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Evaluation of uncertainty in the measurement of the stress-optic coefficient

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ABSTRACT

In this study, a method for evaluating the uncertainty of stress-optic coefficient measurements of photoelastic materials on a uniaxial tension/compression specimen is presented. The same problem is also approached in other previously used methods, in which multiple data fitted with ordinary least squares are considered. However, only the repeatability contribution to uncertainty is considered in those other methods, which are therefore not consistent with the ISO–Guide to the expression of Uncertainty in Measurement. All possible contributions to uncertainty can be taken into account using the General Least Squares-Lagrange Multipliers (GLS-LM) method presented here. The application of the method is illustrated with an example, from which it can be seen that uncertainty has been underestimated in the other methods used to date. As well as the estimate of the stress-optic coefficient and its corresponding standard uncertainty, the method also provides a data consistency test and an outlier identification tool.

1. Introduction

Since the first studies on birefringence [1] and its subsequent application through photoelasticity [2], one of the first steps for the use of this physical phenomenon in stress/strain measurements is to obtain the stress-optic coefficient of photoelastic materials. This coefficient (also known as the material's photoelastic constant and denoted by C) relates the relative retardation between the two polarized light rays that emerge from the illuminated photoelastic material to the principal stress difference within it[1]:

$$\sigma_1 - \sigma_2 = \frac{N\lambda}{hC} \tag{1}$$

In this equation, known as the stress-optic law, $\sigma_1 - \sigma_2$ is the principal stress difference at a given point, N is the fringe order at that point, *i.e.*, the retardation expressed in multiples of λ (the wavelength of the light in use), and h is the thickness of the material through which the light passes.

The stress-optic coefficient is a property that varies with time and from batch to batch, so it is necessary to measure its value at the time of use. Normal procedure to obtain the stress-optic coefficient for a given material is to measure the fringe order on a specimen for which the stress field is known and use Eq. (1). The uniaxial tension/compression loading specimen is the most widely used, due to its simplicity [3–16]. If we review the published works in which this measurement procedure has been employed, we can see that there is no clear and metrologically correct way to report a value of the stress-optic coefficient. A measurement must have an associated uncertainty for it to be valid, comparable, and reproducible. Thus, the ISO/IEC 17025:2005 standard [17] that relates to the competence of laboratories states that in order to accredit their technical competence 'testing laboratories shall have and shall apply procedures for estimating uncertainty of measurement'. The 'Guide to the expression of Uncertainty in Measurement' (GUM) [18], first published in 1993, is the standard that describes the internationally accepted method for its evaluation. In some of the aforementioned –mostly pre-GUM– works, measurement uncertainties are not reported [3,6,8], or uncertainty is simply estimated as a percentage/last significant figure [5,9,10]. In the other –mostly post-GUM– works, measurement uncertainties are expressed, but they are not always evaluated in the same way. In those works, uncertainties are sometimes calculated as the standard deviation of repeated measurements in different trials or on different specimens [7,12], and sometimes through the uncertainty of a linear regression on measurements made under different applied loads [4,11,13–16]. As we will see below, none of these methods for evaluating the uncertainty of measurements is correct; both underestimate the uncertainties. The purpose of this work is to provide a correct method, consistent with the GUM, to evaluate the stress-optic coefficient measurements and their uncertainty obtained by means of a uniaxial tension/compression specimen. Thus, these measurements will comply with international metrological requirements.

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[1] Photoelastic materials exhibit uniaxial birefringence under mechanical stress and are optically isotropic in the non-stress state.

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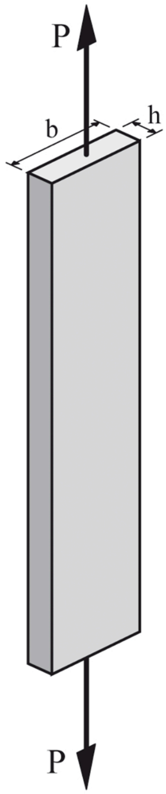

Fig. 1. Rectangular specimen subjected to uniaxial tension.

