J. Math. Anal. Appl. 540 (2024) 128573

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

# On spaces of integrable functions associated to vector measures and limiting real interpolation

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A R T I C L E   I N F O

*Article history:*
Received 29 March 2024
Available online 3 June 2024
Submitted by H. Frankowska

*Keywords:*
Extreme interpolation spaces
Vector measures
Lorentz-Zygmund spaces
Optimal domain
$p$-th power factorable operators
Bidual $(p, q)$-power-concave operators

A B S T R A C T

We investigate which spaces are obtained when considering the limiting class of real interpolation spaces $(0, q; J)$ for ordered Banach couples of spaces of (scalar) integrable functions with respect to a vector measure $m$, defined on a $\sigma$-algebra, with values in a Banach space. If $m$ is in particular a finite positive scalar measure, previous known results are derived from ours. Furthermore, we study the interpolation of $p$-th power factorable operators by the extreme real interpolation method $(1, q; K)$. We also deduce interpolation results for the $(1, q; K)$-method that apply to other related classes of operators to $p$-th power factorable operators, such as bidual $(p, q)$-power-concave operators and $q$-concave operators.



## 1. Introduction

Among all interpolation methods, the *real method* $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$ and $1 \leq q \leq \infty$, has been the most studied. A reason for this is, probably, its flexible construction. Thus, there exist different equivalent definitions of this method, being specially prominent those ones using the $K$- and $J$-functionals of Peetre (see [3], [4], [5] and [45]).

Moreover, the definition of the real method $(A_0, A_1)_{\theta,q}$ has been generalized in several directions. For instance, an extension of $(A_0, A_1)_{\theta,q}$ consists in replacing in its construction the function $t^\theta$ by a more general function $f(t)$ (see [38], [39] and [44]). To illustrate an advantage of the *real method with a function parameter* $(A_0, A_1)_{f,q}$ we will mention that working with a couple of Lebesgue spaces one obtains a larger class of spaces than Lebesgue and Lorentz spaces provided by the real method $(A_0, A_1)_{\theta,q}$. In fact, if $(\Omega, \Sigma, \mu)$ is a positive

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1 The second named author was supported in part by UCM Grant PR3/23-30811.

https://doi.org/10.1016/j.jmaa.2024.128573

$\sigma$-finite measure space, applying the real method with a function parameter to the couple $(L^\infty, L^1)$ it holds that $\left(L^\infty, L^1\right)_{f,q} = L^{p,q}(\log L)^\alpha$, when $f(t) = t^{\frac{1}{p}}(1 + |\log t|)^{-\alpha}$, $1 < p < \infty, 1 \le q \le \infty$ and $\alpha \in \mathbb{R}$.

Some authors have investigated (see [38], [31] and [32]) the *logarithmic interpolation spaces* $(A_0, A_1)_{\theta,q,\mathbb{A}}$ that correspond to the special case where $f(t) = t^\theta \ell^{\mathbb{A}}(t)$, being $\ell^{\mathbb{A}}(t)$ a broken logarithmic function, that is, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell(t) = 1 + |\log t|$ and

$$\ell^{\mathbb{A}}(t) := \begin{cases} \ell^{\alpha_0}(t), & \text{if} \quad 0 < t \le 1, \\ \ell^{\alpha_\infty}(t), & \text{if} \quad 1 < t < \infty. \end{cases}$$

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \le q \le \infty$ and not only $0 < \theta < 1$, but $\theta$ even taking the values 0 and 1. For these limiting values $\theta = 0$ and $\theta = 1$, the extra function $\ell^{\mathbb{A}}(t)$ is essential to get a meaningful definition of $(A_0, A_1)_{\theta,q,\mathbb{A}}$.

However, when $(A_0, A_1)$ is an ordered Banach couple, that is when $A_0 \subseteq A_1$ (where "$\subseteq$" means continuous inclusion), then just making a natural modification in the definition of the real method $(A_0, A_1)_{\theta,q}$, it is possible to define two limiting classes of real interpolation spaces, $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$, without involving any auxiliary function (see Section 2 for precise definitions).

The spaces $(A_0, A_1)_{1,q;K}$ were investigated for the first time by Gomez and Milman [37]. More recently, the extreme interpolation methods $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$ have been studied by Cobos, Fernández-Cabrera, Kühn and Ullrich [11], Cobos, Fernández-Cabrera and Mastyło [12], Cobos and Kühn [6] and Cobos, Fernández-Cabrera and Martínez [13] (see also [10], [14], [15], [7], [8] and [9]). It is important to point out that $(A_0, A_1)_{0,1;J} = A_0$ and for the rest of the values $1 < q \le \infty$, the space $(A_0, A_1)_{0,q;J}$ is very close to $A_0$. On the other hand, $(A_0, A_1)_{1,\infty;K} = A_1$ and for other values $1 \le q < \infty$ the space $(A_0, A_1)_{1,q;K}$ is very near $A_1$. These facts constitute an important difference in the theory.

As an intrinsic problem related to interpolation theory it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. In the recent years, this question has awakened a lot of attention when the couple is formed by spaces of (scalar) integrable functions associated to vector measures (see, for instance, [34], [22] and [24] for the complex interpolation method; [35] and [25] for the real method; [27], [28] and [42] for the real method with a function parameter; and [21], [26], [29] and [30], among others, for related questions). This interest can be better understood by some applications that such spaces have.

Thus, it is worthy to mention that the *space* $L^1(m)$*, of scalar integrable functions with respect to a vector measure* $m$*,* defines a broad class of Banach function spaces. More precisely, every order continuous Banach lattice $X$ with a weak order unit can be represented (order isometrically) as an $L^1(m)$ space of a vector measure $m$ defined on a $\sigma$-algebra, as proved by Curbera [16, Theorem 8]. If the order continuity fails but $X$ has a Fatou type property and a weak order unit belonging to its order continuous part, then $X$ can be identified (order isometrically) with a *space* $L^1_w(m)$ *of weakly integrable functions with respect to a vector measure* $m$ on a $\sigma$-algebra (see [18, Theorem 2.5]). For a $p$-convex Banach lattice $X$, analogous representations as an $L^p(m)$ or an $L^p_w(m)$ space hold, depending on similar assumptions considered above (see [33] and [19]).

Another significant application of integrable function spaces with respect to a vector measure refers to the so-called optimal domain of an operator $T$ acting from a Banach function space $X$ into a Banach space $E$. Namely, if $X$ is an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$ and $T$ satisfies a natural condition which makes $m_T$ be a vector measure ($m_T$ is associated to $T$ by $m_T(A) := T(\chi_A)$, where $\chi_A$ is the characteristic function of $A \in \Sigma$), the space $L^1(m_T)$ is *the optimal domain* for $T$ within the class of order continuous Banach function spaces, that is, $L^1(m_T)$ is the largest space (in that class of spaces) to which $T$ can be extended as a continuous operator, still with values in $E$ (see [17] or [43, Theorem 4.14]).

