ON SOME MEASURES OF NON-COMPACTNESS ASSOCIATED TO BANACH OPERATOR IDEALS

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Abstract. We study two variants of measures of non-compactness of operators associated to a Banach operator ideal in the sense of Pietsch. These measures are motivated by the notions of surjective-ideal-compactness and injective-ideal-compactness, defined respectively by Carl and Stephani and by Stephani. Interpolation results on these measures in the cases of Banach couples generated by a single Banach space are given. As an application, we obtain interpolation theorems on p-compact operators and quasi p-nuclear operators.

1. Introduction and background

Based on the well-known characterization given by Grothendieck [19] in 1955 (see also [26, p. 30]) for relatively compact sets in a Banach space X ($K \subset X$ is relatively compact if and only if $K \subset \{\sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_1}\}$ for some sequence $(x_n) \in c_0(X)$), Sinha and Karn [32, p. 19–20] introduced in 2002 a strengthened form of compactness in Banach spaces. Namely, if $1 \le p \le \infty$ (and p' satisfies that $1/p + 1/p' = 1$) a subset K in X is said to be *relatively p-compact* if and only if $K \subset p\text{-co}(x_n) := \{\sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_{p'}}\}$ for some sequence $(x_n) \in \ell_p(X)$, where the following conventions are understood: $(a_n) \in B_{c_0}$ if $p = 1$, and $(x_n) \in c_0(X)$ when $p = \infty$. Thus, relatively compact sets may be referred to as relatively ∞-compact sets. Note that $p\text{-co}(x_n)$ is a relatively compact set when $(x_n) \in \ell_p(X)$ and so relatively p-compact sets ($1 \le p < \infty$) are relatively compact. If compact sets are viewed as ∞-compact sets, then every p-compact set is a q-compact set, for $1 \le p < q \le \infty$.

The definition of relatively p-compact set leads to the notion of p-compact operator (in the sense of Sinha and Karn): a bounded linear operator $T \in \mathcal{L}(X, Y)$ is called *p-compact operator* if $T(B_X)$ is a relatively p-compact set in Y. Let $\mathcal{K}_p(X, Y) := \{T \in \mathcal{L}(X, Y);\ T \text{ is } p\text{-compact}\}$. It is well-known that $[\mathcal{K}_p, k_p]$ is a Banach operator ideal (see [32, Theorem 4.2] and [13, Proposition 3.15]). This kind of p-compactness for operators is different from the notion of p-compact operator due to Fourie and Swart [17] and, independently, to Pietsch [29] (see [28] and [1]).

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The relationships of the ideal $\mathcal{K}_p$ with other classical ideals were studied for the first time in [32], where it is shown for $T \in \mathcal{L}(X, Y)$ that (see [32, Proposition 5.3]):

- If T is p-compact, then T^* is p-summing.
- When T is p-nuclear, T^* is p-compact.
- If T^* is p-compact, then T is p-summing.

This study has been continued by Delgado, Piñeiro and Serrano in [13, Corollary 3.4 and Proposition 3.8] proving that

$$T \text{ is } p\text{-compact} \text{ if and only if } T^* \text{ is quasi } p\text{-nuclear},$$

and

$$T \text{ is quasi } p\text{-nuclear} \text{ if and only if } T^* \text{ is } p\text{-compact}. \tag{1}$$

The isometric counterparts of each one of these characterizations were given in [18, Theorem 2.8].

It is worth noting that the research of different properties (such as approximation, duality or factorization) in connection with p-compact sets and p-compact operators, as well as certain extensions of this form of compactness, has attracted the interest in the recent years (see, e.g., the articles [1], [2], [10], [13], [14], [21], [28] and [30]). A more general approach based on the notions of surjective $\mathcal{A}$-compactness and injective $\mathcal{A}$-compactness, defined respectively by Carl and Stephani [5] and by Stephani [33], allows the study of some of these questions under this wider framework (see [11], [12], [23] and [24] and references therein). This approach is followed by Delgado and Piñeiro [12] when considering two measures of non-$\mathcal{A}$-compactness of an operator, $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$, associated to a Banach operator ideal $\mathcal{A}$, which the authors use to provide a quantitative version of (1) (see [12, Corollary 3.13]).

In this paper we investigate the measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$. As explained next, the measure $\chi_{\mathcal{A}}$ (respectively, $n_{\mathcal{A}}$) vanishes precisely on the class of surjectively (respectively, injectively) $\mathcal{A}$-compact operators. Let us note that, in the particular case when $\mathcal{A}$ is chosen as the ideal of all bounded linear operators, each of these measures characterizes compactness (or ∞-compactness) of an operator.

Before of giving the precise definition of the measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$, we recall that (see for example [5, Sections 0 and 1] or [12, p. 98–99]) if $\mathcal{A}$ is an operator ideal and X is a Banach space, a subset $D \subset X$ is said to be *$\mathcal{A}$-bounded* if there is a Banach space Z and an operator $S \in \mathcal{A}(Z, X)$ such that $D \subset S(B_Z)$. Analogously, $D \subset X$ is called *relatively $\mathcal{A}$-compact*, or simply *$\mathcal{A}$-compact* (as in [5]), if $D \subset S(K)$ for some compact set $K \subset Z$. Clearly, the class of all $\mathcal{L}$-bounded sets in X is precisely the class of all bounded sets in X. Analogously, if $\mathcal{K}$ stands for the ideal of compact operators, the class of all $\mathcal{K}$-bounded sets coincides with that of all relatively compact sets. On the other hand, the class of $\mathcal{L}$-compact sets coincides with the class of relatively compact sets in X.

The definition of *surjectively $\mathcal{A}$-compact operator* (referred to simply as *$\mathcal{A}$-compact operator* in [5, Definition 2]), generalizes the notion of compact operator and it is the natural: $T \in \mathcal{L}(X, Y)$ is surjectively $\mathcal{A}$-compact if maps every bounded subset in X into an $\mathcal{A}$-compact subset in Y. The class $\mathcal{K}^{\mathcal{A}}$ formed by all surjectively $\mathcal{A}$-compact operators is a surjective operator ideal and $\mathcal{K}^{\mathcal{A}} = \mathcal{A}^{sur} \circ \mathcal{K}$ (see [5, Theorem 2.1]).

When a Banach operator ideal $[\mathcal{A}, \alpha]$ is considered, the notion of $\mathcal{A}$-compactness can be expressed in a similar way to precompactness in a Banach space [5, Theorem 3.1]: An $\mathcal{A}$-bounded set $D \subset X$ is $\mathcal{A}$-compact if and only if for every $\varepsilon > 0$, there are finitely many elements $x_1, \ldots, x_n \in X$, a Banach space Z and an operator $S \in \mathcal{A}(Z, X)$, with $\alpha(S) \leq \varepsilon$, such that

$$D \subset \bigcup_{k=1}^{n} \{x_k + S(B_Z)\}.$$

If $T \in \mathcal{A}^{sur}(X, Y)$ (equivalently $T(B_X)$ is $\mathcal{A}$-bounded) is not surjectively $\mathcal{A}$-compact, it is natural to wonder about the distance between T and $\mathcal{K}^{\mathcal{A}}(X, Y)$. From the aforementioned characterization of the $\mathcal{A}$-compactness, Delgado and Piñeiro [12, Definitions 2.4 and 3.1] introduced the *(outer) measure* $\chi_{\mathcal{A}}$ *of non-$\mathcal{A}$-compactness.* Namely, for $T \in \mathcal{A}^{sur}(X, Y)$,

$$\chi_{\mathcal{A}}(T) := \inf\Big\{\varepsilon > 0;\ T(B_X) \subset \bigcup_{k=1}^{n} \{y_k + S(B_Z)\}\Big\},$$

where the infimum is taken over all possible $y_1, \ldots, y_n \in Y$, Banach spaces Z and operators $S \in \mathcal{A}(Z, Y)$ with $\alpha(S) \leq \varepsilon$. Note that $T \in \mathcal{A}^{sur}(X, Y)$ ensures that in the above definition the infimum is taken on a nonempty set of positive numbers.

In addition, we remark that $\chi_{\mathcal{A}}(T) = \lim_n e_n(T, \mathcal{A})$, where $e_n(T, \mathcal{A})$ stands for the generalized (outer) entropy number, introduced by Carl and Stephani [5, Section 4]. Clearly, if $\mathcal{A} = \mathcal{L}$, then the measure $\chi_{\mathcal{L}}$ coincides with the (ball) measure of non-compactness of an operator. Let us also note that $T \in \mathcal{A}^{sur}(X, Y)$ is surjectively $\mathcal{A}$-compact if and only if $\chi_{\mathcal{A}}(T) = 0$.

Moreover, in [12, Remark 3.3] it is shown that $\chi_{\mathcal{A}}$ is a different notion from the (*outer*) measure $\gamma_{\mathcal{A}}$ related to an operator ideal $\mathcal{A}$, defined by Astala [3] in 1980:

$$\gamma_{\mathcal{A}}(T) := \inf\{\varepsilon > 0;\ T(B_X) \subset \varepsilon B_Y + S(B_Z), \text{ for some Banach space } Z \text{ and operator } S \in \mathcal{A}(Z, Y)\}.$$

Next we focus on the definition of the another function associated to a Banach operator ideal, $n_{\mathcal{A}}$, considered in [12]. Now the aim is to quantify in some sense the degree of inyective non-$\mathcal{A}$-compactness of an operator of the injective hull $\mathcal{A}^{inj}$. First we recall that, given an operator ideal $\mathcal{A}$, an operator $T \in \mathcal{L}(X, Y)$ is said to be *injectively* $\mathcal{A}$*-compact* if there exist a Banach space Z, a sequence $(z_n^*) \in c_0(Z^*)$ and an operator $S \in \mathcal{A}^{inj}(X, Z)$ such that $\|Tx\|_Y \leq \sup_{n\in\mathbb{N}} |\langle z_n^*, Sx\rangle|$ for any $x \in X$ (see [33, Section 1]). By the well-known characterization which says that $T \in \mathcal{L}(X, Y)$ is compact if and only if there is $(x_n^*) \in c_0(X^*)$ such that $\|Tx\|_Y \leq \sup_{n\in\mathbb{N}} |\langle x_n^*, x\rangle|$ for all $x \in X$, it follows that if $\mathcal{A} = \mathcal{L}$, the preceding concept coincides with the notion of compact operator. The class $\mathcal{H}^{\mathcal{A}}$ of all injectively $\mathcal{A}$-compact operators is an injective operator ideal and it can be described in terms of $\mathcal{A}^{inj}$ as $\mathcal{H}^{\mathcal{A}} = \mathcal{K} \circ \mathcal{A}^{inj}$ (see [33, Theorem 1.1(b)]).

On the other hand, when we consider a Banach operator ideal $[\mathcal{A}, \alpha]$, the following characterization for injectively $\mathcal{A}$-compact operators holds (see [12, Theorem

3.9]): An operator $T \in \mathcal{L}(X,Y)$ is injectively $\mathcal{A}$-compact if and only if for every $\varepsilon > 0$, there are $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and an operator $S \in \mathcal{A}(X,Z)$, with $\alpha(S) \le \varepsilon$, such that

$$\|Tx\|_Y \le \sup_{1\le k\le n} |\langle x_k^*, x\rangle| + \|Sx\|_Z, \quad x \in X .$$

The definition of the *(inner) measure $n_{\mathcal{A}}$ of non-$\mathcal{A}$-compactness* is based on this last fact (see [12, Definition 3.10]): for $T \in \mathcal{A}^{inj}(X,Y)$, it is defined

$$n_{\mathcal{A}}(T) := \inf\Big\{\varepsilon > 0\,;\, \|Tx\|_Y \le \sup_{1\le k\le n} |\langle x_k^*, x\rangle| + \|Sx\|_Z,\ x \in X\Big\},$$

where the infimum is taken over all choices of finitely many $x_1^*, \dots, x_n^* \in X^*$, Banach spaces Z and operators $S \in \mathcal{A}(X,Z)$ with $\alpha(S) \le \varepsilon$. The condition $T \in \mathcal{A}^{inj}(X,Y)$ ensures that this infimum is taken over a nonempty set of positive numbers.

