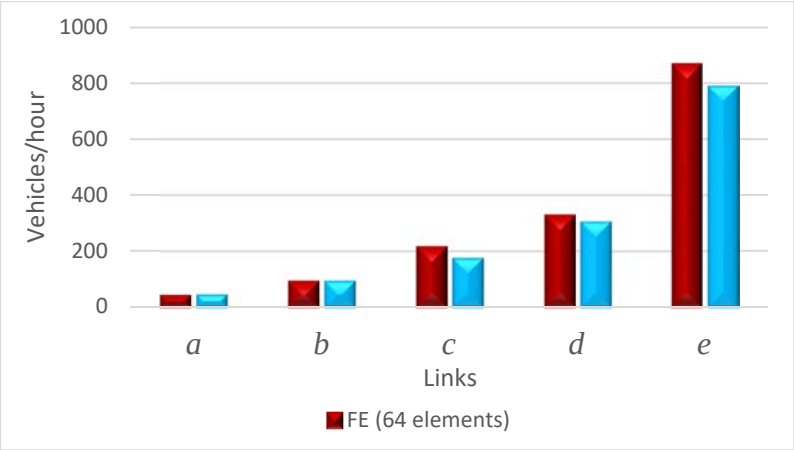

Fig. 8 – Link flows ($\overline{g}_a$) for Path 1.

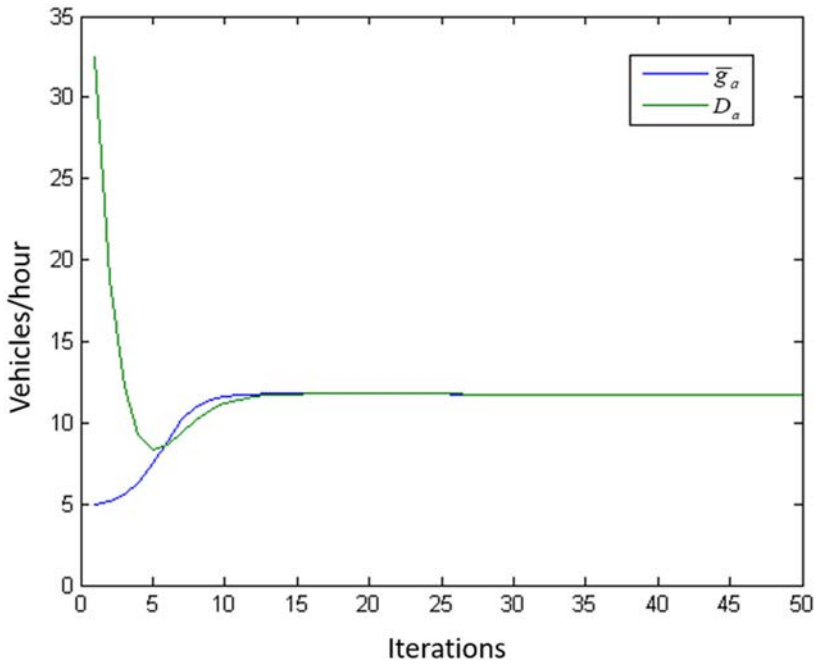

Fig. 9 – Convergence of the methodology in terms of traffic flow on the first link of Path 1 using the 64 FE reduced model.

The iterative procedure converges quickly, as can be observed in Figures 9 and 10. Figure 9 shows the convergence of D_a^d and $\overline{g}_a$ for the first link of Path 1. Convergence is achieved in less than 15 iterations. It is interesting to note that this convergence is reached even when the link is practically decongested. Figure 10 shows a similar information for the last link (*e*) of the same path. Although this link is congested, the convergence is also achieved in the same number of iterations.