2. Conventional method

The conventional method for measuring the stress-optic coefficient of a photoelastic material consists of measuring the fringe order in a rectangular specimen subjected to a given tension or compression (Fig. 1). In the central part of the specimen, the stress state is uniaxial so, at any point within it, σ_2 is zero and Eq. (1) can be recast to evaluate the stress-optic coefficient, C:

$$C = \frac{N\lambda}{h\sigma_1} = \frac{N\lambda b}{P} \tag{2}$$

where, P is the applied load and b is the width of the specimen[2].

The fringe order is measured for different values of the load, and load vs. fringe order is plotted on a graph. Then, a least squares line is fitted to the data, and the slope, N/P, of this line is used in Eq. (2) to determine the stress-optic coefficient. Proceeding in this way, the value obtained will be valid over the entire load application range.

As commented above, the authors of previous works have evaluated measurement uncertainty in two ways. The simplest consists of repeating the measurements, either on the same specimen or on several different ones, and taking the mean and the standard deviation of the results as the stress-optic coefficient value and its standard uncertainty, respectively. As established in the GUM, this method is valid to evaluate the component of the uncertainty that is due to the variability/repeatability (or reproducibility, if different specimens are used) of a direct measurement. The case here is otherwise. The measurement is indirect and multiple other contributions to uncertainty are involved. The measurement uncertainties of all the quantities in Eq. (2) must be considered. These quantities, in turn, have different uncertainty components (calibration of the instruments, their resolution, variability of the measurements, *etc.*). Then, all the uncertainty components should be properly combined, bearing in mind that several measurements of the fringe order are taken for different values of the load and a least squares adjustment is performed to find the N/P term. This process differs greatly from the simplified and incorrect method used in the above-mentioned works.

The second way of evaluating measurement uncertainty used in previous works is also described in the GUM (example H.3), but that does not mean that it can be applied here. Like the former method, many contributions to uncertainty are also ignored in the latter method, in which only the uncertainty of the linear least squares fit is considered. The method consists of evaluating the estimate for the slope parameter, B, of a linear least squares fit, $y = A + Bx$, with $A = 0$, $x = P$, $y = N$, and n pairs of data, as well as its standard uncertainty, $u(B)$ [19]:

$$B = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \tag{3}$$

$$u(B) = \sqrt{\frac{\frac{n}{n-2}\sum_{i=1}^{n}(y_i - A - Bx_i)^2}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}} \tag{4}$$

With these values, the estimate of C and its corresponding standard uncertainty, $u(C)$, are evaluated using Eq. (2), considering that $B = N/P$:

$$C = B\lambda b \tag{5}$$

$$u(C) = u(B)\lambda b \tag{6}$$

The use of these expressions assumes that the measurements of $y = N$ have the same uncertainty (due solely to its variability) and that the uncertainties in the measurements of $x = P$ and the other quantities (b and λ) are negligible. With these assumptions, there are components of uncertainty that are lost, whose contribution to the uncertainty of C should not be ignored. The least squares fitting must be performed using the Generalized Least Squares-Lagrange Multipliers (GLS-LM) method [20,21], to account for all contributions in the evaluation of the uncertainty. This method has proven to be the appropriate tool for evaluating uncertainty through least-squares fitting [22].

3. The GLS-LM method for uncertainty evaluation

It is necessary to consider the uncertainties of all the quantities involved, and to take into account the relationships between them, for uncertainty evaluation in measurements based on least squares fitting. The uncertainties of those quantities and the relations between them are considered in the GLS-LM method, because it incorporates observation errors in all input quantities through the objective function to be minimized, and possible relationships between quantities are considered through the mathematical strategy of Lagrange multipliers.