The space $L^p(m_T)$ has also an important optimality property when $T$ is a *p-th power factorable operator* from $X$ into $E$ (see [43, Chapter 5]), in the sense that $L^p(m_T)$ is maximal among all order continuous Banach function spaces $Y$ that continuously contain $X$ and such that $T$ has an $E$-valued extension from $Y$ into $E$ which is itself $p$-th power factorable (see [43, Theorem 5.11]).

The class of $p$-th power factorable operators coincides with the class of operators that can be extended to the *p-th power space* $X_{[p]}$ of the Banach function space $X$ where the operator acts. These operators have turned out to be useful for analyzing some factorization properties of operators between Banach function spaces (see [43, Chapter 6] and [36]). In addition, relevant operators coming from Fourier analysis as convolution operators and the Fourier transform become $p$-th power factorable in certain cases (see [43, Section 7.5]).

As far as we know, none description has been given in the literature regarding the spaces obtained when applying limiting interpolation methods to Banach couples formed by function spaces associated to a vector measure. Nothing is known either about interpolation of $p$-th power factorable operators by limiting methods. In this paper we start the research on these questions. Namely, after this introduction and some necessary preliminaries (Section 2), we establish results for the $(0, q; J)$-method in the case of the first question (Section 3), and theorems that apply to the $(1, q; K)$-method in relation to the second issue (Section 4).

## 2. Notation and basic definitions

Throughout the paper all functions will be $\mathbb{R}$-valued and all Banach spaces will be over $\mathbb{R}$. If $X$ and $Y$ are Banach spaces, we will put $X = Y$ whenever $X$ and $Y$ are equal in the algebraic and topological sense (their norms are equivalent). By $X \subseteq Y$ we will mean that $X$ is continuously embedded into $Y$. As usual, given two quantities $\nu$, $\tau$ depending on certain parameters, we will write $\nu \preccurlyeq \tau$ if there exists a constant $c > 0$ independent of appropriate parameters such that $\nu \leq c\tau$. When $\nu \preccurlyeq \tau$ and also $\tau \preccurlyeq \nu$, we will put $\nu \simeq \tau$.

We will consider ordered Banach couples, that is to say couples $\bar{A} = (A_0, A_1)$ such that $A_0$, $A_1$ are Banach spaces with $A_0 \subseteq A_1$. We recall that, for each $t > 0$, *Peetre's $K$- and $J$-functionals* are defined by

$$K(t, a) = K(t, a; A_0, A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\},\ a \in A_1,$$

and

$$J(t, a) = J(t, a; A_0, A_1) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in A_0.$$

The *space* $\bar{A}_{0,q;J} = (A_0, A_1)_{0,q;J}$, $1 \leq q \leq \infty$, is formed (see [11, Definition 3.1]) by all the elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0$ such that $a = \int_1^\infty u(t)\frac{dt}{t}$ (convergence in $A_1$) and

$$\left(\int_1^\infty J(t, u(t))^q \,\frac{dt}{t}\right)^{\frac{1}{q}} < \infty \quad \left(\sup_{t>1}\{J(t, u(t))\} < \infty,\ \text{if } q = \infty\right).$$

We set $\|a\|_{\bar{A}_{0,q;J}}$ as the infimum of the last quantity over all possible representations $u(t)$ of $a$ satisfying the above conditions.

Note that $(A_0, A_1)_{0,q;J}$ is an intermediate Banach space with respect to $(A_0, A_1)$ for any $1 \leq q \leq \infty$ and $(A_0, A_1)_{0,1,J} = A_0$ (see [11, Lemma 3.2]). Furthermore, $(A_0, A_1)_{0,q;J}$ coincides (with the usual modification if $q = \infty$) with the collection of all those $a \in A_1$ such that $a = \sum_{n=1}^\infty u_n$ (convergence in $A_1$), with $(u_n) \subset A_0$ and $(\sum_{n=1}^\infty J(2^n, u_n)^q)^{\frac{1}{q}} < \infty$, being the norm $\|a\|_{\bar{A}_{0,q;J}}$ equivalent to $\|a\|_{0,q} :=$

$\inf\left\{\left(\sum_{n=1}^{\infty} J(2^n,u_n)^q\right)^{\frac{1}{q}} : a=\sum_{n=1}^{\infty} u_n\right\}$. As a straightforward consequence, $(0,q;J)$-spaces are ordered in the sense that $(A_0,A_1)_{0,p;J} \subseteq (A_0,A_1)_{0,q;J}$ if $p \le q$.

We also recall that the *space* $\bar{A}_{1,q;K} = (A_0,A_1)_{1,q;K}$, $1 \le q \le \infty$, consists of those $a \in A_1$ which have a finite norm

$$\|a\|_{1,q;K} := \left(\int_1^\infty \left[\frac{K(t,a)}{t}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \quad \left(\|a\|_{1,q;K} := \sup_{t>1}\left\{\frac{K(t,a)}{t}\right\},\ \text{if } q=\infty\right).$$

The space $(A_0,A_1)_{1,q;K}$ is an intermediate Banach space with respect to $(A_0,A_1)$ for every $1 \le q \le \infty$ and $(A_0,A_1)_{1,\infty,K} = A_1$ (see [37] and [11, Section 7]). Furthermore, $(A_0,A_1)_{1,q;K}$ coincides (with the usual modification if $q=\infty$) with the set of elements $a \in A_1$ such that $\|a\|_{1,q} := \left(\sum_{n=1}^{\infty} \left[2^{-n}K(2^n,a)\right]^q\right)^{\frac{1}{q}}$ is finite. In addition, the norm $\|a\|_{1,q;K}$ is equivalent to $\|a\|_{1,q}$. Hence, it follows that $(1,q;K)$-spaces are increasing with $q$, that is, $(A_0,A_1)_{1,p;K} \subseteq (A_0,A_1)_{1,q;K}$ if $p \le q$.

It is worth mentioning that $(0,q;J)$-spaces have an equivalent description by using the $K$-functional, as it is shown in [11, Theorem 4.2]. Namely, it holds by [11, Theorem 4.2] that

$$(A_0,A_1)_{0,q;J} = (A_0,A_1)_{\log,q;K}, \quad 1<q\le\infty, \tag{1}$$

where (see [11, Definition 4.1]) the space $\bar{A}_{\log,q;K} = (A_0,A_1)_{\log,q;K}$, $1<q\le\infty$, consists of all elements $a \in A_1$ for which the following norm is finite

$$\|a\|_{\log,q;K} := \begin{cases} \left(\displaystyle\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1<q<\infty, \\ \displaystyle\sup_{t>1}\left\{\frac{K(t,a)}{1+\log t}\right\}, & q=\infty. \end{cases}$$

Analogously, the space $(A_0,A_1)_{1,q;K}$ can be described by means of the $J$-functional (see [11, Theorem 7.6]).