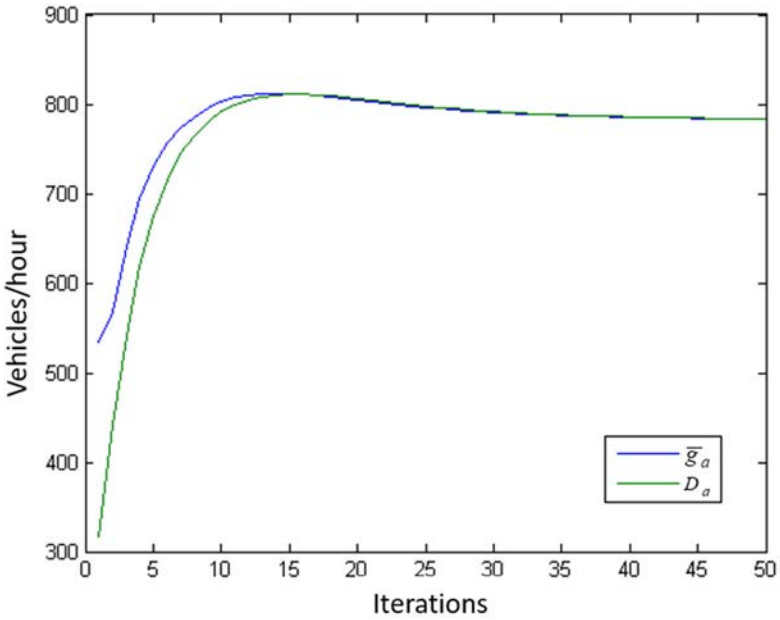

Fig. 10 – Convergence of the methodology in terms of traffic flow on the last link (*e*) of Path 1 using the 64 FE reduced model.

Finally, in Figure 11, a comparison of the convergence behaviour shown by Physarum, 16 FE and 64 FE reduced models, is shown in terms of the travel time from node 1 of Path 3 until the destination. As can be seen, the three approaches show a similar convergence behaviour.

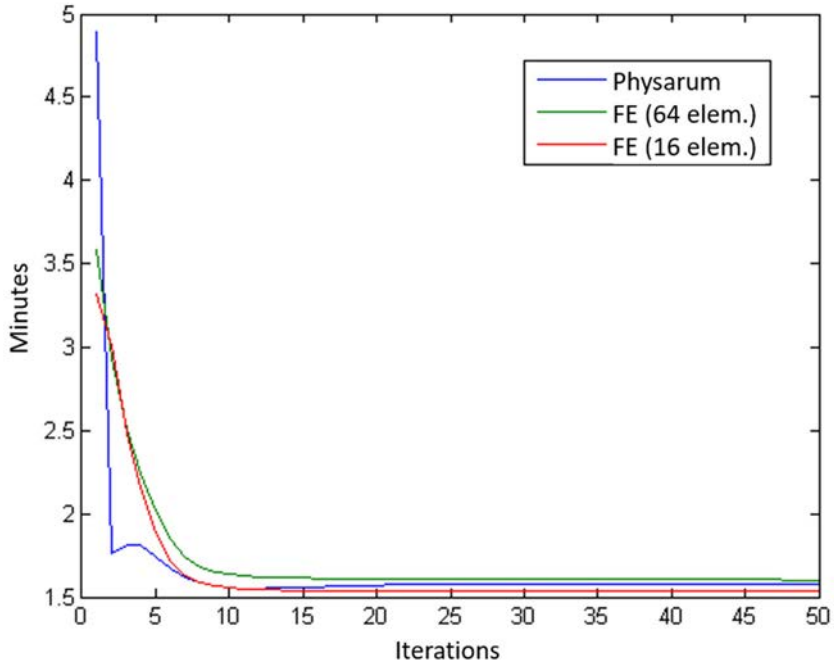

Fig. 11 – Convergence behaviour of travel time at node 1 of Path 3 for Physarum, 16 FE and 64 FE reduced models.

7. CONCLUSIONS

A finite element approach for reducing TAP unknowns is presented. This methodology employs an efficient iterative technique (Physarum analogy) to solve the governing non-linear equation system taking travel times as basic unknowns. This procedure works appropriately, even in presence of links with very low flow (avoiding indeterminacy of the system). The present approach allows a notable reduction of unknowns (more than 100 times in the analysed example), maintaining an accuracy similar to the full model, with an important reduction of the computing time. The methodology can be easily programmed in a context of parallel computing.