The objective function in the GLS-LM method, known as the chi-square function, is the sum of squares and cross products of the differences between all observed data involved in the measurement process and their predicted values, weighted by the inverse of the covariance matrix of the observations:

$$\chi^2(\hat{\boldsymbol{x}};\boldsymbol{x}) = (\boldsymbol{x} - \hat{\boldsymbol{x}})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \hat{\boldsymbol{x}}) \tag{7}$$

where, the terms $\boldsymbol{x} = (x_1, \ldots, x_M)^T$ are the estimates of the input quantities $X = (X_1, \ldots, X_M)^T$ of the measurement process, the terms $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_M)^T$ are their refined values as a result of least-squares fitting,

[2] Note that the specimen thickness, *h,* is not necessary to evaluate the stress-optic coefficient. The reason is that the relative retardation is proportional to *h*, but for a given force, *P*, the stress is inversely proportional to *h*. The net effect is a result for *C* that is independent of *h*.

and $\boldsymbol{\Sigma}^{-1}$ is the inverse of the covariance matrix of the input estimates. Its elements are the variances, $u^2(x_j)$, and covariances, $u(x_i,x_j)$, associated with the input estimates, the values of which are known prior to the fitting process:

$$\boldsymbol{\Sigma}=\begin{pmatrix} u^2(x_1) & u(x_1,x_2) & \dots & u(x_1,x_M) \\ u(x_2,x_1) & u^2(x_2) & \dots & u(x_2,x_M) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ u(x_M,x_1) & u(x_M,x_2) & \dots & u^2(x_M) \end{pmatrix} \tag{8}$$

In the stress-optic measurements process, the input quantities, $\boldsymbol{X}$ (those for which prior information is available either from direct measurements or from other sources), are: b, λ, and the applied force, P, and the fringe order, N, at each step of the process. Therefore, if there are n steps in the specimen loading process, the total number of input quantities is $M=2+2{\cdot}n$. In the application case that will be presented in the next section, the fringe order is measured at $n=20$ values of the load; there will therefore be $M=42$ input quantities:

$$\boldsymbol{X}=\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \dots \\ X_{41} \\ X_{42} \end{pmatrix}=\begin{pmatrix} b \\ \lambda \\ P_1 \\ N_1 \\ \dots \\ P_{20} \\ N_{20} \end{pmatrix} \tag{9}$$

In the measurement process, the output quantities, $\boldsymbol{Y}$, are those for which no prior information is available. In this case, there is only one output quantity, the stress-optic coefficient, C.

The estimate of the output quantity, $\boldsymbol{y}$, and the refined estimates of the input quantities, $\widehat{\boldsymbol{x}}$, are related by n constraints, from the use of Eq. (2) at each load step, so there are n constraints or model functions. In the application case of the next section, $n=20$:

$$\mathbf{h}(\boldsymbol{y},\widehat{\boldsymbol{x}})=\begin{pmatrix} \lambda bN_1-CP_1 \\ \lambda bN_2-CP_2 \\ \dots \\ \lambda bN_{20}-CP_{20} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

The objective function (7) is minimized through the strategy of Lagrange multipliers for which an auxiliary function, the Lagrangian Function, is defined that incorporates the constraints (10) into Eq. (7)[3]:

$$\mathrm{L}(\boldsymbol{y},\widehat{\boldsymbol{x}},\boldsymbol{\lambda};\boldsymbol{x})=(\boldsymbol{x}-\widehat{\boldsymbol{x}})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\widehat{\boldsymbol{x}})+2\boldsymbol{\beta}^T\mathbf{h}(\boldsymbol{y},\widehat{\boldsymbol{x}}) \tag{11}$$

where, $\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)^T$ are a set of n parameters –as many as the constraints– known as Lagrange multipliers. With these, the minimization problem involves solving the gradient equations

$$\nabla_{(\boldsymbol{y},\widehat{\boldsymbol{x}},\boldsymbol{\beta})}\mathrm{L}(\boldsymbol{y},\widehat{\boldsymbol{x}},\boldsymbol{\beta};\boldsymbol{x})=\mathbf{0} \tag{12}$$

This system of $1+M+n=3(n+1)$ nonlinear equations with $3(n+1)$ unknowns is iteratively solved with the Gauss-Newton algorithm. Hence, denoting the iteration number by the superscript $l=1,2,\dots$, the estimates of the output quantities are refined by successive approximations:

$$\begin{aligned} \boldsymbol{y}^{l+1}&=\boldsymbol{y}^l+\Delta\boldsymbol{y}^l \\ \widehat{\boldsymbol{x}}^{l+1}&=\widehat{\boldsymbol{x}}^l+\Delta\widehat{\boldsymbol{x}}^l \end{aligned} \tag{13}$$