In what follows, by a *Banach function space* $X$ on a given finite measure space $(\Omega,\Sigma,\mu)$, or on $\mu$ for short, we mean that $X$ is a lattice ideal of the space of (equivalence classes of) measurable functions $L^0(\mu)$, endowed with a complete norm $\|\cdot\|_X$ that is compatible with the $\mu$-a.e. order and such that $L^\infty(\mu) \subseteq X$. A Banach function space $X$ is said to be *order continuous* if for every sequence $(f_n)$ in $X$ such that $0 \le f_n \downarrow 0$ pointwise, it holds that $\|f_n\|_X \downarrow 0$.

Next we will recall some basic definitions and results on integration with respect to vector measures. Let $E$ be a Banach space and $m:\Sigma \to E$ be a countably additive vector measure, where $\Sigma$ is a $\sigma$-algebra of subsets of some nonempty set $\Omega$. Let $E^*$ denote the dual space of $E$. The *semivariation* of $m$ is the set function $\|m\|:\Sigma \longrightarrow [0,\infty)$ defined by

$$\|m\|(A) := \sup\{|\langle m,x^*\rangle|(A) : \|x^*\|_{E^*} \le 1\}, \quad A \in \Sigma,$$

where $|\langle m,x^*\rangle|$ is the total variation measure of the scalar measure $\langle m,x^*\rangle$, given by $\langle m,x^*\rangle(A) := \langle m(A),x^*\rangle$ for $A \in \Sigma$. It is well-known that

$$\frac{1}{2}\|m\|(A) \le \sup\left\{\|m(B)\|_E : B \subseteq A, B \in \Sigma\right\} \le \|m\|(A),$$

for every set $A \in \Sigma$. For the particular case when $m$ is a finite positive scalar measure, the semivariation $\|m\|$ and the measure $m$ coincide.

A set $A \in \Sigma$ is called *m-null* if $\|m\|(A) = 0$. A *Rybakov (control) measure* for $m$ is a measure defined as $|\langle m, x^*\rangle|$ for some $x^* \in E^*$, satisfying that $|\langle m, x^*\rangle|(A) = 0$ if and only if $A$ is a $m$-null set. Such a measure always exists (see for example [43, p. 108]).

Let $L^0(m)$ denote the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$. Two functions $f, g \in L^0(m)$ will be identified if are equal $m$-a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is an $m$-null set. We also recall that $f \in L^0(m)$ is said to be *weakly integrable* (with respect to $m$) if $f \in L^1(|\langle m, x^*\rangle|)$ for all $x^* \in E^*$. The *space $L^1_w(m)$ of all (m-a.e. equivalence classes of) weakly integrable functions* is a Banach function space on every Rybakov (control) measure for $m$, when endowed with the norm

$$\|f\|_{L^1_w(m)} := \sup\left\{\int_\Omega |f|\, d|\langle m, x^*\rangle| \;:\; \|x^*\|_{E^*} \leq 1\right\}.$$

A weakly integrable function $f$ is said to be *integrable* (with respect to $m$) if for every $A \in \Sigma$ there exists an element of $E$ denoted by $\int_A f dm$ (called *integral of $f$ over $A$*) such that $\left\langle \int_A f dm, x^*\right\rangle = \int_A f d\langle m, x^*\rangle$ for all $x^* \in E^*$. The *space $L^1(m)$ of all (m-a.e. equivalence classes of) integrable functions* is an order continuous closed ideal of $L^1_w(m)$, and in general $L^1(m) \subsetneq L^1_w(m)$.

If $1 < p < \infty$, a function $f \in L^0(m)$ is said to be *weakly p-integrable* (with respect to $m$) if $|f|^p \in L^1_w(m)$, and *p-integrable* (with respect to $m$) if $|f|^p \in L^1(m)$. We will denote by $L^p_w(m)$ the *space of (m-a.e. equivalence classes of) weakly p-integrable functions* and by $L^p(m)$ the *space of (m-a.e. equivalence classes of) p-integrable functions*. Obviously we have that $L^p(m) \subseteq L^p_w(m)$. The natural norm for both spaces is given by

$$\|f\|_{L^p_w(m)} := \sup\left\{\left(\int_\Omega |f|^p\, d|\langle m, x^*\rangle|\right)^{\frac{1}{p}} : \|x^*\|_{E^*} \leq 1\right\}, \quad f \in L^p_w(m).$$

We remark that, when $m$ is a finite positive scalar measure, both spaces $L^p_w(m)$ and $L^p(m)$ coincide with the classical (scalar) Lebesgue space $L^p$.

In addition, let $L^\infty(m)$ be the Banach space of all ($m$-a.e. equivalence classes of) essentially bounded functions equipped with the supremum norm. It holds that $L^\infty(m) \subseteq L^p(m)$, for any $1 \leq p < \infty$. We refer to the paper [33] and the monograph [43] for much more additional information on spaces $L^p(m)$ and $L^p_w(m)$.

Finally, we recall the definition of the *Lorentz-Zygmund space with respect to a vector measure m*. For $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$, the space $L^{p,q}(\log L)^\alpha(\|m\|)$ is defined as those functions $f \in L^0(m)$ for which the quantity

$$\|f\|_{L^{p,q}(\log L)^\alpha(\|m\|)} := \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{\frac{1}{p}}(1 + |\log t|)^\alpha f_*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup\limits_{t>0} \left\{t^{\frac{1}{p}}(1 + |\log t|)^\alpha f_*(t)\right\}, & q = \infty, \end{cases}$$

is finite. Here $f_*$ stands for the *decreasing rearrangement of $f$ with respect to the vector measure $m$*, given by $f_*(t) := \inf\{s > 0 : \|m\|_f(s) \leq t\}$, for $t > 0$, where $\|m\|_f(s) := \|m\|(\{w \in \Omega : |f(w)| > s\})$. Note that the function $f_*$ is non-increasing, right-continuous, and also $f_*(t) = 0$ for any $t \geq \|m\|(\Omega)$.

If $\alpha = 0$, the space $L^{p,q}(\log L)^\alpha(\|m\|)$ coincides with the space $L^{p,q}(\|m\|)$, introduced in [35]. For $p = q$, the space $L^{p,p}(\|m\|)$ is simply denoted by $L^p(\|m\|)$. As it has been pointed out in [35], in general, the spaces $L^p(\|m\|)$ and $L^p(m)$ do not coincide. However, for every $1 \leq p < \infty$, all the following continuous inclusion hold (see [35, Proposition 7])

$$L^{\infty}(m) \subseteq L^{p,1}(\|m\|) \subseteq L^{p}(\|m\|) \subseteq L^{p}(m) \subseteq L^{p}_{w}(m) \subseteq L^{p,\infty}(\|m\|).$$

## 3. Interpolation of integrable function spaces with respect to vector measures by limiting $(0, q; J)$-methods

We begin by remembering two estimates for the $K$-functional that we will use in this section. In the next proposition inequality (2) (respectively, inequality (3)) is a consequence of [35, Proposition 8 and Remark 9] (respectively, [35, Proposition 10 and Lemma 3]) and [3, Chapter 5, Proposition 1.2].