We note that $n_{\mathcal{A}}(T) = \lim_n c_n(T,\mathcal{A})$, where $c_n(T,\mathcal{A})$ denotes the generalized Gelfand number defined by Stephani [33, Section 4]. Then, when in particular $\mathcal{A} = \mathcal{L}$, $n_{\mathcal{L}}$ coincides with the seminorm $\|\cdot\|_m$ studied by Lebow and Schechter [25], and so $\chi_{\mathcal{L}}(T)/2 \le n_{\mathcal{L}}(T) \le 2\chi_{\mathcal{L}}(T)$ (see [25, Theorem 3.1]). Observe that $T \in \mathcal{A}^{inj}(X,Y)$ is injectively $\mathcal{A}$-compact if and only if $n_{\mathcal{A}}(T) = 0$.

In [12, Remark 3.11] it is shown that $n_{\mathcal{A}}$ is not the same concept that the *(inner) measure $\beta_{\mathcal{A}}$ related to an operator ideal $\mathcal{A}$*, introduced by Tylli [35] in 1995:

$$\beta_{\mathcal{A}}(T) := \inf\{\varepsilon > 0;\ \text{there are a Banach space } Z \text{ and an operator } S \in \mathcal{A}(X,Z) \\ \text{such that } \|Tx\|_Y \le \varepsilon\|x\|_X + \|Sx\|_Z, \text{ for any } x \in X\}.$$

Using [5, Theorem 2.1] and the surjectivity of the dual ideal of the ideal Π_p of p-summing operators, it holds that $\mathcal{K}^{\Pi_p^d} = \Pi_p^d \circ \mathcal{K}$. Since also $\mathcal{K}_p = \Pi_p^d \circ \mathcal{K}$ (see for example [1, Corollary 4.9]), it follows that $\mathcal{K}^{\Pi_p^d} = \mathcal{K}_p$. Then, we have

$$T \text{ is } p\text{-compact} \iff T \text{ is surjectively } \Pi_p^d\text{-compact} \iff \chi_{\Pi_p^d}(T) = 0\,.$$

Analogously, as a consequence of [33, Theorem 1.1(b)] and the injectivity of the ideal Π_p, it holds that $\mathcal{H}^{\Pi_p} = \mathcal{K} \circ \Pi_p$. Moreover $\mathcal{QN}_p = \mathcal{K} \circ \Pi_p$ (see [33, p. 255]) and so $\mathcal{H}^{\Pi_p} = \mathcal{QN}_p$. Therefore,

$$T \text{ is quasi } p\text{-nuclear} \iff T \text{ is injectively } \Pi_p\text{-compact} \iff n_{\Pi_p}(T) = 0\,.$$

We finish this section pointing out that these facts and [12, Corollary 3.13], where it is proved that

$$n_{\Pi_p}(T) = \chi_{\Pi_p^d}(T^*) \ \text{ for } T \in \Pi_p(X,Y)\,,$$

allow Delgado and Piñeiro to obtain a quantitative version of (1).

As we have mentioned before, our aim is the study of the measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$. We do this after this introduction and the preliminary Section 2. On a hand, in Section 3 we establish results on different properties of $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ which extend that known about them in [12]. On the other hand, in Section 4 we investigate the behaviour under interpolation of these measures of non-$\mathcal{A}$-compactness. As far as we know there is no result in the literature in this sense. As a consequence of our interpolation formulas for $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$, we establish results on interpolation of surjective $\mathcal{A}$-compactness and injective $\mathcal{A}$-compactness, for an arbitrary Banach operator ideal $\mathcal{A}$, in the cases in which one of the Banach couples reduces to a single Banach space. In particular, we deduce interpolation theorems on p-compact operators and quasi p-nuclear operators.

2. Notation and basic definitions

Throughout the paper we will use standard notation. Given a Banach space X, we denote the closed unit ball of X by B_X and the dual space of X by X^*. If X and Y are Banach spaces, $\mathcal{L}(X,Y)$ stands for the Banach space of all bounded linear operators T from X into Y equipped with the operator norm $\|T\| = \sup_{x\in B_X}\|Tx\|$.

Let $\ell_1(B_X)$ be the Banach space of all absolutely summable families of scalars (λ_x) indexed by elements of B_X. We denote by $Q_X\colon \ell_1(B_X)\to X$ the *metric surjection* defined by $Q_X(\lambda_x)_{x\in B_X} := \sum_{x\in B_X}\lambda_x x$. On the other hand, let $\ell_\infty(B_{X^*})$ be the Banach space of all bounded families of scalars indexed by elements of B_{X^*}. By $J_X\colon X\to \ell_\infty(B_{X^*})$ we mean the *metric injection* given by $J_X x := (\langle x^*, x\rangle)_{x^*\in B_{X^*}}$.

Given two Banach spaces Z_0 and Z_1, let $(Z_0\oplus Z_1)_\infty$ (resp., $(Z_0\oplus Z_1)_1$) be the direct sum of the Banach spaces Z_0 and Z_1 endowed with the norm $\|(z_0,z_1)\| = \max\{\|z_0\|_{Z_0}, \|z_1\|_{Z_1}\}$ (resp., $\|(z_0,z_1)\| = \|z_0\|_{Z_0}+\|z_1\|_{Z_1}$), for $(z_0,z_1)\in Z_0\times Z_1$.

An *operator ideal* $\mathcal{A}$ is defined as a method of ascribing to each pair of Banach spaces (X,Y) a linear subspace $\mathcal{A}(X,Y)$ of $\mathcal{L}(X,Y)$ such that the following properties are satisfied:

(I1) The operator $x^*\otimes y := \langle x^*,\cdot\rangle y \in \mathcal{A}(X,Y)$, for any $x^*\in X^*, y\in Y$;
(I2) If $S\in\mathcal{L}(U,X)$, $T\in\mathcal{A}(X,Y)$ and $R\in\mathcal{L}(Y,V)$, then $R\circ T\circ S\in\mathcal{A}(U,V)$.

If in addition, for every (X,Y), the space $\mathcal{A}(X,Y)$ is supplied with a norm α in such a way that:

(N1) $\alpha(x^*\otimes y) = \|x^*\|\cdot\|y\|$, for all $x^*\in X^*, y\in Y$;
(N2) $\alpha(R\circ T\circ S) \le \|R\|\cdot\alpha(T)\cdot\|S\|$, whenever U and V are Banach spaces and $S\in\mathcal{L}(U,X)$, $T\in\mathcal{A}(X,Y)$ and $R\in\mathcal{L}(Y,V)$;
(N3) $(\mathcal{A}(X,Y),\alpha)$ is a Banach space;

then $[\mathcal{A},\alpha]$ is called a *Banach operator ideal.* Familiar examples of Banach operator ideals are the ideals $[\mathcal{L},\|\cdot\|]$ of all bounded linear operators, $[\mathcal{K},\|\cdot\|]$ of all compact operators and $[\mathcal{W},\|\cdot\|]$ of all weakly compact operators, where $\|\cdot\|$ is the usual operator norm.

As usual $\mathcal{A}^d$ stands for the *dual ideal* of an operator ideal $\mathcal{A}$, that is $\mathcal{A}^d(X,Y) = \{T\in\mathcal{L}(X,Y);\, T^*\in\mathcal{A}(Y^*,X^*)\}$. If $[\mathcal{A},\alpha]$ is a Banach operator ideal, $[\mathcal{A}^d,\alpha^d]$ becomes a Banach operator ideal, with $\alpha^d(T) := \alpha(T^*)$ for $T\in\mathcal{A}^d(X,Y)$.

We also recall that an operator ideal $\mathcal{A}$ is said to be *surjective* whenever $\mathcal{A} = \mathcal{A}^{sur}$, where $\mathcal{A}^{sur}$ is the (*surjective hull*) ideal whose components are

$$\mathcal{A}^{sur}(X,Y) := \{T\in\mathcal{L}(X,Y);\, T(B_X)\subset S(B_Z), S\in\mathcal{A}(Z,Y)\}\,.$$

Analogously, an operator ideal $\mathcal{A}$ is called *injective* when $\mathcal{A} = \mathcal{A}^{inj}$, where $\mathcal{A}^{inj}$ is the (*injective hull*) ideal whose components are

$$\mathcal{A}^{inj}(X,Y) := \{T\in\mathcal{L}(X,Y);\, \|Tx\|_Y \le \|Sx\|_Z \text{ for } x\in X,\ S\in\mathcal{A}(X,Z)\}\,.$$

For a Banach operator ideal $[\mathcal{A}, \alpha]$ it holds that $[\mathcal{A}^{sur}, \alpha^{sur}]$ and $[\mathcal{A}^{inj}, \alpha^{inj}]$ are also Banach operator ideals, where $\alpha^{sur}(T) := \inf\{\alpha(S);\, T(B_X) \subset S(B_Z), S \in \mathcal{A}(Z,Y)\} = \alpha(T \circ Q_X)$ and $\alpha^{inj}(T) := \inf\{\alpha(S);\, \|Tx\|_Y \le \|Sx\|_Z$ for $x \in X$, $S \in \mathcal{A}(X,Z)\} = \alpha(J_Y \circ T)$.

We conclude this section recalling the definition of two classes of operators that are important in this paper, such as the ideal Π_p of p-summing operators and the ideal $\mathcal{QN}_p$ of quasi p-nuclear operators. Given $1 \le p < \infty$, an operator $T \in \mathcal{L}(X,Y)$ is a *p-summing* if T maps weakly p-summable sequences in X into p-summable sequences in Y (for the theory of p-summing operators, we refer to [15, Chapter 2]). On the other hand, $T \in \mathcal{L}(X,Y)$ is called *quasi p-nuclear*, $1 \le p < \infty$, if there exists a sequence $(x_n^*) \in \ell_p(X^*)$ such that $\|Tx\|_Y \le \big(\sum_{n=1}^{\infty} |\langle x_n^*, x\rangle|^p\big)^{1/p}$, for any $x \in X$ (see for example [13, p. 293]).

An exhaustive study of operator theory can be carried out in the classical books [15], [20] and [29]. We refer to these monographs for wide information, in particular, about the Banach operator ideals $[\Pi_p, \pi_p]$ and $[\mathcal{QN}_p, \nu_p^Q]$.

3. Some properties of $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$

In this section we prove several properties of $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ that complement those studied in [12]. As we have pointed out in Section 1, $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ are different from the measures $\gamma_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$, defined by Astala [3] and Tylli [35] respectively. However, they share certain similar properties (see [7] and references therein for the main properties of $\gamma_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$). For example, it is easy to check that if $T \in \mathcal{A}^{sur}(X,Y)$ (resp., $T \in \mathcal{A}^{inj}(X,Y)$), $R \in \mathcal{L}(Y,Y_0)$ and $S \in \mathcal{L}(X_0,X)$, then

$$\chi_{\mathcal{A}}(R \circ T \circ S) \le \|R\|\chi_{\mathcal{A}}(T)\|S\| \quad (\text{resp., } n_{\mathcal{A}}(R \circ T \circ S) \le \|R\| n_{\mathcal{A}}(T)\|S\|)\,.$$

The measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ are also submultiplicative.