At each iteration the model functions are linearized by approximation to a first-order Taylor series expansion around $\boldsymbol{y}^l$ and $\widehat{\boldsymbol{x}}^l$, which transforms Eq. (12) into a linear system [21][4]:

$$\begin{pmatrix} \mathbf{0}^{(1,1)} & \mathbf{0}^{(1,2+2n)} & \left[\nabla_y\mathbf{h}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)\right]^T \\ \mathbf{0}^{(2+2n,1)} & \sum^{-1} & \left[\nabla_{\widehat{x}}\mathbf{h}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)\right]^T \\ \left[\nabla_y\mathbf{h}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)\right] & \left[\nabla_{\widehat{x}}\mathbf{h}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)\right] & \mathbf{0}^{(n,n)} \end{pmatrix}\begin{pmatrix} \Delta\boldsymbol{y}^l \\ \Delta\widehat{\boldsymbol{x}}^l \\ \boldsymbol{\beta}^{l+1} \end{pmatrix}$$

$$=\begin{pmatrix} \mathbf{0}^{(1,1)} \\ \sum^{-1}(\boldsymbol{x}-\widehat{\boldsymbol{x}}^l) \\ -\mathbf{h}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l) \end{pmatrix} \tag{14}$$

The values obtained for $\Delta\boldsymbol{y}^l$ and $\Delta\widehat{\boldsymbol{x}}^l$ are used in (13) to improve the estimates of the quantities for the next iteration, stopping the process with an appropriate convergence criterion.

The left-hand side matrix in (14) is called $\mathbf{D}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)$ and it may be demonstrated [21] that its inverse contains, in its upper left $(3+2n)\times(3+2n)$ submatrix, the estimated covariances associated with the estimates of the quantities:

$$\mathbf{D}(\boldsymbol{y}^l,\widehat{\boldsymbol{x}}^l)^{-1}=\begin{pmatrix} u(y,y) & u(y,\widehat{x}_1) & \cdots & u(y,\widehat{x}_{2+2n}) & ()^{(1,n)} \\ u(\widehat{x}_1,y) & u(\widehat{x}_1,\widehat{x}_1) & \cdots & u(\widehat{x}_1,\widehat{x}_{2+2n}) & \\ \vdots & \vdots & \ddots & \cdots & ()^{(2+2n,n)} \\ u(\widehat{x}_{2+2n},y) & u(\widehat{x}_{2+2n},\widehat{x}_1) & \cdots & u(\widehat{x}_{2+2n},\widehat{x}_{2+2n}) & \\ ()^{(n,1)} & & ()^{(n,2+2n)} & & ()^{(n,n)} \end{pmatrix} \tag{15}$$

Then, in addition to the estimates $\boldsymbol{y}$ and $\widehat{\boldsymbol{x}}$, their associated standard uncertainties can also be calculated, considering that

$$\begin{aligned} u(y)&=\sqrt{u(y,y)} \\ u(\widehat{x}_i)&=\sqrt{u(\widehat{x}_i,\widehat{x}_i)} \end{aligned} \tag{16}$$

4. Application case

We will calculate the measurement uncertainties in an application case, to illustrate how these uncertainties are evaluated using the GLS-LM method. A rectangular specimen was cut to nominal dimensions of $160\times40\times3.05mm$ from a Vishay PS-1 photoelastic sheet [23]. The specimen was annealed at 150 °C for an hour to remove any internal stresses that might have been introduced. It was then loaded in a universal testing machine, so that it was subjected to an increasing tensile load from zero to 2000 N, in steps of approximately 100 N. The testing machine was placed in a dark-field circular polariscope illuminated by monochromatic light from a sodium lamp ($\lambda=589.3nm$). The fringe order was measured at the central part of the specimen, through a Babinet–Soleil compensator, for each load step, so that $n=20$ pairs of data load vs. fringe order were recorded.

The following considerations were taken into account when defining the input quantities of the measurement process:

- The width, b, of the specimen was measured at different points using a Vernier caliper, and the mean and standard deviation of its values were obtained. According to the GUM [18], and considering the uncertainties due to resolution ($0.02mm$) and calibration of the

[3] To improve the final appearance of the resulting system of Eq. (14), the second term of the well-known Lagrangian function has been multiplied by 2.