**Proposition 3.1.** *If $f \in L^1_w(m)$, then*

$$f_*(t) \preccurlyeq K\left(t^{-1}, f; L^{\infty}(m), L^1_w(m)\right). \tag{2}$$

*On the other hand, if $f \in L^1(m)$, then*

$$K\left(t, f; L^{\infty}(m), L^1(m)\right) \preccurlyeq t \int_0^{t^{-1}} f_*(s) ds. \tag{3}$$

Our first theorem shows that when interpolating the couples $\left(L^{\infty}(m), L^1(m)\right)$ and $\left(L^{\infty}(m), L^1_w(m)\right)$ by the $(0, q; J)$-method, we obtain the Lorentz-Zygmund space $L^{\infty,q}(\log L)^{-1}(\|m\|)$. The proof uses some ideas of [27, Theorem 3] and also equality (1) and Proposition 3.1. Theorem 3.2 provides a version of [11, Corollary 4.3] in the setting of vector measures.

**Theorem 3.2.** *For $1 < q \leq \infty$, it holds that*

$$\left(L^{\infty}(m), L^1(m)\right)_{0,q;J} = \left(L^{\infty}(m), L^1_w(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

**Proof.** We set $m_{\Omega} := \|m\|(\Omega)$. Since $L^1(m) \subseteq L^1_w(m)$, it follows that

$$\left(L^{\infty}(m), L^1(m)\right)_{0,q;J} \subseteq \left(L^{\infty}(m), L^1_w(m)\right)_{0,q;J}.$$

In order to see that

$$\left(L^{\infty}(m), L^1_w(m)\right)_{0,q;J} \subseteq L^{\infty,q}(\log L)^{-1}(\|m\|), \tag{4}$$

we assume first that $1 < q < \infty$. Namely, take any $f \in \left(L^{\infty}(m), L^1_w(m)\right)_{0,q;J}$. According to inequality (2) in Proposition 3.1, we have that

$$\begin{aligned}
\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)} &= \left(\int_0^{m_{\Omega}} \left[(1 + |\log t|)^{-1} f_*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_0^{m_{\Omega}} \left[(1 + |\log t|)^{-1} K(t^{-1}, f; L^{\infty}(m), L^1_w(m))\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}.
\end{aligned}$$

The change of variables $u = \dfrac{m_{\Omega}}{t}$ gives that

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)} \preccurlyeq \left(\int_1^\infty \left[\left(1+\left|\log\frac{m_\Omega}{u}\right|\right)^{-1} K\left(\frac{u}{m_\Omega}, f; L^\infty(m), L^1_w(m)\right)\right]^q \frac{du}{u}\right)^{\frac{1}{q}}$$

$$\leq \max\left\{1, \frac{1}{m_\Omega}\right\} \left(\int_1^\infty \left[\left(1+\left|\log\frac{m_\Omega}{u}\right|\right)^{-1} K(u, f; L^\infty(m), L^1_w(m))\right]^q \frac{du}{u}\right)^{\frac{1}{q}}.$$

In addition,

$$1+|\log u| = 1+\left|\log\left(\frac{u}{m_\Omega}\cdot m_\Omega\right)\right| \leq 1+\left|\log\frac{u}{m_\Omega}\right| + |\log m_\Omega| \leq \left(1+\left|\log\frac{u}{m_\Omega}\right|\right)(1+|\log m_\Omega|)$$

and so

$$\left(1+\left|\log\left(\frac{m_\Omega}{u}\right)\right|\right)^{-1} = \left(1+\left|\log\left(\frac{u}{m_\Omega}\right)\right|\right)^{-1} \leq \frac{1+|\log m_\Omega|}{1+|\log u|}. \tag{5}$$

Therefore, (5) and (1) yield

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)} \preccurlyeq \max\left\{1, \frac{1}{m_\Omega}\right\}(1+|\log m_\Omega|)\left(\int_1^\infty \left[\frac{K\left(u, f; L^\infty(m), L^1_w(m)\right)}{1+\log u}\right]^q \frac{du}{u}\right)^{\frac{1}{q}}$$

$$= \max\left\{1, \frac{1}{m_\Omega}\right\}(1+|\log m_\Omega|)\,\|f\|_{(L^\infty(m),L^1_w(m))_{\log,q;K}} \simeq \|f\|_{(L^\infty(m),L^1_w(m))_{0,q;J}}.$$

When $q=\infty$ the inclusion (4) can be established by using the same reasoning with menor modifications.

Now we will prove that if $1<q\leq\infty$,

$$L^{\infty,q}(\log L)^{-1}(\|m\|) \subseteq \left(L^\infty(m), L^1(m)\right)_{0,q;J}. \tag{6}$$

First we suppose that $1<q<\infty$. Define the function $W(t) := \dfrac{1}{t(1+|\log t|)^q}$. In order to use Hardy inequality for non-increasing functions (see [1, Theorem 1.7] and [40, Chapter 10]), let us check that

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{q-1}{r^q}\int_0^r W(t)dt, \quad r>0. \tag{7}$$

Since $t\mapsto t(1+|\log t|)$ is a non-decreasing function on $(0,+\infty)$, we have that

$$\int_r^\infty \frac{W(t)}{t^q}dt = \int_r^\infty \frac{1}{t^{q+1}(1+|\log t|)^q}dt \leq \frac{1}{r^{q-1}(1+|\log r|)^{q-1}}\int_r^\infty \frac{1}{t^2(1+|\log t|)}dt$$

$$\leq \frac{1}{r^{q-1}(1+|\log r|)^{q-1}}\int_r^\infty \frac{1}{t^2}dt = \frac{1}{r^q(1+|\log r|)^{q-1}}.$$

In other words,

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{1}{r^q(1+|\log r|)^{q-1}}, \text{ for every } r>0. \tag{8}$$

On the other hand, we note that

$$\int_0^r W(t)dt \geq \frac{1}{q-1} \cdot \frac{1}{(1+|\log r|)^{q-1}}, \text{ for each } r>0. \tag{9}$$

In fact, if $0<r\leq 1$, it is clear that

$$\int_0^r W(t)dt = \int_0^r \frac{1}{t(1+|\log t|)^q}dt = \frac{1}{q-1} \cdot \frac{1}{(1+|\log r|)^{q-1}},$$

and, for $r>1$,

$$\int_0^r W(t)dt = \int_0^r \frac{1}{t(1+|\log t|)^q}dt \geq \int_0^1 \frac{1}{t(1+|\log t|)^q}dt = \frac{1}{q-1}.$$