ACKNOWLEDGEMENTS

The present study is part of the projects “Mathematical models in environmental engineering: optimal design and control” (Secyt, Universidad Tecnológica Nacional) and “Optimization problems in civil engineering. Part III” (Departamento de Ingeniería, Universidad Nacional del Sur).

REFERENCES

CHANDRUPATLA, D.R and BELEGUNDU, A.D. (2012). Introduction to finite elements in engineering, 4th ed. Ed. Pearson.

CORTÍNEZ, V.H. and DOMINGUEZ, P.N. (2013). Un modelo de difusión anisótropa para el estudio del tráfico urbano. Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería, 29(1), pp. 1-11.

CORTÍNEZ, V.H. and DOMINGUEZ, P.N. (2017). An anisotropic continuum model for traffic assignment in mixed transportation networks. Applied Mathematical Modelling 50, pp. 340-353.

CORTÍNEZ, V.H. and DOMINGUEZ, P.N. (2018). Una nueva interpretación del modelo Physarum para el problema de asignación de tráfico en equilibrio de usuario. Mecánica Computacional XXXVI, AMCA (ISSN: 2591-3522), pp. 2089-2098.

DOMINGUEZ, P. N. (2013). Un nuevo modelo continuo de asignación de tráfico para el diseño óptimo de redes de transporte urbano, Doctoral Thesis, Universidad Nacional del Sur.

FLORIAN, M., CONSTANTIN, I and FLORIAN, D. (2009) A new look at projected gradient method for equilibrium assignment. J. Trans. Res. Board 2090, pp. 10-16.

HEARN, D.W; LAWPHONGPANICH, S. and VENTURA, J. (1985). Finiteness in restricted simplicial decomposition. Oper. Res. Lett. (4), pp. 125–130.

HO, H.; WONG, S. and LOO, B. (2006). Combined distribution and assignment model for a continuum traffic equilibrium problem with multiple user classes. Transportation Research, part B 40, pp. 633-650.

JAFARI, E., PANDEY, V. and BOYLES, S. D. (2017). A decomposition approach to the static traffic assignment problem, Transportation Research part B 105, pp. 270-296.

LARSSON, T. and PATRIKSSON, M. (1992). Simplicial decomposition with disaggregated representation for the traffic assignment problem, Transp. Sci. 26, pp. 4–17.

LOTITO, P. (2006). Issues in the implementation of the DSD algorithm for the traffic assignment problem. European Journal of the Operation Research 175, pp. 1577-1587.

MITRADJIEVA, M. and LINDBERG, P.O. (2013). The stiff is moving conjugate direction Frank–Wolfe methods with applications to traffic assignment. Transp. Sci. 47, pp. 280–293.

RAADSEN, M.P.H., BLIEMER, M.C.J. and BELL, M.G.H. (2020). Aggregation, disaggregation and decomposition methods in traffic assignment: historical perspectives and new trends. Transportation Research part B 139, pp.199-223.

SASAKI, T., IIDA, Y. and YANG, H. (1990). User equilibrium traffic assignment by continuum approximation of network flow. Actas del 11th International Symposium of Transportation and Traffic Theory, Yokohama, Japan, pp. 233-252.

SHEFFI, Y. (1984). Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods, Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632.

TERO, A. KOBAYASHI, R. and NAKAGAKI, T. A. (2007). Mathematical model for adaptive transport network in path finding by true slime mold. Journal of theoretical biology 244, pp. 553-564.

XU, M., CHEN, A. and GAO, Z. (2008). An improved origin-based algorithm for solving the combined distribution and assignment problem. European Journal of Operation Research 188, pp. 354-369.

XU, S., JIANG, W., DENG, X. and SHOU, Y. (2018). A modified Physarum-inspired model for the user equilibrium traffic assignment problem. Applied Mathematical Modelling 55, pp. 340-353.

ZHANG, X. and MAHADEVAN, S. (2018). A bio-inspired approach to traffic network equilibrium assignment problem. IEEE Transactions on cybernetics, pp.1304-1315.