[4] The notations $\mathbf{0}^{(a,b)}$ and $()^{(a,b)}$ are used to indicate an $a\times b$ zero submatrix and an $a\times b$ submatrix with no relevant information, respectively.

Table 1
Uncertainty budget in the measurement of the width b of the specimen.

Source of Uncertainty	Probability distribution	Estimate (mm)	Standard uncertainty	Sensitivity coefficient	Contribution (mm)
Repeated observations	t-student	39.93	0.3182	1	0.3182
Resolution	rectangular	0.00	$0.02/\sqrt{12}$	1	0.0058
Calibration	Gaussian	0.00	0.0263	1	0.0263
		39.93			0.32

caliper (in the certificate of calibration), a standard uncertainty $u(b) = 0.32mm$ was obtained for an estimated value, $b = 39.93mm$ (Table 1).

- Regarding the wavelength of the light, sodium-vapor discharge lamps have two primary emission wavelengths, 589.0 nm and 589.6 nm. Then, according to the GUM [18], this quantity can be described by a rectangular distribution over its range, which provides a mean value $\lambda = 589.3nm$ with a standard uncertainty $u(\lambda) = 0.7/\sqrt{12}nm$.
- The measurement of the load P was carried out at the time of the test, so it was decided to adjust a calibration line with which a single uncertainty value is obtained for the entire range of application. In accordance with the GUM [18], a standard uncertainty $u(P) = 0.97N$ was obtained by combining, the variability, calibration and resolution uncertainty components for the load cell of the testing machine (Table 2).

- Finally, calibration of the measurement of the fringe order followed the draft standard proposed by the Standardization Project for Optical Techniques of Strain Measurement [24,25]. This calibration uncertainty was combined, for each measurement, with the components of the uncertainty due to the resolution of the compensator (0.01*fringes*) and due to the repeatability of measurements (fringe order measurements were repeated ten times at each load step). Residual birefringence was corrected. As a sample, Table 3 provides the uncertainty budget for one of the fringe order measurements. All data were assumed to be uncorrelated.

The GLS-LM method involves applying an iterative process to a large system of equations that is, moreover, near singular. Its implementation in MATLAB [22] makes it easy to apply, simply by introducing the necessary data: number, name, and starting values of the output quantities; number, name, estimates, and covariance matrix of the input quantities; and number and equations of the constraints. When the program was executed, the GLS-LM method quickly converged, and the estimates, $\hat{x}$ and y, and their associated standard uncertainties were obtained. They are shown in Table 4 under the row headings $\hat{x}_j$, y_i, $u(\hat{x}_j)$, and $u(y_i)$.

The stress-optic coefficient estimate was $C = 82.55 \cdot 10^{-12} m^2/N$ with an associated standard uncertainty of $u(C) = 0.67 \cdot 10^{-12} m^2/N$. This measurement[5] was in fairly close agreement with the data provided in the photoelastic material data sheet [23] (its strain optical coefficient, K, elastic modulus, E, and Poisson's ratio, ν):

$$C = \frac{1+\nu}{E}K = \frac{1+0.38}{2500 \cdot 10^6 N/m^2} \cdot 0.150 = 82.8 \cdot 10^{-12} m^2/N \tag{17}$$

The GLS-LM method provides not only an estimate of the quantity to be measured and its corresponding standard uncertainty, but also a fitted estimate and the standard uncertainty of the value of each input quantity. From those estimates and standard uncertainties, both the chi-square function (7) and the normalized deviations between the input estimates and their adjusted values can be evaluated, and the consistency of data can be tested. The chi-square function yielded an observed value of $\chi^2(\hat{x}; x) = 14.87$. The probability that this value is derived from a χ^2 distribution with $n-1 = 19$ degrees of freedom is $p = \text{Prob}\{\chi^2(19) > 14.87\} = 73.11\%$, greater than $\alpha = 5\%$, so the data were consistent with the constraints. This consistency was confirmed by the normalized deviations, given in Table 1 under the row headings, d_j, given that they all satisfied the criterion $|d_j| < 2$ (there were no outliers).