Then, from (8) and (9), we have that

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{1}{r^q(1+|\log r|)^{q-1}} \leq \frac{q-1}{r^q}\int_0^r W(t)dt, \quad r>0.$$

Hardy inequality with $W(t)$ implies that, for every $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$,

$$\left(\int_0^\infty \frac{1}{t(1+|\log t|)^q}\left[\frac{1}{t}\int_0^t f_*(s)ds\right]^q dt\right)^{\frac{1}{q}} \preccurlyeq \left(\int_0^\infty \left[\frac{f_*(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}. \tag{10}$$

This in particular yields that the function $\frac{1}{t}\int_0^t f_*(s)ds$ is finite a.e. Hence, it can be easily deduced that $f \in L^1(\|m\|)$ and so $f \in L^1(m)$. Thus, for any $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$, it follows from (1), inequality (3) in Proposition 3.1, and (10) that

$$\begin{aligned}
\|f\|_{(L^\infty(m),L^1(m))_{0,q;J}} &\simeq \|f\|_{(L^\infty(m),L^1(m))_{\log,q;K}} = \left(\int_1^\infty \left[\frac{K\left(t,f;L^\infty(m),L^1(m)\right)}{1+\log t}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_1^\infty \left[\frac{t\int_0^{t^{-1}} f_*(s)ds}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^1 \frac{1}{u(1-\log u)^q}\left[\frac{1}{u}\int_0^u f_*(s)ds\right]^q du\right)^{\frac{1}{q}} \\
&\preccurlyeq \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}.
\end{aligned}$$

Then (6) is proved when $1<q<\infty$.

Finally, we will see that (6) also holds if $q=\infty$. Observe that

$$\int_0^u (1+|\log s|)ds \leq 2u(1+|\log u|), \text{ for each } u>0. \tag{11}$$

Given $f \in L^{\infty,\infty}(\log L)^{-1}(\|m\|)$, from (11) it follows that

$$\|f\|_{L^1(\|m\|)} = \int_0^{m_\Omega} f_*(t)dt = \int_0^{m_\Omega} \frac{f_*(t)}{1+|\log t|}(1+|\log t|)dt$$
$$\leq 2m_\Omega\,(1+|\log m_\Omega|)\,\|f\|_{L^{\infty,\infty}(\log L)^{-1}(\|m\|)},$$

and so $f \in L^1(\|m\|) \subseteq L^1(m)$. By (1), inequality (3) in Proposition 3.1, and (11), we have that

$$\|f\|_{(L^\infty(m),L^1(m))_{0,\infty;J}} \simeq \|f\|_{(L^\infty(m),L^1(m))_{\log,\infty;K}} = \sup_{t>1}\left\{\frac{K\left(t,f;L^\infty(m),L^1(m)\right)}{1+\log t}\right\}$$
$$\preccurlyeq \sup_{t>1}\left\{\frac{t\int_0^{t^{-1}} f_*(s)ds}{1+\log t}\right\} = \sup_{0<u<1}\left\{\frac{\int_0^u f_*(s)ds}{u(1-\log u)}\right\}$$
$$= \sup_{0<u<1}\left\{\frac{\int_0^u \frac{f_*(s)}{1-\log s}(1-\log s)ds}{u(1-\log u)}\right\}$$
$$\leq \sup_{0<u<1}\left\{\frac{\int_0^u(1-\log s)ds}{u(1-\log u)}\right\}\|f\|_{L^{\infty,\infty}(\log L)^{-1}(\|m\|)}$$
$$\leq 2\,\|f\|_{L^{\infty,\infty}(\log L)^{-1}(\|m\|)},$$

and the validity of (6) is proved for $q=\infty$. $\square$

**Remark 3.3.** Regarding the inclusion (6) in the proof of Theorem 3.2, we note that applying [2, Theorem 6.4 (6.7)] with $\lambda = 1$, $a = q$ and $\alpha = -1$, it follows that

$$\int_0^1 \left[\frac{1}{t(1-\log t)}\int_0^t f_*(s)ds\right]^q \frac{dt}{t} \preccurlyeq \int_0^1 \left[\frac{1}{1-\log t} f_*(t)\right]^q \frac{dt}{t} \leq \|f\|^q_{L^{\infty,q}(\log L)^{-1}(\|m\|)},$$

for any $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$. As a consequence, it is also possible to deduce that $f \in L^1(\|m\|) \subseteq L^1(m)$. Then, reasoning as in the proof of Theorem 3.2, it can be proved that $\|f\|_{(L^\infty(m),L^1(m))_{0,q;J}} \preccurlyeq \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}$. However, we have preferred in Theorem 3.2's proof to show the concrete calculations for the validity of (7) with the function $W(t) = \dfrac{1}{t(1+|\log t|)^q}$, which leads to the validity of (10).

**Remark 3.4.** The next result extends some known results in the case of finite positive scalar measures (see [11, Corollaries 4.7 and 3.8]). In order to establish its assertion c), we will use that the norm

$$|||a|||_{0,q} := \left(\sum_{n=1}^\infty \left[n^{-\frac{1}{q'}}2^{-\theta_0 n}K(2^n,a;A_0,A_1)\right]^q\right)^{\frac{1}{q}}$$

considered in [11, Theorem 3.7] is equivalent to the quantity

$$\left(\int_1^\infty \left[t^{-\theta_0}(1+\log t)^{-\frac{1}{q'}}K(t,a;A_0,A_1)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

**Corollary 3.5.** *Let* $1 < r < p < \infty$, $1 < q \leq \infty$ *and* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. *It holds that:*

a) $\left(L^{p,q}(\|m\|), L^1(m)\right)_{0,q;J} = \left(L^{p,q}(\|m\|), L^1_w(m)\right)_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

b) $(L^\infty(m), L^{p,q}(\|m\|))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|)$.

c) $(L^{p,q}(\|m\|), L^{r,q}(\|m\|))_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

**Proof.** a) Note that [27, Corollary 2] (see also [4, Theorem 3.4.1 (a)]) gives that

$$L^{p,q}(\|m\|) = (L^\infty(m), L^1(m))_{\frac{1}{p},q} = (L^\infty(m), L^1_w(m))_{\frac{1}{p},q}.$$

In addition, by [11, Theorem 4.6 (a)], $\left(L^{p,q}(\|m\|), L^1(m)\right)_{0,q;J} = \Gamma^{p,q}_{L^1}(m)$, where

$$\Gamma^{p,q}_{L^1}(m) := \left\{ f \in L^1(m) : \left( \int_1^\infty \left[ \frac{K\left(t, f; L^\infty(m), L^1(m)\right)}{t^{\frac{1}{p}}(1+\log t)^{\frac{1}{q'}}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