Lemma 1. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal.*

(i) *Assume that $T \in \mathcal{A}^{sur}(X,Y)$ and $S \in \mathcal{A}^{sur}(Y,Z)$. Then,*

$$\chi_{\mathcal{A}}(S \circ T) \le \chi_{\mathcal{A}}(S)\chi_{\mathcal{A}}(T)\,.$$

(ii) *Assume that $T \in \mathcal{A}^{inj}(X,Y)$ and $S \in \mathcal{A}^{inj}(Y,Z)$. Then,*

$$n_{\mathcal{A}}(S \circ T) \le n_{\mathcal{A}}(S) n_{\mathcal{A}}(T)\,.$$

Proof. We just prove (ii) (part (i) has been established in [12, Proposition 3.5(6)]). Fix $\beta > n_{\mathcal{A}}(T)$. Then there exist $x_1^*, \dots, x_m^* \in X^*$, a Banach space H and an operator $P \in \mathcal{A}(X,H)$, with $\alpha(P) \le \beta$, such that

$$\|Tx\|_Y \le \sup_{1 \le i \le m} |\langle x_i^*, x\rangle| + \|Px\|_H, \quad x \in X\,.$$

Similarly, let $\gamma > n_{\mathcal{A}}(S)$, then there are $y_1^*, \dots, y_n^* \in Y^*$, a Banach space K and an operator $Q \in \mathcal{A}(Y,K)$, with $\alpha(Q) \le \gamma$ such that

$$\|Sy\|_Z \le \sup_{1 \le j \le n} |\langle y_j^*, y\rangle| + \|Qy\|_K, \quad y \in Y\,.$$

Hence, for every $x \in X$,

$$\|STx\|_Z \leq \sup_{1\leq j\leq n} |\langle y_j^*, Tx\rangle| + \|QTx\|_K \leq \sup_{1\leq j\leq n} |\langle T^* y_j^*, x\rangle| + \|Q\|\|Tx\|_Y$$
$$\leq \sup_{1\leq j\leq n} |\langle T^* y_j^*, x\rangle| + \|Q\|\Big[\sup_{1\leq i\leq m} |\langle x_i^*, x\rangle| + \|Px\|_H\Big].$$

Combining the above, we get

$$\|STx\|_Z \leq \sup_{1\leq k\leq r} |\langle \hat{x}_k^*, x\rangle| + \|\Phi x\|_H, \quad x \in X,$$

where $\hat{x}_1^*, \dots, \hat{x}_r^* \in X^*$ and $\Phi := \|Q\|P \in \mathcal{A}(X,H)$. Clearly, $\alpha(\Phi) = \|Q\|\alpha(P) \leq \alpha(Q)\alpha(P)$ and so $n_{\mathcal{A}}(S\circ T) \leq \gamma\beta$. This yields

$$n_{\mathcal{A}}(S\circ T) \leq n_{\mathcal{A}}(S) n_{\mathcal{A}}(T).$$

□

Lemma 2. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal and let X be a Banach space.*

(i) *If $\mathrm{Id}_X \in \mathcal{A}^{sur}(X,X)$, then $\chi_{\mathcal{A}}(\mathrm{Id}_X) = 0$ if and only if X is finite dimensional. In addition, $\chi_{\mathcal{A}}(\mathrm{Id}_X) \neq 0$ implies that $\chi_{\mathcal{A}}(\mathrm{Id}_X) \geq 1$.*

(ii) *If $\mathrm{Id}_X \in \mathcal{A}^{inj}(X,X)$, then $n_{\mathcal{A}}(\mathrm{Id}_X) = 0$ if and only if X is finite dimensional. In addition, $n_{\mathcal{A}}(\mathrm{Id}_X) \neq 0$ implies that $n_{\mathcal{A}}(\mathrm{Id}_X) \geq 1$.*

Proof. The statement (i) is given in [12, Proposition 3.5(7)]. To show (ii), observe that $n_{\mathcal{A}}(\mathrm{Id}_X) = 0$ implies that Id_X is injectively $\mathcal{A}$-compact and therefore it is compact. Thus, X is finite dimensional.

Now assume that X is finite dimensional. Then, Id_X is a finite rank operator. In particular, Id_X belongs to the ideal of injectively $\mathcal{A}$-compact operators, which means that $n_{\mathcal{A}}(\mathrm{Id}_X) = 0$.

Finally note that if $n_{\mathcal{A}}(\mathrm{Id}_X) > 0$, by Lemma 1(ii), we obtain that

$$n_{\mathcal{A}}(\mathrm{Id}_X) = n_{\mathcal{A}}(\mathrm{Id}_X \circ \mathrm{Id}_X) \leq (n_{\mathcal{A}}(\mathrm{Id}_X))^2,$$

which completes the proof. □

Remark 3. *As a direct consequence of the definition of $\chi_{\mathcal{A}}$, for every metric surjection $q\colon X_0 \to X$ and $T \in \mathcal{A}^{sur}(X,Y)$, it follows that $\chi_{\mathcal{A}}(T) = \chi_{\mathcal{A}}(T\circ q)$. Analogously, for each metric injection $j\colon Y \to Y_0$ and $T \in \mathcal{A}^{inj}(X,Y)$, it holds that $n_{\mathcal{A}}(T) = n_{\mathcal{A}}(j\circ T)$.*

The following minimal properties occur (similarly to those in [7, Section 2] for the measures $\gamma_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$). We include a proof for the sake of completeness.

Proposition 4. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal.*

(i) *For any operator $T \in \mathcal{A}^{sur}(X,Y)$ one has*

$$\chi_{\mathcal{A}}(J_Y \circ T) = \min\{\chi_{\mathcal{A}}(j\circ T);\ j\colon Y \to Y_0 \text{ a metric injection}\}.$$

(ii) *For any operator $T \in \mathcal{A}^{inj}(X,Y)$ one has*

$$n_{\mathcal{A}}(T\circ Q_X) = \min\{n_{\mathcal{A}}(T\circ q);\ q\colon X_0 \to X \text{ a metric surjection}\}.$$

Proof. (i) We claim that, for any metric injection $j\colon Y \to Y_0$, one has

$$\chi_{\mathcal{A}}(J_Y \circ T) \le \chi_{\mathcal{A}}(j \circ T)\,.$$

By the metric extension property of $\ell_\infty(B_{Y^*})$, we can find an operator $S \in \mathcal{L}(Y_0, \ell_\infty(B_{Y^*}))$ such that $S \circ i_{j(Y)} = J_Y \circ j^{-1}_{|_{j(Y)}}$ and $\|S\| = \|J_Y \circ j^{-1}_{|_{j(Y)}}\|$, where $i_{j(Y)}$ is the inclusion of $j(Y)$ in Y_0. Thus, we get

$$\chi_{\mathcal{A}}(J_Y \circ T) = \chi_{\mathcal{A}}(S \circ i_{j(Y)} \circ j \circ T) \le \|S\|\chi_{\mathcal{A}}(j \circ T) = \chi_{\mathcal{A}}(j \circ T)$$

and this proves our claim. Since the reverse inequality in the statement of (i) is obvious, the proof is complete.

(ii) It is enough to establish $n_{\mathcal{A}}(T \circ Q_X) \le n_{\mathcal{A}}(T \circ q)$, for each metric surjection $q\colon X_0 \to X$. Let $R\colon X_0/Kerq \to X$ be the isometric isomorphism induced by q. If $\phi_{Kerq}\colon X_0 \to X_0/Kerq$ denotes the canonical quotient map, we have that $q = R \circ \phi_{Kerq}$. By the metric lifting property of $\ell_1(B_X)$, for every $\varepsilon > 0$, there exists an operator $S \in \mathcal{L}(\ell_1(B_X), X_0)$ such that $\phi_{Kerq} \circ S = R^{-1} \circ Q_X$ and $\|S\| \le (1+\varepsilon)\|R^{-1} \circ Q_X\| = 1 + \varepsilon$. Then,

$$\begin{aligned} n_{\mathcal{A}}(T \circ Q_X) &= n_{\mathcal{A}}(T \circ R \circ \phi_{Kerq} \circ S) \le n_{\mathcal{A}}(T \circ q \circ S) \\ &\le n_{\mathcal{A}}(T \circ q)\|S\| \le (1+\varepsilon) n_{\mathcal{A}}(T \circ q)\,. \end{aligned}$$

This implies that $n_{\mathcal{A}}(T \circ Q_X) \le n_{\mathcal{A}}(T \circ q)$ and the proof finishes. □

As it is pointed out in [12, p. 100], the following result follows straightaway from [9, Proposition 5].

Proposition 5. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Then, for any operator $T \in \mathcal{A}^{sur}(X, Y)$ the following formula holds*:

$$\chi_{\mathcal{A}}(T) = \inf\{\alpha(S);\, T(B_X) \subset R(B_F) + S(B_G)\}\,,$$

where the infimum is taken over all Banach spaces F and G and operators $R \in \mathcal{K}^{\mathcal{A}}(F, Y)$ and $S \in \mathcal{A}(G, Y)$.

In a similar fashion, we show a characterization of $n_{\mathcal{A}}$.

Proposition 6. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Then for any operator $T \in \mathcal{A}^{inj}(X, Y)$ the following formula holds*:

$$n_{\mathcal{A}}(T) = \inf\{\alpha(S);\, \|Tx\|_Y \le \|Rx\|_F + \|Sx\|_G,\ \text{for all } x \in X\}\,, \tag{2}$$

where the infimum is taken over all Banach spaces F and G and operators $R \in \mathcal{H}^{\mathcal{A}}(X, F)$ and $S \in \mathcal{A}(X, G)$.

Proof. Let $r_{\mathcal{A}}(T)$ denote the right-hand side of (2). If $\varepsilon > r_{\mathcal{A}}(T)$, then we can find Banach spaces F and G and operators $R \in \mathcal{H}^{\mathcal{A}}(X, F)$ and $S \in \mathcal{A}(X, G)$ with $\alpha(S) \le \varepsilon$, such that

$$\|Tx\|_Y \le \|Rx\|_F + \|Sx\|_G, \quad x \in X\,.$$

By [33, Theorem 1.1(c)], there exist a Banach space Z, a sequence $(z_n^*)_{n\in\mathbb{N}} \in c_0(Z^*)$ and $Q \in \mathcal{A}^{inj}(X, Z)$ so that

$$\|Rx\|_F \le \sup_{n\in\mathbb{N}} |\langle z_n^*, Qx\rangle|, \quad x \in X\,.$$

Since $Q \in \mathcal{A}^{inj}(X,Z)$, there are a Banach space V and an operator $P \in \mathcal{A}(X,V)$ with $\|Qx\|_Z \le \|Px\|_V$, for any $x \in X$. Moreover, given any $\delta > 0$, there is $n_0 \in \mathbb{N}$ such that, for each $n \ge n_0$ and every $x \in X$, we have $|\langle z_n^*, Qx\rangle| \le \delta\|Qx\|_Z$. Hence

$$\begin{aligned}\|Rx\|_F &\le \sup_{n\in\mathbb{N}} |\langle z_n^*, Qx\rangle| \le \max\Big\{\sup_{1\le n\le n_0} |\langle z_n^*, Qx\rangle|, \delta\|Qx\|_Z\Big\} \\ &\le \sup_{1\le n\le n_0} |\langle z_n^*, Qx\rangle| + \delta\|Px\|_V\,.\end{aligned}$$

Combining the above, we get

$$\|Tx\|_Y \le \sup_{1\le n\le n_0} |\langle z_n^*, Qx\rangle| + \|(\delta P)x\|_V + \|Sx\|_G, \quad x \in X\,.$$

If we define $K := (V \oplus G)_1$ and $L := \psi_V \circ (\delta P) + \psi_G \circ S$, where $\psi_V : V \to K$ and $\psi_V : G \to K$ are the natural inclusions of V and G into K, respectively, it is clear that $\alpha(L) \le \delta\alpha(P) + \varepsilon$ and

$$\|Tx\|_Y \le \sup_{1\le n\le n_0} |\langle z_n^*, Qx\rangle| + \|Lx\|_K, \quad x \in X\,.$$

This implies $n_{\mathcal{A}}(T) \le \delta\alpha(P) + \varepsilon$ and so $n_{\mathcal{A}}(T) \le r_{\mathcal{A}}(T)$.