5. Discussion

The results obtained with the GLS-LM method may be compared with those obtained through the procedures used in previous works. The first procedure is simply to repeat measurements and to calculate their mean and standard deviation; these values are taken, respectively, as the estimate of the measurement and its corresponding standard uncertainty. It has previously been commented that this procedure to evaluate the uncertainty of an indirect measurement is not valid, as there are more components of uncertainty apart from any variability with repetition. If we repeat the conventional method of measuring the stress-optic coefficient ten times, the results of Table 5 are obtained.

As can be seen, while the estimated value obtained with this procedure is consistent with the value provided by the GLS-LM method[6], the standard uncertainty is clearly underestimated. In the method used in previous works only the contribution to uncertainty due to part of the variability of the measurements is considered. As we can see, this portion of uncertainty (0.11 B) represents only 16% of the total uncertainty (0.67 B). Therefore, this method of evaluating the measurement uncertainty is incorrect because it is incomplete. Much of the variability has been lost because it has not been analyzed separately when measuring each input quantity. On the other hand, the uncertainty components related with the resolution and calibration of the devices and procedures used to measure the variables involved in the process (width of the specimen, wavelength of light, applied load and photoelastic fringe order) have not been considered. This is only possible to do with the GLS-LM method.

The second procedure is, basically, the same as the previous one; but it has the advantage that it is not necessary to correct the possible residual birefringence existing in the photoelastic material in each fringe order measurement. It automatically corrects the residual birefringence through the direct measurement of the N/P ratio. The estimated and the standard uncertainty for the slope parameter in the linear least squares fit load vs. fringe order Fig. 2) are first calculated with Eqs. (3) and ((4) and the estimate of the stress-optic coefficient and its corresponding standard uncertainty are then calculated with Eqs. (5) and (6).

The values obtained were $C = 82.43 \cdot 10^{-12} m^2/N$ and $u(C) = 0.11 \cdot 10^{-12} m^2/N$, which are the same as those obtained with the previous method of repeating the measurement process. Again, the uncertainties of measurement are underestimated. In both this method and

[5] Sometimes the stress-optic coefficient is expressed in Brewsters (B), a measure of the susceptibility of the material to photoelasticity: $B = 10^{-12} m^2/N$.

[6] The difference between both estimates is because the least squares fit is weighted by the covariance matrix in the GLS-LM method, while there is no weighting in the conventional method. They are nevertheless coherent values, since the difference between them is several times less than the standard uncertainty.

Table 2
Uncertainty budget in the measurement of the load P.

Source of uncertainty	Probability distribution	Estimate (N)	Standard uncertainty	Sensitivity coefficient	Contribution (N)
Variability	t-student	0–2000	0.9595	1	0.9595
Resolution	rectangular	0	$0.2/\sqrt{12}$	1	0.0577
Calibration	gaussian	0	0.0948	1	0.0948
		0–2000			0.97

Table 3
Uncertainty budget in the measurement of N_{14}.

Source of uncertainty	Probability distribution	Estimate (*fringes*)	Standard uncertainty	Sensitivity coefficient	Contribution (*fringes*)
Repeated observations	t-student	4.930	0.0141	1	0.0082
Resolution	rectangular	0.00	$0.01/\sqrt{12}$	1	0.0029
Calibration	Gaussian	0.00	0.0043	1	0.0118
		4.930			0.015

Table 4
Estimates and standard uncertainties before and after the GLS-LM fitting.