Analogously, by [11, Theorem 4.6 (a)], $\left(L^{p,q}(\|m\|), L^1_w(m)\right)_{0,\, q;\, J} = \Gamma^{p,q}_{L^1_w}(m)$, with

$$\Gamma^{p,q}_{L^1_w}(m) := \left\{ f \in L^1_w(m) : \left( \int_1^\infty \left[ \frac{K\left(t, f; L^\infty(m), L^1_w(m)\right)}{t^{\frac{1}{p}}(1+\log t)^{\frac{1}{q'}}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

Due to $L^1(m) \subseteq L^1_w(m)$, it holds that $\Gamma^{p,q}_{L^1}(m) \subseteq \Gamma^{p,q}_{L^1_w}(m)$. Therefore, it is enough to check that

$$L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|) \subseteq \Gamma^{p,q}_{L^1}(m) \text{ and } \Gamma^{p,q}_{L^1_w}(m) \subseteq L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|).$$

The inclusion $\Gamma^{p,q}_{L^1_w}(m) \subseteq L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$ follows by using the same argument that in the proof of (4) in Theorem 3.2 with minimal changes.

On the other hand, the inclusion $L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|) \subseteq \Gamma^{p,q}_{L^1}(m)$ can be established by a similar reasoning to that used to see (6) in Theorem 3.2 (see also Remark 3.3). For the sake of completeness, we next give the main details. Given $f \in L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$, using [2, Theorem 6.4 (6.7)] with $\lambda = 1 - \frac{1}{p}$, $a = q$ and $\alpha = -\frac{1}{q'}$,

$$\int_0^1 \left[ \frac{t^{\frac{1}{p}-1}}{(1-\log t)^{\frac{1}{q'}}} \int_0^t f_*(s)ds \right]^q \frac{dt}{t} \preccurlyeq \int_0^1 \left[ \frac{t^{\frac{1}{p}}}{(1-\log t)^{\frac{1}{q'}}} f_*(t) \right]^q \frac{dt}{t} \leq \|f\|^q_{L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)}. \tag{12}$$

Hence, it follows that $f \in L^1(\|m\|) \subseteq L^1(m)$. By (1), inequality (3) in Proposition 3.1, and (12), we obtain that

$$\begin{aligned} \|f\|_{\Gamma^{p,q}_{L^1}(m)} &= \left( \int_1^\infty \left[ \frac{K\left(t, f; L^\infty(m), L^1(m)\right)}{t^{\frac{1}{p}}(1+\log t)^{\frac{1}{q'}}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} \preccurlyeq \left( \int_1^\infty \left[ \frac{t^{1-\frac{1}{p}}}{(1+\log t)^{\frac{1}{q'}}} \int_0^{t^{-1}} f_*(s)ds \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \left( \int_0^1 \left[ \frac{u^{\frac{1}{p}-1}}{(1-\log u)^{\frac{1}{q'}}} \int_0^u f_*(s)ds \right]^q \frac{du}{u} \right)^{\frac{1}{q}} \preccurlyeq \|f\|_{L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)}. \end{aligned}$$

b) It follows as a straightforward consequence of [27, Corollary 2], [11, Theorem 4.6 (b)] and Theorem 3.2.

c) By [27, Corollary 2], we have that

$$L^{p,q}(\|m\|) = \big(L^\infty(m), L^1_w(m)\big)_{\frac{1}{p},q} \text{ and } L^{r,q}(\|m\|) = \big(L^\infty(m), L^1_w(m)\big)_{\frac{1}{r},q}.$$

Then, applying [11, Theorem 3.7] (see also Remark 3.4),

$$(L^{p,q}(\|m\|), L^{r,q}(\|m\|))_{0,\,q;\,J} = \Gamma^{p;q}_{L^1_w}(m).$$

And, as shown in the proof of a), $\Gamma^{p;q}_{L^1_w}(m) = L^{p,q}(\log L)^{-\frac{1}{q}'}(\|m\|)$. □

**Corollary 3.6.** *For any $1 < p, q < \infty$,*

$$(L^\infty(m), L^p(m))_{0,q;J} = (L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

**Proof.** If $1 < q \le p < \infty$, taking into account that $L^{p,q}(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^1(m)$ (see [35, Propositions 5.1) and 7] and [33, Corollary 3.2]), it follows from Corollary 3.5.b) and Theorem 3.2 that

$$\begin{aligned} L^{\infty,q}(\log L)^{-1}(\|m\|) &= (L^\infty(m), L^{p,q}(\|m\|))_{0,\,q;\,J} \subseteq (L^\infty(m), L^p(\|m\|))_{0,q;J} \\ &\subseteq (L^\infty(m), L^p(m))_{0,q;J} \subseteq \big(L^\infty(m), L^1(m)\big)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|). \end{aligned}$$

Analogously, for $1 < p < q < \infty$, since $L^q(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^1(m)$ (see [35, Propositions 5.2) and 7] and [33, Corollary 3.2]), we have by applying Corollary 3.5.b) and Theorem 3.2 that

$$\begin{aligned} L^{\infty,q}(\log L)^{-1}(\|m\|) &= (L^\infty(m), L^q(\|m\|))_{0,q;J} \subseteq (L^\infty(m), L^p(\|m\|))_{0,q;J} \\ &\subseteq (L^\infty(m), L^p(m))_{0,q;J} \subseteq \big(L^\infty(m), L^1(m)\big)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|). \end{aligned}$$

Therefore $(L^\infty(m), L^p(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|)$, for $1 < p, q < \infty$. Similar arguments can be used to obtain that

$$(L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|), \quad 1 < p, q < \infty. \quad \square$$

## 4. Interpolation of $p$-th power factorable operators by limiting $(1, q; K)$-methods

We start recalling some basic information that we will need in this section. Given a Banach function space $X$ on a finite measure space $(\Omega, \Sigma, \mu)$ and a Banach space $E$, an operator $T : X \longrightarrow E$ is said to be *p-th power factorable,* $1 \le p < \infty$, if there is a constant $K > 0$ such that

$$\|T(f)\|_E \le K \left\| |f|^{\frac{1}{p}} \right\|_X^p, \quad f \in X.$$

Observe that the collection of all $E$-valued 1-th power factorable operators on $X$ is exactly the class of continuous operators from $X$ into $E$. We refer to [43, Chapters 5, 6 and 7], [36], [23] and [29] for wide information about $p$-th power factorable operators and other related questions.

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of function spaces on the same finite measure space $(\Omega, \Sigma, \mu)$ such that $X_0 \subseteq X_1$. Let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that $T$ is an *admissible operator* between the couples $\bar{X}$ and $\bar{E}$ (that is, a continuous operator $T : X_1 \to E_1$ whose restriction to $X_0$ defines a continuous operator from $X_0$ to $E_0$). Let $T_i := T_{|X_i} : X_i \longrightarrow E_i$, $i = 0, 1$. In addition, let $T_{1,q;K} : (X_0, X_1)_{1,q;K} \longrightarrow (E_0, E_1)_{1,q;K}$ be the interpolated operator by the extreme real method $(1, q; K)$.