Now assume that $\varepsilon > n_{\mathcal{A}}(T)$. There are finitely many functionals $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and an operator $S \in \mathcal{A}(X,Z)$, with $\alpha(S) \le \varepsilon$, such that

$$\|Tx\|_Y \le \sup_{1\le k\le n} |\langle x_k^*, x\rangle| + \|Sx\|_Z, \quad x \in X\,.$$

The operator $R\colon X \to \ell_\infty^n$ defined by $Rx := (\langle x_k^*, x\rangle)_{1\le k\le n}$ has a finite rank, and so $R \in \mathcal{H}^{\mathcal{A}}(X, \ell_\infty^n)$, and also

$$\|Tx\|_Y \le \|Rx\|_{\ell_\infty^n} + \|Sx\|_Z, \quad x \in X\,.$$

This yields that $r_{\mathcal{A}}(T) \le n_{\mathcal{A}}(T)$. □

Theorem 7. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal.*

(i) *If $T \in \mathcal{A}^{inj}(X,Y)$, then $\chi_{\mathcal{A}}(J_Y \circ T) \le n_{\mathcal{A}}(T)$.*

(ii) *If $T \in \mathcal{A}^{sur}(X,Y)$, then $n_{\mathcal{A}}(T \circ Q_X) \le \chi_{\mathcal{A}}(T)$.*

Proof. (i) Fix $\varepsilon > n_{\mathcal{A}}(T)$. By Remark 3 one has that $n_{\mathcal{A}}(T) = n_{\mathcal{A}}(J_Y \circ T)$, and so we can find functionals $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and an operator $S \in \mathcal{A}(X,Z)$, with $\alpha(S) \le \varepsilon$, such that

$$\|J_Y Tx\|_Y \le \sup_{1\le k\le n} |\langle x_k^*, x\rangle| + \|Sx\|_Z, \quad x \in X\,.$$

Put $\widetilde{Z} = (\ell_\infty^n \oplus Z)_1$ and let $\psi_{\ell_\infty^n} : \ell_\infty^n \to \widetilde{Z}$ and $\psi_Z \colon Z \to \widetilde{Z}$ be the natural inclusions of ℓ_∞^n and Z into $\widetilde{Z}$. We define $P\colon X \to \widetilde{Z}$ as

$$P := \psi_{\ell_\infty^n} \circ S_\infty^n + \psi_Z \circ S\,,$$

where $S_\infty^n \colon X \to \ell_\infty^n$ is given by $S_\infty^n x := (\langle x_k^*, x\rangle)_{1\le k\le n}$. It is clear that

$$\|J_Y Tx\|_Y \le \|Px\|_{\widetilde{Z}},\; x \in X\,.$$

Then the operator $R\colon P(X) \to \ell_\infty(B_{Y^*})$ defined as $Rv = J_Y Tx$, if $v = Px$, has norm less than or equal to 1. Let $\overline{R}$ denote the extension of R to $\overline{P(X)}$. By the metric extension property of $\ell_\infty(B_{Y^*})$, it follows that we can find an operator $\widetilde{R} \in \mathcal{L}(\widetilde{Z}, \ell_\infty(B_{Y^*}))$ with $\|\widetilde{R}\| = \|\overline{R}\| \le 1$, and $\widetilde{R}v = \overline{R}v$ if $v \in \overline{P(X)}$.

Taking into account the definition of P, we get

$$J_Y(T(B_X)) \subset \widetilde{R}(P(B_X)) \subset \widetilde{R}\big(\psi_{\ell_\infty^n}(S_\infty^n(B_X)) + \psi_Z(S(B_X))\big)$$
$$\subset \widetilde{R}(\psi_{\ell_\infty^n}(S_\infty^n(B_X))) + \widetilde{R}(\psi_Z(S(B_X)))\,.$$

Since $\widetilde{R} \circ \psi_{\ell_\infty^n} \circ S_\infty^n$ is a finite rank operator, it holds that $\widetilde{R} \circ \psi_{\ell_\infty^n} \circ S_\infty^n \in \mathcal{K}^{\mathcal{A}}(X, \ell_\infty(B_{Y^*}))$. On the other hand, $\widetilde{R} \circ \psi_Z \circ S \in \mathcal{A}(X, \ell_\infty(B_{Y^*}))$ with $\alpha(\widetilde{R} \circ \psi_Z \circ S) \le \|\widetilde{R} \circ \psi_Z\| \alpha(S) \le \varepsilon$. Proposition 5 allows to conclude that $\chi_{\mathcal{A}}(J_Y \circ T) \le n_{\mathcal{A}}(T)$ and this completes the proof.

(ii) Suppose that $\varepsilon > \chi_{\mathcal{A}}(T)$. Using Remark 3 we have that $\chi_{\mathcal{A}}(T) = \chi_{\mathcal{A}}(T \circ Q_X)$, and then there exist finitely many elements $y_1, \dots, y_n \in Y$, a Banach space Z and an operator $S \in \mathcal{A}(Z, Y)$, with $\alpha(S) \le \varepsilon$, such that

$$T(Q_X(B_{\ell_1(B_X)})) \subset \bigcup_{k=1}^{n} \{y_k + S(B_Z)\}\,. \tag{3}$$

Let $M = ([y_1, \dots, y_n], \|\cdot\|_M)$, where $\|\cdot\|_M := \|\cdot\|_Y / \max\{\|y_1\|_Y, \dots, \|y_n\|_Y\}$. Consider $\widetilde{Z} = (M \oplus Z)_\infty$ and let $\widetilde{S}\colon \widetilde{Z} \to Y$ be the operator defined as $\widetilde{S} := i_M \circ \mathrm{Id}_M \circ \phi_M + S \circ \phi_Z$, where $i_M\colon M \to Y$ is the canonical embedding of M into Y, and $\phi_M\colon \widetilde{Z} \to M$ and $\phi_Z\colon \widetilde{Z} \to Z$ are the natural projections of $\widetilde{Z}$ onto M and Z, respectively. By (3), we get

$$T(Q_X(B_{\ell_1(B_X)})) \subset \widetilde{S}(B_{\widetilde{Z}})\,.$$

Then using the metric lifting property of $\ell_1(B_X)$, namely [29, Lemma 8.5.4], for any $\delta > 0$ it is possible to construct $R \in \mathcal{L}(\ell_1(B_X), \widetilde{Z})$ such that $T \circ Q_X = \widetilde{S} \circ R$ and $\|R\| \le 1 + \delta$. Hence

$$\|TQ_X(\lambda_x)\|_Y = \|\widetilde{S}R(\lambda_x)\|_Y \le \|i_M \mathrm{Id}_M \phi_M R(\lambda_x)\|_Y + \|S\phi_Z R(\lambda_x)\|_Y\,,$$

for every $(\lambda_x)_{x\in B_X} \in \ell_1(B_X)$. Since $i_M \circ \mathrm{Id}_M \circ \phi_M \circ R$ is a finite rank operator, it follows that $i_M \circ \mathrm{Id}_M \circ \phi_M \circ R \in \mathcal{H}^{\mathcal{A}}(\ell_1(B_X), Y)$. On the other hand, $S \circ \phi_Z \circ R \in \mathcal{A}(\ell_1(B_X), Y)$ and $\alpha(S \circ \phi_Z \circ R) \le \alpha(S)\|\phi_Z \circ R\| \le \varepsilon(1 + \delta)$. From Proposition 6 we conclude that $n_{\mathcal{A}}(T \circ Q_X) \le \chi_{\mathcal{A}}(T)$. □

4. Interpolation formulas for $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$

A natural question is to study the behaviour under interpolation of characteristics for operators acting between Banach spaces. In this section we show interpolation estimates for both measures of non-$\mathcal{A}$-compactness of operators, $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$, associated to a Banach operator ideal $\mathcal{A}$. We will use some techniques inspired by the papers [6] and [8]. Before of establishing our results concerning this matter, we recall some basic definitions on interpolation theory.

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, A_0 and A_1 are two Banach spaces which are continuously embedded in some Hausdorff topological vector space. The sum $A_0 + A_1$ and the intersection $A_0 \cap A_1$ of A_0 and A_1 become Banach spaces when endowed with the norms $K(1, \cdot; \bar{A})$ and $J(1, \cdot; \bar{A})$, respectively, where the K- and J-*functionals* are defined, for $t > 0$, by

$$K(t, a) = K(t, a; \bar{A}) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1};\ a = a_0 + a_1,\ a_i \in A_i\},\ a \in A_0 + A_1.$$
$$J(t, a) = J(t, a; \bar{A}) := \max\{\|a\|_{A_0},\ t\|a\|_{A_1}\},\ a \in A_0 \cap A_1\,.$$

A Banach space A is called an *intermediate space* with respect to $\bar{A} = (A_0, A_1)$ if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$, where "$\hookrightarrow$" means continuous inclusion. Given an intermediate space A with respect to a couple $\bar{A} = (A_0, A_1)$, it is possible in some sense to describe the "position" of A within the couple $\bar{A}$ by means of the following functions:

$$\psi_A(t) = \psi_A(t; \bar{A}) := \sup\{K(t, a);\ \|a\|_A = 1\}$$

and

$$\rho_A(t) = \rho_A(t; \bar{A}) := \inf\{J(t, a);\ a \in A_0 \cap A_1, \|a\|_A = 1\}\,.$$

These functions are variants of functions studied, e.g., in [16], [27] and [31]. Clearly, the functions $\psi_A(t)$ and $\rho_A(t)$ are strictly positive and non-decreasing, and the functions $\psi_A(t)/t$ and $\rho_A(t)/t$ are non-increasing.