Quantity	b (mm)	λ (mm)	P_1 (N)	N_1 (fr.)	P_2 (N)	N_2 (fr.)
x_j	39.93	$589.3{\bullet}10^{-6}$	100.12	0.360	200.53	0.710
$u(x_j)$	0.32	$0.202{\bullet}10^{-6}$	0.97	0.013	0.97	0.012
$\hat{x}_j$	39.93	$589.3{\bullet}10^{-6}$	100.27	0.352	200.65	0.705
$u(\hat{x}_j)$	0.32	$0.202{\bullet}10^{-6}$	0.94	0.003	0.93	0.003
d_j	0.00	0.00	–0.63	0.63	–0.47	0.47
Quantity	P_3 (N)	N_3 (fr.)	P_4 (N)	N_4 (fr.)	P_5 (N)	N_5 (fr.)
x_j	299.78	1.070	400.23	1.420	499.97	1.770
$u(x_j)$	0.97	0.014	0.97	0.013	0.97	0.013
$\hat{x}_j$	300.06	1.054	400.50	1.406	500.23	1.757
$u(\hat{x}_j)$	0.94	0.003	0.94	0.003	0.094	0.004
d_j	–1.21	1.20	–1.09	1.09	–1.07	1.07
Quantity	P_6 (N)	N_6 (fr.)	P_7 (N)	N_7 (fr.)	P_8 (N)	N_8 (fr.)
x_j	600.19	2.120	700.23	2.460	799.69	2.800
$u(x_j)$	0.97	0.013	0.97	0.014	0.97	0.014
$\hat{x}_j$	600.42	2.108	700.25	2.459	799.56	2.808
$u(\hat{x}_j)$	0.94	0.004	0.94	0.004	0.94	0.004
d_j	–0.93	0.93	–0.08	0.08	0.57	–0.57
Quantity	P_9 (N)	N_9 (fr.)	P_{10} (N)	N_{10} (fr.)	P_{11} (N)	N_{11} (fr.)
x_j	900.09	3.150	1000.21	3.510	1100.05	3.850
$u(x_j)$	0.97	0.013	0.97	0.015	0.97	0.014
$\hat{x}_j$	899.89	3.160	1000.18	3.512	1099.84	3.862
$u(\hat{x}_j)$	0.94	0.004	0.95	0.004	0.94	0.004
d_j	0.81	–0.81	0.15	–0.15	0.92	–0.91
Quantity	P_{12} (N)	N_{12} (fr.)	P_{13} (N)	N_{13} (fr.)	P_{14} (N)	N_{14} (fr.)
x_j	1199.96	4.220	1299.92	4.580	1400.10	4.930
$u(x_j)$	0.97	0.013	0.97	0.014	0.97	0.015
$\hat{x}_j$	1200.08	4.214	1300.16	4.566	1400.29	4.917
$u(\hat{x}_j)$	0.94	0.005	0.94	0.005	0.95	0.005
d_j	–0.48	0.48	–1.10	1.09	–0.91	0.91
Quantity	P_{15} (N)	N_{15} (fr.)	P_{16} (N)	N_{16} (fr.)	P_{17} (N)	N_{17} (fr.)
x_j	1500.06	5.250	1600.20	5.630	1699.89	5.990
$u(x_j)$	0.97	0.014	0.97	0.015	0.97	0.015
$\hat{x}_j$	1499.78	5.267	1600.35	5.620	1700.18	5.970
$u(\hat{x}_j)$	0.94	0.005	0.95	0.005	0.95	0.006
d_j	1.27	–1.27	–0.74	0.73	–1.42	1.42
Quantity	P_{18} (N)	N_{18} (fr.)	P_{19} (N)	N_{19} (fr.)	P_{20} (N)	N_{20} (fr.)
x_j	1800.11	6.310	1899.91	6.660	2000.12	7.010
$u(x_j)$	0.97	0.014	0.97	0.015	0.97	0.015
$\hat{x}_j$	1799.93	6.321	1899.75	6.671	1999.93	7.023
$u(\hat{x}_j)$	0.95	0.006	0.95	0.006	0.95	0.006
d_j	0.83	–0.83	0.81	–0.80	0.94	–0.94
Quantity	C					
(units)	(m^2/N)					
y_i	$82.55{\bullet}10^{-12}$					
$u(y_j)$	$0.67{\bullet}10^{-12}$					

Table 5
Mean and standard deviation of ten measurements of the stress-optic coefficient.