Inspired by certain results in [29] (see also [23]), we will establish that if $T_0$ and $T_1$ are $p$-th power factorable for some $1 < p < \infty$ and $X_0$ and $X_1$ are order continuous then, $T_{1,q;K}$ is $p$-th power factorable for any $1 \leq q < \infty$.

Before continuing, let us remember what is the optimal domain of an operator. Given an order continuous Banach function space $X$ on a finite measure space $(\Omega, \Sigma, \mu)$ and a Banach space $E$, an operator $T : X \longrightarrow E$ is called *$\mu$-determined* if the measures $\mu$ and $m_T$ have exactly the same null sets, where $m_T : \Sigma \to E$ is the *vector measure associated* to $T$ defined by $m_T(A) := T(\chi_A)$. When $T$ is $\mu$-determined, the space $L^1(m_T)$ is an order continuous Banach function space on $(\Omega, \Sigma, \mu)$, $X$ is continuously embedded into $L^1(m_T)$ via the natural inclusion $J_T : f \in X \longrightarrow J_T(f) := f \in L^1(m_T)$, and the integration operator

$$I_{m_T} : f \in L^1(m_T) \longrightarrow I_{m_T}(f) := \int_\Omega f dm_T \in E$$

is the unique continuous linear extension of $T$ satisfying that $T = I_{m_T} \circ J_T$ (see [17] or [43, Proposition 4.4]). Thus, if $Y$ is another order continuous Banach function space such that $X \subseteq Y$ and $T : Y \longrightarrow E$ is a continuous linear extension of $T$, then $Y \subseteq L^1(m_T)$. In this sense, it is said that $L^1(m_T)$ is the (order continuous) *optimal domain* for the operator $T$.

Let us observe that if $T_1$ is a $\mu$-determined operator, then so are the operators $T_0$ and $T_{1,q;K}$. Denoting by $m_i := m_{T_i}$ the vector measure associated to $T_i$, $i = 0, 1$, and by $m_{1,q;K} := m_{T_{1,q;K}}$ the vector measure associated to $T_{1,q;K}$ we have the optimal domains $L^1(m_0)$, $L^1(m_1)$ and $L^1(m_{1,q;K})$. Moreover, let us remark that if $X_0$ and $X_1$ are order continuous Banach function spaces and $q < \infty$, then $(X_0, X_1)_{1,q;K}$ is order continuous and so, in particular, $\big(L^1(m_0), L^1(m_1)\big)_{1,q;K}$ is order continuous for $1 \leq q < \infty$. Note that similar arguments for the classical real method are still valid (see [20, Remarks 1.9 and 1.10, p. 17]).

Our first result relates the $(1, q; K)$-space $\big(L^1(m_0), L^1(m_1)\big)_{1,q;K}$ of the optimal domains of $T_0$ and $T_1$ with the optimal domain $L^1(m_{1,q;K})$ of the interpolated operator $T_{1,q;K}$.

**Theorem 4.1.** *Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$. Let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that $T$ is an admissible operator between the couples $\bar{X}$ and $\bar{E}$, and suppose that $T_1$ is $\mu$-determined. For every $1 \leq q < \infty$,*

$$\big(L^1(m_0), L^1(m_1)\big)_{1,q;K} \subseteq L^1(m_{1,q;K}). \tag{13}$$

**Proof.** It is clear that $L^1(m_0) \subseteq L^1(m_1)$ and thus $\big(L^1(m_0), L^1(m_1)\big)$ is an ordered Banach couple. Furthermore, $\big(L^1(m_0), L^1(m_1)\big)_{1,q;K}$ is an order continuous Banach function space since $L^1(m_0)$ and $L^1(m_1)$ are order continuous.

Note also that the integration maps $I_{m_0}$ and $I_{m_1}$ coincide on $L^1(m_0)$, that is, $I_{m_0}(f) = I_{m_1}(f)$ for each $f \in L^1(m_0)$. Thus, the operator $\widehat{T} : f \in L^1(m_1) \longrightarrow \widehat{T}(f) := I_{m_1}(f) \in E_1$ verifies that $\widehat{T}_{|L^1(m_0)} = I_{m_0}$ and $\widehat{T}_{|X_1} = T_1$. In addition, the restriction of the interpolated operator

$$\widehat{T}_{1,q;K} : (L^1(m_0), L^1(m_1))_{1,q;K} \longrightarrow (E_0, E_1)_{1,q;K}$$

to the interpolated space $(X_0, X_1)_{1,q;K}$ coincides with the operator

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \longrightarrow (E_0, E_1)_{1,q;K}.$$

Therefore, $\widehat{T}_{1,q;K}$ is a continuous linear extension of $T_{1,q;K}$ to the order continuous Banach function space $\big(L^1(m_0), L^1(m_1)\big)_{1,q;K}$. Then, the inclusion (13) follows by the optimality of the domain $L^1(m_{1,q;K})$ for the operator $T_{1,q;K}$ (see [17] or [43, Theorem 4.14]). □

**Theorem 4.2.** *Under the same hypotheses as Theorem 4.1, for any $1 \leq q < \infty$ and $1 \leq p < \infty$, it holds that* $(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq L^p(m_{1,q;K})$.

**Proof.** We may suppose that $p > 1$ because the case $p = 1$ is established in Theorem 4.1.

Using the fact that $(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq (L^p(m_0), L^p(m_1))_{1,p\,q;K}$, it is sufficient to check that $(L^p(m_0), L^p(m_1))_{1,p\,q;K} \subseteq L^p(m_{1,q;K})$. To do this, we will see that $|f|^p \in L^1(m_{1,q;K})$, for an arbitrary $f \in (L^p(m_0), L^p(m_1))_{1,p\,q;K}$. In fact, by Theorem 4.1, we must only prove that $|f|^p \in \left(L^1(m_0), L^1(m_1)\right)_{1,q;K}$ when $f \in (L^p(m_0), L^p(m_1))_{1,p\,q;K}$. Applying [41, Theorem 1 and Lemma 1] we obtain that

$$\begin{aligned}
\||f|^p\|^q_{(L^1(m_0),L^1(m_1))_{1,q;K}} &= \int_1^\infty \left[\frac{K\left(t, |f|^p; L^1(m_0), L^1(m_1)\right)}{t}\right]^q \frac{dt}{t} \\
&= p \int_1^\infty \left[\frac{K\left(s^p, |f|^p; L^1(m_0), L^1(m_1)\right)}{s^p}\right]^q \frac{ds}{s} \\
&\simeq \int_1^\infty \left[\frac{K\left(s, |f|; L^p(m_0), L^p(m_1)\right)^p}{s^p}\right]^q \frac{ds}{s} \\
&= \|f\|^{p\,q}_{(L^p(m_0),L^p(m_1))_{1,p\,q;K}},
\end{aligned}$$

and the proof is concluded. □

As a consequence of Theorem 4.2 and a well-known characterization of $p$-th power factorable operators [43, Theorem 5.7 (iii)], we deduce the following result on interpolation of $p$-th power factorable operators by the $(1, q; K)$-method.