Examples of intermediate spaces that will be relevant in this paper are the spaces A_i°, that is, the closure of $A_0 \cap A_1$ in A_i endowed with the norm of A_i ($i = 0, 1$). Other important intermediate spaces are the Gagliardo completion $A_i^\sim$ of A_i ($i = 0, 1$) in $A_0 + A_1$. The space $A_i^\sim$ consists of all those $a \in A_0 + A_1$ for which there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A_i such that

$$\sup_{n\in\mathbb{N}} \|a_n\|_{A_i} < \infty \quad \text{and} \quad \lim_{n\to\infty} \|a - a_n\|_{A_0+A_1} = 0\,. \tag{4}$$

The norm in $A_i^\sim$ is given by

$$\|a\|_{A_i^\sim} = \inf\Big\{\sup_{n\in\mathbb{N}} \|a_n\|_{A_i};\ (a_n)_{n\in\mathbb{N}} \text{ satisfies (4)}\Big\}\,.$$

An intermediate space A with respect to $\bar{A} = (A_0, A_1)$ is called an *interpolation space* if for any operator $T\colon \bar{A} \to \bar{A}$ (that is, T is a bounded linear operator from $A_0 + A_1$ into $A_0 + A_1$ whose restriction to each A_i defines a bounded operator from A_i into A_i for $i = 0, 1$), the restriction $T\colon A \to A$ is a bounded operator. In that case, there is a constant $C = C(A, \bar{A})$ such that

$$\|T\|_{A,A} \le C\|T\|_{\bar{A},\bar{A}}, \quad \text{for all } T\colon \bar{A} \to \bar{A}\,, \tag{5}$$

where $\|T\|_{\bar{A},\bar{A}} := \max\{\|T\|_{A_0,A_0}, \|T\|_{A_1,A_1}\}$. We say that an intermediate space A is a *rank-one interpolation space* if inequality (5) is fulfilled for all rank-one operators $T\colon \bar{A} \to \bar{A}$. In some papers (see [16] and [31]), rank-one interpolation spaces are also referred to as *partly interpolation spaces*. An example of an intermediate space with respect to the couple (L_1, L_∞) which is not an interpolation space can be found in [22, p. 122]. Nevertheless, such a space is a rank-one interpolation space because, according to [16] and [31], any space lying between the Lorentz and the Marcinkiewicz spaces with the same fundamental function is a rank-one interpolation space.

We also recall that an intermediate space A with respect to $\bar{A} = (A_0, A_1)$ is said to be *of class* $C_K(\theta; \bar{A})$, where $0 < \theta < 1$, if there is a constant $C > 0$ such that, for all $t > 0$ and $a \in A$,

$$K(t, a) \le Ct^\theta \|a\|_A\,.$$

Analogously, A is called *of class* $C_J(\theta; \bar{A})$, with $0 < \theta < 1$, if there exists a constant $C > 0$ such that, for all $t > 0$ and $a \in A_0 \cap A_1$,

$$\|a\|_A \le Ct^{-\theta} J(t, a)\,.$$

An intermediate space A is said to be *of class* $C(\theta;\bar{A})$ whenever it is of class $C_K(\theta;\bar{A})$ and of class $C_J(\theta;\bar{A})$. The real interpolation space $(A_0,A_1)_{\theta,q}$ and the complex interpolation space $(A_0,A_1)_{[\theta]}$ are important examples of spaces of class $C(\theta;\bar{A})$.

Remark 8. *If A is of class $C_K(\theta;\bar{A})$, then*

$$\lim_{t\to 0}\psi_A(t)=\lim_{t\to\infty}\frac{\psi_A(t)}{t}=0\,.$$

On the other hand, if A is of class $C_J(\theta;\bar{A})$, then we get

$$\lim_{t\to 0}\frac{t}{\rho_A(t)}=\lim_{t\to\infty}\frac{1}{\rho_A(t)}=0\,.$$

We refer to the books [4], [34] for the fundamentals of interpolation theory, and to the papers [6], [8] for further information about the functions ψ_A and ρ_A.

Theorem 9. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal. Let $\bar{X}=(X_0,X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. For any $T\in\mathcal{A}^{sur}(X_0+X_1,Y)$, we have the following*:

(i) *If $\chi_{\mathcal{A}}(T_{X_0,Y})=0$,*

$$\chi_{\mathcal{A}}(T_{X,Y})\le\chi_{\mathcal{A}}(T_{X_1,Y})\cdot\lim_{t\to\infty}\frac{\psi_X(t)}{t}\,.$$

(ii) *If $\chi_{\mathcal{A}}(T_{X_1,Y})=0$,*

$$\chi_{\mathcal{A}}(T_{X,Y})\le\chi_{\mathcal{A}}(T_{X_0,Y})\cdot\lim_{t\to 0}\psi_X(t)\,.$$

(iii) *If $\chi_{\mathcal{A}}(T_{X_i,Y})>0$ for $i=0,1$, then*

$$\chi_{\mathcal{A}}(T_{X,Y})\le 2\chi_{\mathcal{A}}(T_{X_0,Y})\cdot\psi_X\left(\frac{\chi_{\mathcal{A}}(T_{X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0,Y})}\right)\,.$$

Proof. Let $\varepsilon_i>\chi_{\mathcal{A}}(T_{X_i,Y}), i=0,1$. There are finitely many elements $y_1^i,\dots,y_{n_i}^i\in Y$, Banach spaces Z_i and operators $S_i\in\mathcal{A}(Z_i,Y)$, with $\alpha(S_i)\le\varepsilon_i$, such that

$$T(B_{X_i})\subset\bigcup_{k=1}^{n_i}\Big\{y_k^i+S_i(B_{Z_i})\Big\},\quad i=0,1\,.$$

Take $\varepsilon>0$ and $t>0$ arbitrarily. Given any $x\in B_X$, since $K(t,x)\le\psi_X(t)$, we can find $x_i\in X_i$ such that $x=x_0+x_1$ and

$$\|x_0\|_{X_0}+t\|x_1\|_{X_1}\le\psi_X(t)+\varepsilon\,,$$

and so

$$\|x_i\|_{X_i}\le(\psi_X(t)+\varepsilon)t^{-i},\quad i=0,1\,.$$

Then,

$$\begin{aligned}T(B_X)&\subset(\psi_X(t)+\varepsilon)T(B_{X_0})+(\psi_X(t)+\varepsilon)t^{-1}T(B_{X_1})\\&\subset\bigcup_{k=1}^{n_0}\Big\{(\psi_X(t)+\varepsilon)y_k^0+(\psi_X(t)+\varepsilon)S_0(B_{Z_0})\Big\}\\&\quad+\bigcup_{k=1}^{n_1}\Big\{(\psi_X(t)+\varepsilon)t^{-1}y_k^1+(\psi_X(t)+\varepsilon)t^{-1}S_1(B_{Z_1})\Big\}\,.\end{aligned}$$

Hence, we have a covering

$$T(B_X) \subset \bigcup_{k=1}^{n} \Big\{ y_k + S(B_Z) \Big\} ,$$

with $y_1, \dots, y_n \in Y$, $Z = (Z_0 \oplus Z_1)_\infty$ and $S\colon Z \to Y$ the operator defined as

$$S(z_0, z_1) = (\psi_X(t) + \varepsilon) S_0 z_0 + (\psi_X(t) + \varepsilon) t^{-1} S_1 z_1 ,$$

that is, $S = (\psi_X(t)+\varepsilon)(S_0 \circ \phi_0) + (\psi_X(t)+\varepsilon)t^{-1}(S_1 \circ \phi_1)$, where $\phi_i\colon Z \to Z_i$ is the natural projection ($i = 0, 1$). Thus, $S \in \mathcal{A}(Z, Y)$ and $\alpha(S) \le (\psi_X(t)+\varepsilon)\varepsilon_0 + (\psi_X(t)+\varepsilon)t^{-1}\varepsilon_1 = (\psi_X(t)+\varepsilon)(\varepsilon_0 + t^{-1}\varepsilon_1)$. It gives that, for any $\varepsilon > 0$ and $t > 0$,

$$\chi_{\mathcal{A}}(T_{X,Y}) \le (\psi_X(t) + \varepsilon)(\varepsilon_0 + t^{-1}\varepsilon_1) .$$

Therefore,

$$\chi_{\mathcal{A}}(T_{X,Y}) \le \psi_X(t) \Big[\chi_{\mathcal{A}}(T_{X_0,Y}) + t^{-1} \chi_{\mathcal{A}}(T_{X_1,Y}) \Big], \quad t > 0 . \tag{6}$$

When $\chi_{\mathcal{A}}(T_{X_i,Y}) = 0$ for $i = 0$ or $i = 1$, taking into account (6) and that $\psi_X(t)/t$ is non-increasing and $\psi_X(t)$ is non-decreasing, we deduce (i) and (ii), respectively. On the other hand, the case (iii) is obtained by choosing $t := \chi_{\mathcal{A}}(T_{X_1,Y})/\chi_{\mathcal{A}}(T_{X_0,Y})$ in (6). □

Remark 10. *Writing down Theorem 9 for $\mathcal{A} = \mathcal{L}$, we obtain* [6, Theorem 3.1]*, and so* (*see Remark 8*) *we also deduce* [4, Theorem 3.8.1(i)].

Corollary 11. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. Given $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$, it follows that $T\colon X \to Y$ is a surjectively $\mathcal{A}$-compact operator whenever one of the following assertions holds*:

- ⋄ *$T\colon X_0 \to Y$ and $T\colon X_1 \to Y$ are surjectively $\mathcal{A}$-compact operators.*
- ⋄ *$T\colon X_0 \to Y$ is surjectively $\mathcal{A}$-compact and* $\lim_{t\to\infty} \frac{\psi_X(t)}{t} = 0$.
- ⋄ *$T\colon X_1 \to Y$ is surjectively $\mathcal{A}$-compact and* $\lim_{t\to 0} \psi_X(t) = 0$.

As an application, we obtain results on interpolation of p-compact operators. The next corollary is an example.

Corollary 12. *Let $1 \le p < \infty$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that $T \in \Pi_p^d(X_0 + X_1, Y)$. When X is an intermediate space of class $C_K(\theta; \bar{X}), 0 < \theta < 1$, then $T\colon X \to Y$ is a p-compact operator if either $T\colon X_0 \to Y$ or $T\colon X_1 \to Y$ is p-compact.*

The following result complements Theorem 9. Its first part states, in particular, that if $T\colon X_0 \to Y$ is a surjectively $\mathcal{A}$-compact operator, then every rank-one interpolation space X for which $T\colon X \to Y$ is not surjectively $\mathcal{A}$-compact must necessarily verify that $X_1^\circ \hookrightarrow X$. The second part shows that the sufficient conditions obtained in Theorem 9(i) are also necessary under a suitable additional hypothesis on the Banach couple $\bar{X}$, namely when $X_1^\circ = X_1$ holds. The proof of Theorem 13 can be established by means of similar arguments to those used in the proofs of [6, Theorem 3.9 and Corollary 3.11]. For the sake of completeness, we include the details.

Theorem 13. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose that $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$ and X is a rank-one interpolation space with respect to $\bar{X}$. If $T\colon X_0 \to Y$ is a surjectively $\mathcal{A}$-compact operator, then at least one of the following conditions is fulfilled:*

(i) $T\colon X \to Y$ *is surjectively* $\mathcal{A}$*-compact.*
(ii) $X_1^\circ \hookrightarrow X$.

Furthermore, if $X_1^\circ = X_1$, the operator $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact if and only if at least one of the next conditions holds:

(i') $T\colon X_1 \to Y$ *is surjectively* $\mathcal{A}$*-compact.*
(ii') $\lim_{t\to\infty} \frac{\psi_X(t)}{t} = 0$.

Proof. By Theorem 9(i), we know that

$$\chi_{\mathcal{A}}(T_{X,Y}) \le \chi_{\mathcal{A}}(T_{X_1,Y}) \cdot \lim_{t\to\infty} \frac{\psi_X(t)}{t} .$$

Then either $\chi_{\mathcal{A}}(T_{X,Y}) = 0$, equivalently $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact, or $\chi_{\mathcal{A}}(T_{X,Y}) > 0$. In this latter case, we have that $\lim_{t\to\infty} \frac{\psi_X(t)}{t} > 0$ and so $X_1^\circ \hookrightarrow X$ (see [6, Lemma 3.7(ii)]).