C (10^{-12} m^2/N)	82.55	82.31	82.58	82.28	82.51	82.51	82.52	82.39	82.30	82.33
$\overline{C}$ (10^{-12} m^2/N)	82.43									
$u(\overline{C})$ (10^{-12} m^2/N)	0.11									

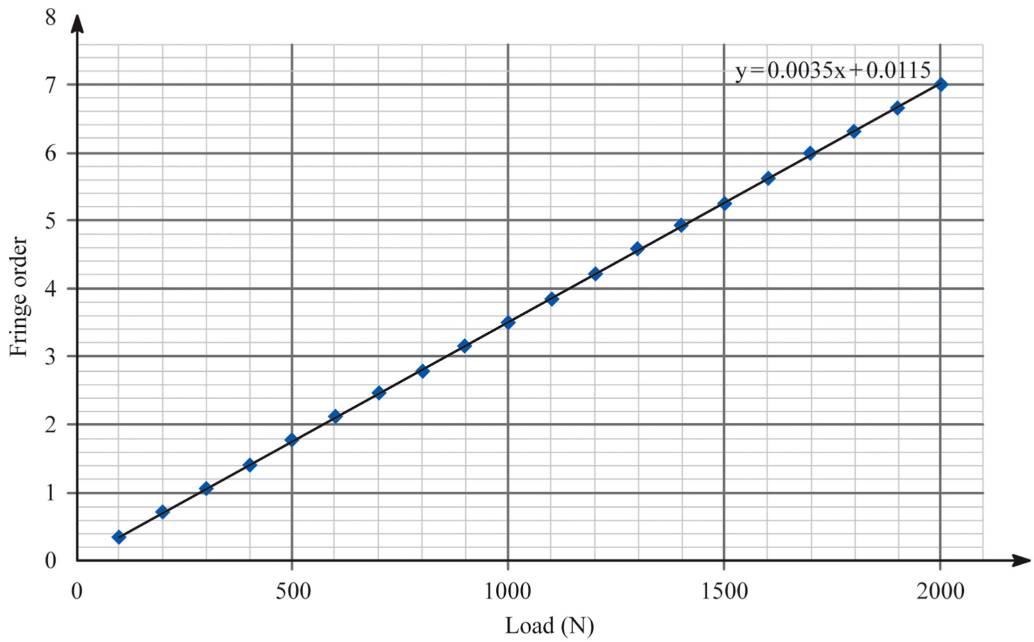

Fig. 2. Best-fit straight line for the evaluation of the slope N/P.

the previous one there are contributions to uncertainty that are being wrongly ignored.

The idea of repeating measurements or replicating them under different load conditions is a good idea, since it is the way to detect uncertainty due to variability/repeatability. However, both calculating the standard deviation of the observations and performing a conventional least squares adjustment are procedures that do not allow considering the multiple contributions to the uncertainty of each of the magnitudes involved. For this, it is necessary to use the GLS-LM method.

6. Conclusions

Metrological activities are essential to guarantee the quality of scientific and industrial activities. The main metrological condition to be fulfilled by measurements is that they must be expressed with their corresponding uncertainty, which must be evaluated in accordance with the international standards included in the ISO standard GUM. In the present work, it has been shown that current procedures for the evaluation of the uncertainty in the measurement of the optical-stress coefficient of photoelastic materials are incomplete. Those procedures are not consistent with the GUM, because the only component of the uncertainty under consideration is the uncertainty detected in the form of global repeatability. The GLS-LM method is the appropriate procedure with which the other contributions to uncertainty may be considered, such as the uncertainty in the measurement of the width of the specimen or the uncertainties of resolution and calibration of the measuring devices.

The GLS-LM method is the most general variant of the least squares method. In it, the function to be minimized is the weighted sum of squares of the differences between all observed data involved in the measurement process and their fitted values. Since the fit involves all the variables and the weighting is based on the inverse of their covariance matrix, the measurement uncertainties of all the variables can be considered and, as a result of the process, their uncertainties after the fitting can be obtained. Complementary to the estimate of the optical-stress coefficient and its corresponding standard uncertainty, the GLS-LM method can also, through the chi-square function, to test whether data are consistent with the theoretical model, and through the normalized deviations, to detect outliers.

The use of the GLS-LM method to measure the stress-optic coefficient of photoelastic materials guarantees a correct evaluation of its uncertainty, in accordance with international standards, and ensures the validity, comparability and reproducibility of the measurements.

CRediT authorship contribution statement

M. Solaguren-Beascoa Fernández: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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