**Corollary 4.3.** *Under the same assumptions that Theorem 4.1, if $T_0 : X_0 \longrightarrow E_0$ and $T_1 : X_1 \longrightarrow E_1$ are $p$-th power factorable operators for some $1 < p < \infty$, then the operator $T_{1,q;K} : (X_0, X_1)_{1,q;K} \longrightarrow (E_0, E_1)_{1,q;K}$ is also $p$-th power factorable for every $1 \leq q < \infty$.*

**Proof.** Due to $T_i$ is $p$-th power factorable, it holds that $X_i \subseteq L^p(m_i)$, $i = 0, 1$. Then, by Theorem 4.2, we obtain that $(X_0, X_1)_{1,q;K} \subseteq (L^p(m_0), L^p(m_1))_{1,q;K} \subseteq L^p(m_{1,q;K})$ and so $T_{1,q;K}$ is a $p$-th power factorable operator. □

Next let us apply again Theorem 4.2 to deduce interpolation results for another class of operators, such as bidual $(p, q)$-power-concave operators. These operators were introduced in [43, Chapter 6] in connection with the Maurey-Rosenthal factorization theory. Consider a Banach function space $X$ over a finite measure $(\Omega, \Sigma, \mu)$. For $1 \leq p, q < \infty$, a $\mu$-determined operator $T : X \longrightarrow E$, with values into a Banach space $E$, is said to be *bidual $(p, q)$-power-concave* if there exists a weight $0 < w \in L^1(\mu)$ such that $X \subseteq L^q(w\,d\mu) \subseteq L^p(m_T)$ (see [43, Theorem 6.9 and Remark 6.10 (I)]). Bidual $(1, q)$-power-concave operators are of particular relevance. They are known also as *bidual $q$-concave* operators. We recall that a bidual $q$-concave operator is, in particular, $q$-concave (see [43, Proposition 6.2 (i)] with $p = 1$).

**Remark 4.4.** In the definition of a bidual $(p, q)$-power-concave operator is essential the weighted Lebesgue space $L^q(w\,d\mu)$. This space is the Lebesgue $L^q$-space for the finite measure with density $w$ given by $A \mapsto \int_A w\,d\mu$, that is, a function $f \in L^0(\mu)$ is in $L^q(w\,d\mu)$ if and only if $\int_\Omega |f|^q w d\mu < \infty$. In $L^q(w\,d\mu)$ we consider the norm $\|f\|_{L^q(w\,d\mu)} := \||f|^q w\|^{\frac{1}{q}}_{L^1(\mu)}$. It is clear that $L^q(w\,d\mu)$ is the same (isometrically) as the space

$L_q(w^{\frac{1}{q}})$ considered in [11]. Then, by using [11, Theorem 7.4] we get, for two weights $w_0 \geq w_1 > 0$ $\mu$-a.e., that

$$(L^q(w_0\, d\mu), L^q(w_1\, d\mu))_{1,q;K} = L^q(w\, d\mu), \tag{14}$$

where $w := w_1 \left(1 + \frac{1}{q} \log \frac{w_0}{w_1}\right)$ and $1 \leq q < \infty$.

It is not difficult to check that $0 < w \leq \frac{1}{q} w_0 + \left(1 - \frac{1}{q}\right) w_1$ $\mu$-a.e, and then $w \in L^1(\mu)$ if $w_0$ and $w_1$ are both in $L^1(\mu)$.

**Corollary 4.5.** *Under the same assumptions that Theorem 4.1, if $T_0 : X_0 \longrightarrow E_0$ and $T_1 : X_1 \longrightarrow E_1$ are bidual $(p,q)$-power-concave operators for some $1 \leq p < \infty$ and $1 \leq q < \infty$, then the interpolated operator $T_{1,q;K} : (X_0, X_1)_{1,q;K} \longrightarrow (E_0, E_1)_{1,q;K}$ is also bidual $(p,q)$-power-concave.*

**Proof.** We must check that, for some function $0 < w$ in $L^1(\mu)$, it holds that $(X_0, X_1)_{1,q;K} \subseteq L^q(w\, d\mu) \subseteq L^p(m_{1,q;K})$. Since $T_0$ and $T_1$ are bidual $(p,q)$-power-concave, there exist two weights $0 < w_0$ and $0 < w_1$ in $L^1(\mu)$ such that $X_0 \subseteq L^q(w_0\, d\mu) \subseteq L^p(m_0)$ and $X_1 \subseteq L^q(w_1\, d\mu) \subseteq L^p(m_1)$. We can assume that $w_0 \geq w_1$ $\mu$-a.e. If this is not the case, it would be enough to replace $w_0$ by $\max\{w_0, w_1\}$. Recall that $L^q(w_0\, d\mu) \cap L^q(w_1\, d\mu) = L^q(\max\{w_0, w_1\}\, d\mu)$. Therefore, we have the inclusions

$$(X_0, X_1)_{1,q;K} \subseteq (L^q(w_0\, d\mu), L^q(w_1\, d\mu))_{1,q;K} \subseteq (L^p(m_0), L^p(m_1))_{1,q;K}$$

with $w_0 \geq w_1$ $\mu$-a.e. Now, by (14) the equality $(L^q(w_0\, d\mu), L^q(w_1\, d\mu))_{1,q;K} = L^q(w\, d\mu)$ holds, where $0 < w$ belongs to $L^1(\mu)$. From Theorem 4.2 we know that $(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq L^p(m_{1,q;K})$, and then

$$(X_0, X_1)_{1,q;K} \subseteq L^q(w\, d\mu) \subseteq L^p(m_{1,q;K}),$$

as we wanted to prove. □

**Corollary 4.6.** *Assume the same hypotheses as Theorem 4.1, and also suppose that $X_0$ and $X_1$ are $q$-convex Banach function spaces and $T_0$ and $T_1$ are $q$-concave operators. Then $T_{1,q;K}$ is $q$-concave for all $1 \leq q < \infty$.*

**Proof.** By [43, Proposition 6.2 (iv) and equality (6.6)] it follows that $T_0$ and $T_1$ are bidual $q$-concave operators. Then so is $T_{1,q;K}$, by Corollary 4.5 with $p = 1$. Applying [43, Proposition 6.2 (i)] we conclude that $T_{1,q;K}$ is $q$-concave. □

## Acknowledgments

The authors would like to thank the referee for his/her comments.

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