On the other hand, note that Theorem 9(i) ensures that either (i') or (i") is sufficient to obtain that the operator $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact.
Now assume that $X_1^\circ = X_1$ and $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact. If (i') is not true, that is, $T\colon X_1 \to Y$ is not surjectively $\mathcal{A}$-compact, it necessarily implies that $\lim_{t\to\infty} \frac{\psi_X(t)}{t} = 0$. In other case $X_1^\circ \hookrightarrow X$ holds (see [6, Lemma 3.7(ii)]) and, since $X_1^\circ = X_1$, the operator $T\colon X_1 \to Y$ would be surjectively $\mathcal{A}$-compact, which is a contradiction. Analogously, if (ii') does not hold, then $X_1^\circ \hookrightarrow X$ and so $T\colon X_1^\circ \to Y$ is surjectively $\mathcal{A}$-compact. Taking into account that $X_1^\circ = X_1$, we obtain that (i') is fulfilled. □

Now we focus on the injective $\mathcal{A}$-compactness and the measure $n_{\mathcal{A}}$.

Theorem 14. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$. For any $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$, we have the following*:

(i) *If* $n_{\mathcal{A}}(T_{X,Y_0}) = 0$,

$$n_{\mathcal{A}}(T_{X,Y}) \le n_{\mathcal{A}}(T_{X,Y_1}) \cdot \lim_{t\to 0} \frac{t}{\rho_Y(t)} .$$

(ii) *If* $n_{\mathcal{A}}(T_{X,Y_1}) = 0$,

$$n_{\mathcal{A}}(T_{X,Y}) \le n_{\mathcal{A}}(T_{X,Y_0}) \cdot \lim_{t\to\infty} \frac{1}{\rho_Y(t)} .$$

(iii) *If* $n_{\mathcal{A}}(T_{X,Y_i}) > 0$ *for* $i = 0, 1$, *then*

$$n_{\mathcal{A}}(T_{X,Y}) \le \frac{2 n_{\mathcal{A}}(T_{X,Y_0})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0}) / n_{\mathcal{A}}(T_{X,Y_1})\Big)} .$$

Proof. Let $\varepsilon_i > n_{\mathcal{A}}(T_{X,Y_i}), i = 0, 1$. Then there exist finitely many functionals $f_1^i, \ldots, f_{n_i}^i \in X^*$, Banach spaces Z_i and operators $S_i \in \mathcal{A}(X, Z_i)$, with $\alpha(S_i) \le \varepsilon_i$, such that for both $i = 0$ and $i = 1$, we have

$$\|Tx\|_{Y_i} \le \sup_{1\le k\le n_i} |\langle f_k^i, x\rangle| + \|S_i x\|_{Z_i}, \quad x \in X\,.$$

Let $t > 0$. Put $Z = (Z_0 \oplus Z_1)_1$ and let $S\colon X \to Z$ be the operator given by

$$Sx = \frac{1}{\rho_Y(t)}(S_0 x, tS_1 x)\,.$$

that is, $S = \frac{1}{\rho_Y(t)}(\varphi_0 \circ S_0) + \frac{t}{\rho_Y(t)}(\varphi_1 \circ S_1)$, where $\varphi_i\colon Z_i \to Z$ is the natural inclusion ($i = 0, 1$). Hence, $S \in \mathcal{A}(X, Z)$ and

$$\alpha(S) \le \frac{\varepsilon_0}{\rho_Y(t)} + \frac{\varepsilon_1 t}{\rho_Y(t)} = \frac{1}{\rho_Y(t)}(\varepsilon_0 + t\varepsilon_1)\,.$$

Clearly, for each $y \in Y_0 \cap Y_1$, we have $\|y\|_Y \le J(t, y)/\rho_Y(t)$. Hence, for all $x \in X$,

$$\begin{aligned}
\|Tx\|_Y &\le \frac{J(t,Tx)}{\rho_Y(t)} = \frac{1}{\rho_Y(t)} \max\Big\{\|Tx\|_{Y_0}, t\|Tx\|_{Y_1}\Big\} \\
&\le \frac{1}{\rho_Y(t)} \max_{i=0,1}\Big\{t^i \sup_{1\le k\le n_i} |\langle f_k^i, x\rangle| + t^i\|S_i x\|_{Z_i}\Big\} \\
&\le \sup_{1\le k\le n_i, i=0,1} \Big|\langle \frac{t^i}{\rho_Y(t)} f_k^i, x\rangle\Big| + \frac{1}{\rho_Y(t)}\|S_0 x\|_{Z_0} + \frac{t}{\rho_Y(t)}\|S_1 x\|_{Z_1} \\
&= \sup_{1\le k\le n_i, i=0,1} \Big|\langle \frac{t^i}{\rho_Y(t)} f_k^i, x\rangle\Big| + \|Sx\|_Z.
\end{aligned}$$

This implies that, for any $t > 0$,

$$n_{\mathcal{A}}(T_{X,Y}) \le \frac{1}{\rho_Y(t)}(\varepsilon_0 + t\varepsilon_1)\,,$$

and so

$$n_{\mathcal{A}}(T_{X,Y}) \le \frac{1}{\rho_Y(t)}\Big[n_{\mathcal{A}}(T_{X,Y_0}) + t n_{\mathcal{A}}(T_{X,Y_1})\Big], \quad t > 0\,. \tag{7}$$

If $n_{\mathcal{A}}(T_{X,Y_i}) = 0$ for $i = 0$ or $i = 1$, replacing this information in (7) and keeping in mind that $t/\rho_Y(t)$ is non-decreasing and $1/\rho_Y(t)$ is non-increasing, the statements (i) and (ii), respectively, are proved. When $n_{\mathcal{A}}(T_{X,Y_i}) > 0$ for $i = 0, 1$, we conclude the proof by substituting in (7) the value $t := n_{\mathcal{A}}(T_{X,Y_0})/n_{\mathcal{A}}(T_{X,Y_1})$. □

Remark 15. *As a consequence of Theorem 14, for the particular case $\mathcal{A} = \mathcal{L}$, we have a similar estimate to that given in* [6, Theorem 3.2]*, and thus (see Remark 8) we also recover* [4, Theorem 3.8.1(ii)]*.*

Corollary 16. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$. Given $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$, it follows that $T\colon X \to Y$ is an injectively $\mathcal{A}$-compact operator whenever one of the following assertions holds:*

- ⋄ *$T\colon X \to Y_0$ and $T\colon X \to Y_1$ are injectively $\mathcal{A}$-compact operators.*
- ⋄ *$T\colon X \to Y_0$ is injectively $\mathcal{A}$-compact and $\lim_{t\to 0} \frac{t}{\rho_Y(t)} = 0$.*

◇ $T\colon X\to Y_1$ *is injectively* $\mathcal{A}$*-compact and* $\lim_{t\to\infty}\frac{1}{\rho_Y(t)}=0$.

As a consequence, results on interpolation of quasi p-nuclear operators can be obtained. An example of this is the next corollary.

Corollary 17. *Let* $1\le p<\infty$*. Let* X *be a Banach space and let* $\bar{Y}=(Y_0,Y_1)$ *be a Banach couple. Assume that* $T\in\Pi_p(X,Y_0\cap Y_1)$*. When* Y *is an intermediate space of class* $C_J(\theta;\bar{Y}), 0<\theta<1$*, then* $T\colon X\to Y$ *is a quasi* p*-nuclear operator if either* $T\colon X\to Y_0$ *or* $T\colon X\to Y_1$ *is quasi* p*-nuclear.*

We also establish an analogous result to Theorem 13 in the "dual" situation (see now [6, Theorem 3.10 and Corollary 3.12] for the case of compact operators). Namely, we show that if $T\colon X\to Y_0$ is an injectively $\mathcal{A}$-compact operator, then every rank-one interpolation space Y for which $T\colon X\to Y$ is not injectively $\mathcal{A}$-compact must necessarily satisfy that $Y\hookrightarrow Y_1^{\sim}$. Furthermore, it is proved that the sufficient conditions obtained in Theorem 14(i) are also necessary under the additional hypothesis $Y_1^{\sim}=Y_1$.

Theorem 18. *Let* $[\mathcal{A},\alpha]$ *be a Banach operator ideal. Let* X *be a Banach space and let* $\bar{Y}=(Y_0,Y_1)$ *be a Banach couple. Suppose that* $T\in\mathcal{A}^{inj}(X,Y_0\cap Y_1)$ *and* Y *is a rank-one interpolation space with respect to* $\bar{Y}$*. If* $T\colon X\to Y_0$ *is an injectively* $\mathcal{A}$*-compact operator, then at least one of the following conditions is fulfilled*:

(i) $T\colon X\to Y$ *is injectively* $\mathcal{A}$*-compact.*
(ii) $Y\hookrightarrow Y_1^{\sim}$*.*

Moreover, if $Y_1^{\sim}=Y_1$*, the operator* $T\colon X\to Y$ *is injectively* $\mathcal{A}$*-compact if and only if at least one of the next conditions holds*:

(i') $T\colon X\to Y_1$ *is injectively* $\mathcal{A}$*-compact.*
(ii') $\lim_{t\to 0}\frac{t}{\rho_Y(t)}=0$.

Proof. According to Theorem 14(i), whenever $T\colon X\to Y_0$ is injectively $\mathcal{A}$-compact, we have that

$$n_{\mathcal{A}}(T_{X,Y})\le n_{\mathcal{A}}(T_{X,Y_1})\cdot\lim_{t\to 0}\frac{t}{\rho_Y(t)}\,.$$

Then either $n_{\mathcal{A}}(T_{X,Y})=0$, that is, $T\colon X\to Y$ is injectively $\mathcal{A}$-compact, or $n_{\mathcal{A}}(T_{X,Y})>0$. In this case, it must hold that $\lim_{t\to 0}\frac{t}{\rho_Y(t)}>0$ and in consequence $Y\hookrightarrow Y_1^{\sim}$ (see [6, Lemma 3.8(ii)]).

In addition, Theorem 14(i) guarantees that (i'), or (i"), is a sufficient condition to obtain that the operator $T\colon X\to Y$ is injectively $\mathcal{A}$-compact.
Now assume that $Y_1^{\sim}=Y_1$ and $T\colon X\to Y$ is injectively $\mathcal{A}$-compact. If (i') is not fulfilled, equivalently $T\colon X\to Y_1$ is not injectively $\mathcal{A}$-compact, it necessarily follows that $\lim_{t\to 0}\frac{t}{\rho_Y(t)}=0$. If not, $Y\hookrightarrow Y_1^{\sim}$ holds (see [6, Lemma 3.8(ii)]) and, since $Y_1^{\sim}=Y_1$, the operator $T\colon X\to Y_1$ would be injectively $\mathcal{A}$-compact, which is a contradiction. On the other hand, if (ii') does not hold, then $Y\hookrightarrow Y_1^{\sim}$ and so $T\colon X\to Y_1^{\sim}$ is injectively $\mathcal{A}$-compact. Due to $Y_1^{\sim}=Y_1$, (i') is true. □

Next we establish interpolation formulas for the measure of $T\colon X\to Y$ in terms of the measures of the restrictions $T\colon X_0\cap X_1\to Y$ and $T\colon X_0+X_1\to Y$

(respectively $T\colon X \to Y_0 \cap Y_1$ and $T\colon X \to Y_0 + Y_1$), for $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$ (respectively $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$). This is interesting, since sometimes the known information about the operator only refers to such extreme restrictions.

Theorem 19. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. For every $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$, the following statements are true*:

(i) *When $\chi_{\mathcal{A}}(T_{X_0 \cap X_1, Y}) = 0$,*

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq \chi_{\mathcal{A}}(T_{X_0+X_1,Y}) \cdot \left(\lim_{t \to 0} \psi_X(t) + \lim_{t \to \infty} \frac{\psi_X(t)}{t} \right) .$$

(ii) *When $\chi_{\mathcal{A}}(T_{X_0 \cap X_1, Y}) > 0$,*

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq 2 \left(\frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0+X_1,Y})/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})\Big)}{1/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})} + \right.$$
$$\left. + \frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\Big)}{1/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})} \right) .$$

Proof. Given $\sigma > \chi_{\mathcal{A}}(T_{X_0 \cap X_1, Y})$, there are finitely many elements $u_1, \ldots, u_m \in Y$, a Banach space U and an operator $R \in \mathcal{A}(U, Y)$, with $\alpha(R) \leq \sigma$, so that

$$T(B_{X_0 \cap X_1}) \subset \bigcup_{k=1}^{m} \Big\{ u_k + R(B_U) \Big\} . \tag{8}$$

On the other hand, if $\delta > \chi_{\mathcal{A}}(T_{X_0+X_1,Y})$ we can find $v_1, \ldots, v_n \in Y$, a Banach space V and an operator $S \in \mathcal{A}(V, Y)$, with $\alpha(S) \leq \delta$, such that

$$T(B_{X_0 + X_1}) \subset \bigcup_{k=1}^{n} \Big\{ v_k + S(B_V) \Big\} . \tag{9}$$

Let $\varepsilon > 0$ and $0 < t \leq 1$ arbitrarily. Define $W := (U \oplus V)_\infty$ and $P \in \mathcal{L}(W, Y)$ as

$$P(u,v) := (1+\varepsilon) \left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t} \right) Ru + (1+\varepsilon) \left(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}} \right) Sv ,$$

i.e., $P = (1+\varepsilon) \left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t} \right) (R \circ \phi_U) + (1+\varepsilon) \left(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}} \right) (S \circ \phi_V)$, where $\phi_U\colon W \to U$ and $\phi_V\colon W \to V$ are the natural projections. Whence, $P \in \mathcal{A}(W, Y)$ and

$$\alpha(P) \leq (1+\varepsilon) \big(\psi_X(t^{-1}) + \psi_X(t)/t \big)\, \sigma + (1+\varepsilon) \big(\psi_X(t) + \psi_X(t^{-1})/t^{-1} \big)\, \delta .$$

Note that $K(s,x) \leq \psi_X(s) \|x\|_X$, for every $x \in X$ and any $s > 0$. Then, given $x \in B_X$, there are decompositions of x as $x = x_0 + x_1 = x_0' + x_1'$, with $x_i, x_i' \in X_i\,(i = 0,1)$, and

$$\|x_0\|_{X_0} + t\|x_1\|_{X_1} \leq K(t,x) + \varepsilon \psi_X(t) \leq (1+\varepsilon) \psi_X(t) ,$$

$$\|x_0'\|_{X_0} + t^{-1}\|x_1'\|_{X_1} \leq K(t^{-1},x) + \varepsilon \psi_X(t^{-1}) \leq (1+\varepsilon) \psi_X(t^{-1}) .$$

Thus,

$$\|x_i\|_{X_i} \le (1+\varepsilon)\psi_X(t)/t^i, \quad \|x_i'\|_{X_i} \le (1+\varepsilon)\psi_X(t^{-1})/t^{-i}, \quad i=0,1\,. \tag{10}$$

Now let $\widehat{x} := x_0' - x_0 = x_1 - x_1' \in X_0 \cap X_1$. It follows from (10) and $0 < t \le 1$ that

$$\begin{aligned} \|\widehat{x}\|_{X_0\cap X_1} &\le \max\{\|x_0\|_{X_0} + \|x_0'\|_{X_0}, \|x_1\|_{X_1} + \|x_1'\|_{X_1}\} \\ &\le (1+\varepsilon)\max\Big\{\psi_X(t)+\psi_X(t^{-1}), \psi_X(t)/t + \psi_X(t^{-1})/t^{-1}\Big\} \\ &\le (1+\varepsilon)\max\Big\{\psi_X(t)/t+\psi_X(t^{-1}), \psi_X(t)/t + \psi_X(t^{-1})\Big\} \\ &= (1+\varepsilon)\big(\psi_X(t^{-1}) + \psi_X(t)/t\big) \end{aligned} \tag{11}$$

and

$$\|x-\widehat{x}\|_{X_0+X_1} \le \|x_0\|_{X_0} + \|x_1'\|_{X_1} \le (1+\varepsilon)\Big(\psi_X(t) + \psi_X(t^{-1})/t^{-1}\Big)\,. \tag{12}$$

Using (11) and (12), we deduce that

$$B_X \subset (1+\varepsilon)\left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t}\right) B_{X_0\cap X_1} + (1+\varepsilon)\Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big) B_{X_0+X_1}\,.$$

Keeping in mind (8) and (9), it follows that

$$\begin{aligned} T(B_X) &\subset (1+\varepsilon)\left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t}\right) T(B_{X_0\cap X_1}) \\ &\quad + (1+\varepsilon)\Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big) T(B_{X_0+X_1}) \\ &\subset \bigcup_{k=1}^{m}\Big\{(1+\varepsilon)\left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t}\right) u_k \\ &\quad + (1+\varepsilon)\left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t}\right) R(B_U)\Big\} \\ &\quad + \bigcup_{k=1}^{n}\Big\{(1+\varepsilon)\Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big) v_k \\ &\quad + (1+\varepsilon)\Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big) S(B_V)\Big\}. \end{aligned}$$

Therefore, there exist finitely many $w_1, \dots, w_l \in Y$ such that

$$T(B_X) \subset \bigcup_{k=1}^{l}\Big\{w_k + P(B_W)\Big\}\,,$$

with $P \in \mathcal{A}(W,Y)$ and

$$\alpha(P) \le (1+\varepsilon)\big(\psi_X(t^{-1}) + \psi_X(t)/t\big)\,\sigma + (1+\varepsilon)\big(\psi_X(t) + \psi_X(t^{-1})/t^{-1}\big)\,\delta\,.$$

It implies that, for each $0 < t \le 1$,

$$\begin{aligned} \chi_{\mathcal{A}}(T_{X,Y}) &\le \left(\psi_X(t^{-1}) + \frac{\psi_X(t)}{t}\right)\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y}) \\ &\quad + \Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big)\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\,. \end{aligned} \tag{13}$$

When $\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y}) = 0$, it follows that

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq \Big(\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}\Big)\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\,,$$

and using that $\psi_X(t) + \frac{\psi_X(t^{-1})}{t^{-1}}$ is non-decreasing, we have that

$$\begin{aligned}\chi_{\mathcal{A}}(T_{X,Y}) &\leq \chi_{\mathcal{A}}(T_{X_0+X_1,Y}) \cdot \left(\lim_{t\to 0}\psi_X(t) + \lim_{t\to 0}\frac{\psi_X(t^{-1})}{t^{-1}}\right)\\ &= \chi_{\mathcal{A}}(T_{X_0+X_1,Y}) \cdot \left(\lim_{t\to 0}\psi_X(t) + \lim_{t\to\infty}\frac{\psi_X(t)}{t}\right).\end{aligned}$$

Now assume that $\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y}) > 0$. Since $\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y}) \leq \chi_{\mathcal{A}}(T_{X_0+X_1,Y})$,

$$t := \frac{\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0+X_1,Y})} \leq 1\,.$$

Substituting this value in (13) yields that

$$\begin{aligned}\chi_{\mathcal{A}}(T_{X,Y}) \leq &\left(\psi_X\left(\frac{\chi_{\mathcal{A}}(T_{X_0+X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})}\right) + \frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\Big)}{\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})}\right)\cdot\\ &\cdot\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})\\ &+\left(\psi_X\left(\frac{\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0+X_1,Y})}\right) + \frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0+X_1,Y})/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})\Big)}{\chi_{\mathcal{A}}(T_{X_0+X_1,Y})/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})}\right)\cdot\\ &\cdot\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\\ &= 2\left(\frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0+X_1,Y})/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})\Big)}{1/\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})}+\right.\\ &\qquad\left.+\frac{\psi_X\Big(\chi_{\mathcal{A}}(T_{X_0\cap X_1,Y})/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})\Big)}{1/\chi_{\mathcal{A}}(T_{X_0+X_1,Y})}\right).\end{aligned}$$

□

Corollary 20. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$ and $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$. If $T\colon X_0 \cap X_1 \to Y$ is surjectively $\mathcal{A}$-compact and*

$$\lim_{t\to 0}\psi_X(t) = \lim_{t\to\infty}\frac{\psi_X(t)}{t} = 0\,,$$

then $T\colon X \to Y$ is a surjectively $\mathcal{A}$-compact operator.

Corollary 21. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose X is of class $C_K(\theta;\bar{X}), 0 < \theta < 1$. For $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$, it follows that $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact if and only if $T\colon X_0 \cap X_1 \to Y$ is surjectively $\mathcal{A}$-compact.*

Observe that the above corollaries, applied to the Banach operator ideal $[\mathcal{A},\alpha]$ given by the dual ideal of p-summing operators, imply interpolation results on p-compact operators. This fact motivates the study of the dual case and its connections with the measure $n_{\mathcal{A}}$.

Theorem 22. *Let $[\mathcal{A},\alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y}=(Y_0,Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$. For every $T\in\mathcal{A}^{inj}(X,Y_0\cap Y_1)$, the following statements are true*:

(i) *When $n_{\mathcal{A}}(T_{X,Y_0+Y_1})=0$,*

$$n_{\mathcal{A}}(T_{X,Y})\le 2n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\cdot\left(\lim_{t\to 0}\frac{t}{\rho_Y(t)}+\lim_{t\to\infty}\frac{1}{\rho_Y(t)}\right).$$

(ii) *When $n_{\mathcal{A}}(T_{X,Y_0+Y_1})>0$,*

$$n_{\mathcal{A}}(T_{X,Y})\le 3\left(\frac{n_{\mathcal{A}}(T_{X,Y_0+Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0+Y_1})/n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\Big)}\right.$$
$$\left.+\frac{n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})/n_{\mathcal{A}}(T_{X,Y_0+Y_1})\Big)}\right).$$

Proof. Suppose $\sigma>n_{\mathcal{A}}(T_{X,Y_0+Y_1})$. Then, there exist finitely many functionals $f_1^*,\dots,f_m^*\in X^*$, a Banach space H and an operator $R\in\mathcal{A}(X,H)$, with $\alpha(R)\le\sigma$, so that

$$\|Tx\|_{Y_0+Y_1}\le\sup_{1\le k\le m}|\langle f_k^*,x\rangle|+\|Rx\|_H,\quad x\in X. \tag{14}$$

Moreover, if $\delta>n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})$, then there exist functionals $g_1^*,\dots,g_n^*\in X^*$, a Banach space G and an operator $S\in\mathcal{A}(X,G)$, with $\alpha(S)\le\delta$, such that

$$\|Tx\|_{Y_0\cap Y_1}\le\sup_{1\le k\le n}|\langle g_k^*,x\rangle|+\|Sx\|_G,\quad x\in X. \tag{15}$$

Take $\varepsilon>0$ and $t\ge 1$ arbitrarily. Let $Z:=(H\oplus G)_1$ and define $P\in\mathcal{L}(X,Z)$ by

$$Px:=\left((1+\varepsilon)\Big(\frac{1}{\rho_Y(t^{-1})}+\frac{t}{\rho_Y(t)}\Big)Rx,(2+\varepsilon)\Big(\frac{t^{-1}}{\rho_Y(t^{-1})}+\frac{1}{\rho_Y(t)}\Big)Sx\right).$$

Thus, $P=(1+\varepsilon)\Big(\frac{1}{\rho_Y(t^{-1})}+\frac{t}{\rho_Y(t)}\Big)(\varphi_H\circ R)+(2+\varepsilon)\Big(\frac{t^{-1}}{\rho_Y(t^{-1})}+\frac{1}{\rho_Y(t)}\Big)(\varphi_G\circ S)$, where $\varphi_H\colon H\to Z$ and $\varphi_G\colon G\to Z$ are the natural inclusions. Whence $P\in\mathcal{A}(X,Z)$ and

$$\alpha(P)\le(1+\varepsilon)\left(\frac{1}{\rho_Y(t^{-1})}+\frac{t}{\rho_Y(t)}\right)\sigma+(2+\varepsilon)\left(\frac{t^{-1}}{\rho_Y(t^{-1})}+\frac{1}{\rho_Y(t)}\right)\delta.$$

On the other hand, for each $x\in X$, there is a decomposition of Tx as $Tx=y_0+y_1$, with $y_i\in Y_i$ and

$$\|y_i\|_{Y_i}\le\|y_0\|_{Y_0}+\|y_1\|_{Y_1}\le(1+\varepsilon)\|Tx\|_{Y_0+Y_1},\quad i=0,1. \tag{16}$$

By (16) and (14),

$$\|y_i\|_{Y_i}\le(1+\varepsilon)\sup_{1\le k\le m}|\langle f_k^*,x\rangle|+(1+\varepsilon)\|Rx\|_H,\quad i=0,1. \tag{17}$$

Since $y_i \in Y_0 \cap Y_1$ ($i = 0, 1$), it follows from (16) that

$$\begin{aligned}\|y_i\|_{Y_{1-i}} &= \|Tx - y_{1-i}\|_{Y_{1-i}} \leq \|Tx\|_{Y_{1-i}} + \|y_{1-i}\|_{Y_{1-i}} \\ &\leq \|Tx\|_{Y_0\cap Y_1} + (1+\varepsilon)\|Tx\|_{Y_0+Y_1} \leq (2+\varepsilon)\|Tx\|_{Y_0\cap Y_1}\,,\end{aligned}$$

for $i = 0, 1$. Using (15),

$$\|y_i\|_{Y_{1-i}} \leq (2+\varepsilon) \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle| + (2+\varepsilon)\|Sx\|_G, \quad i = 0, 1\,. \tag{18}$$

Observe that, for every $y \in Y_0 \cap Y_1$ and for all $s > 0$, $\|y\|_Y \leq J(s, y)/\rho_Y(s)$. Thus, using (17) and (18), it follows that

$$\begin{aligned}
\|Tx\|_Y &\leq \|y_0\|_Y + \|y_1\|_Y \leq \frac{J(t^{-1}, y_0)}{\rho_Y(t^{-1})} + \frac{J(t, y_1)}{\rho_Y(t)} \\
&\leq \frac{1}{\rho_Y(t^{-1})} \max\left\{\|y_0\|_{Y_0}, t^{-1}\|y_0\|_{Y_1}\right\} + \frac{1}{\rho_Y(t)} \max\left\{\|y_1\|_{Y_0}, t\|y_1\|_{Y_1}\right\} \\
&\leq \frac{1}{\rho_Y(t^{-1})} \max\Big\{(1+\varepsilon) \sup_{1\leq k\leq m} |\langle f_k^*, x\rangle| + (1+\varepsilon)\|Rx\|_H, \\
&\qquad\qquad t^{-1}\Big[(2+\varepsilon) \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle| + (2+\varepsilon)\|Sx\|_G\Big]\Big\} \\
&\quad + \frac{1}{\rho_Y(t)} \max\Big\{(2+\varepsilon) \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle| + (2+\varepsilon)\|Sx\|_G, \\
&\qquad\qquad t\Big[(1+\varepsilon) \sup_{1\leq k\leq m} |\langle f_k^*, x\rangle| + (1+\varepsilon)\|Rx\|_H\Big]\Big\} \\
&\leq \max\Big\{(1+\varepsilon)\frac{1}{\rho_Y(t^{-1})} \sup_{1\leq k\leq m} |\langle f_k^*, x\rangle|, (2+\varepsilon)\frac{t^{-1}}{\rho_Y(t^{-1})} \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle|\Big\} \\
&\qquad + (1+\varepsilon)\frac{1}{\rho_Y(t^{-1})}\|Rx\|_H + (2+\varepsilon)\frac{t^{-1}}{\rho_Y(t^{-1})}\|Sx\|_G \\
&\quad + \max\Big\{(1+\varepsilon)\frac{t}{\rho_Y(t)} \sup_{1\leq k\leq m} |\langle f_k^*, x\rangle|, (2+\varepsilon)\frac{1}{\rho_Y(t)} \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle|\Big\} \\
&\qquad + (1+\varepsilon)\frac{t}{\rho_Y(t)}\|Rx\|_H + (2+\varepsilon)\frac{1}{\rho_Y(t)}\|Sx\|_G \\
&\leq 2\max\Big\{(1+\varepsilon)\Big(\frac{1}{\rho_Y(t^{-1})} + \frac{t}{\rho_Y(t)}\Big) \sup_{1\leq k\leq m} |\langle f_k^*, x\rangle|, \\
&\qquad\qquad (2+\varepsilon)\Big(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\Big) \sup_{1\leq k\leq n} |\langle g_k^*, x\rangle|\Big\} \\
&\quad + (1+\varepsilon)\Big(\frac{1}{\rho_Y(t^{-1})} + \frac{t}{\rho_Y(t)}\Big)\|Rx\|_H + (2+\varepsilon)\Big(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\Big)\|Sx\|_G,
\end{aligned}$$

and so

$$\|Tx\|_Y \leq \sup_{1\leq k\leq l} |\langle h_k^*, x\rangle| + \|Px\|_Z\,,$$

for certain functionals $h_1^*, \ldots, h_l^* \in X^*$ and $P \in \mathcal{A}(X, Z)$, with

$$\alpha(P) \leq (1+\varepsilon)\left(\frac{1}{\rho_Y(t^{-1})} + \frac{t}{\rho_Y(t)}\right)\sigma + (2+\varepsilon)\left(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\right)\delta\,.$$

Hence,

$$n_{\mathcal{A}}(T_{X,Y}) \leq (1+\varepsilon)\Big(\frac{1}{\rho_Y(t^{-1})} + \frac{t}{\rho_Y(t)}\Big)\sigma + (2+\varepsilon)\Big(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\Big)\delta\,.$$

Thus, for any $t \geq 1$,

$$\begin{aligned} n_{\mathcal{A}}(T_{X,Y}) \leq \Big(\frac{1}{\rho_Y(t^{-1})} + \frac{t}{\rho_Y(t)}\Big) n_{\mathcal{A}}(T_{X,Y_0+Y_1}) \\ + 2\Big(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\Big) n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\,. \end{aligned} \tag{19}$$

If $n_{\mathcal{A}}(T_{X,Y_0+Y_1}) = 0$, we obtain that

$$n_{\mathcal{A}}(T_{X,Y}) \leq 2\Big(\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}\Big) n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\,,$$

and keeping in mind that $\frac{t^{-1}}{\rho_Y(t^{-1})} + \frac{1}{\rho_Y(t)}$ is non-increasing, we conclude that

$$\begin{aligned} n_{\mathcal{A}}(T_{X,Y}) &\leq 2n_{\mathcal{A}}(T_{X,Y_0\cap Y_1}) \cdot \Big(\lim_{t\to\infty} \frac{t^{-1}}{\rho_Y(t^{-1})} + \lim_{t\to\infty} \frac{1}{\rho_Y(t)}\Big) \\ &= 2n_{\mathcal{A}}(T_{X,Y_0\cap Y_1}) \cdot \Big(\lim_{t\to 0} \frac{t}{\rho_Y(t)} + \lim_{t\to\infty} \frac{1}{\rho_Y(t)}\Big). \end{aligned}$$

On the other hand, if $n_{\mathcal{A}}(T_{X,Y_0+Y_1}) > 0$, then $n_{\mathcal{A}}(T_{X,Y_0\cap Y_1}) > 0$ too, because $n_{\mathcal{A}}(T_{X,Y_0+Y_1}) \leq n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})$. Then, taking

$$t := \frac{n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})}{n_{\mathcal{A}}(T_{X,Y_0+Y_1})} \geq 1$$

in (19), we deduce that

$$\begin{aligned} &n_{\mathcal{A}}(T_{X,Y}) \leq \\ &\leq \frac{n_{\mathcal{A}}(T_{X,Y_0+Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0+Y_1})/n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\Big)} + \frac{n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})/n_{\mathcal{A}}(T_{X,Y_0+Y_1})\Big)} \\ &+ 2\left(\frac{n_{\mathcal{A}}(T_{X,Y_0+Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0+Y_1})/n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\Big)} + \frac{n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})/n_{\mathcal{A}}(T_{X,Y_0+Y_1})\Big)}\right) \\ &= 3\left(\frac{n_{\mathcal{A}}(T_{X,Y_0+Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0+Y_1})/n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})\Big)} + \frac{n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0\cap Y_1})/n_{\mathcal{A}}(T_{X,Y_0+Y_1})\Big)}\right). \end{aligned}$$

□

We specify two particular cases stated in the following corollaries.

Corollary 23. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$ and $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$. If $T\colon X \to Y_0 + Y_1$ is injectively $\mathcal{A}$-compact and*

$$\lim_{t\to 0} \frac{t}{\rho_Y(t)} = \lim_{t\to\infty} \frac{1}{\rho_Y(t)} = 0\,,$$

then $T\colon X \to Y$ is an injectively $\mathcal{A}$-compact operator.

Corollary 24. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Suppose Y is of class $C_J(\theta; \bar{Y}), 0 < \theta < 1$. For $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$, it follows that $T\colon X \to Y$ is injectively $\mathcal{A}$-compact if and only if $T\colon X \to Y_0 + Y_1$ is injectively $\mathcal{A}$-compact.*

Let us point out that if $[\mathcal{A}, \alpha]$ is the Banach operator ideal of p-summing operators, then the above corollaries yield interpolation results on quasi p-nuclear operators.

We now observe that applying Corollaries 20 and 23 and [6, Lemmata 3.7 and 3.8], respectively, it is possible to establish results in the same line that [8, Corollaries 3.11 and 3.10].

Corollary 25. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose that $T \in \mathcal{A}^{sur}(X_0 + X_1, Y)$ and X is a rank-one interpolation space with respect to $\bar{X}$. If $T\colon X_0 \cap X_1 \to Y$ is a surjectively $\mathcal{A}$-compact operator, then at least one of the following conditions is fulfilled*:

(i) $T\colon X \to Y$ *is surjectively $\mathcal{A}$-compact.*
(ii) $X_0^\circ \hookrightarrow X$.
(iii) $X_1^\circ \hookrightarrow X$.

Corollary 26. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Suppose that $T \in \mathcal{A}^{inj}(X, Y_0 \cap Y_1)$ and Y is a rank-one interpolation space with respect to $\bar{Y}$. If $T\colon X \to Y_0 + Y_1$ is an injectively $\mathcal{A}$-compact operator, then at least one of the following conditions is fulfilled*:

(i) $T\colon X \to Y$ *is injectively $\mathcal{A}$-compact.*
(ii) $Y \hookrightarrow Y_0^\sim$.
(iii) $Y \hookrightarrow Y_1^\sim$.

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