Closed surjective ideals of multilinear operators and interpolation

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Abstract. In this paper we introduce a function for multilinear operators that can be considered as an extension of the so-called outer measure associated to a linear operator ideal. We prove that it allows to characterize the operators that belong to a closed surjective ideal of multilinear operators as those having measure equal to zero. We also obtain some interpolation formulas for this new measure. As a consequence we deduce interpolation results for arbitrary closed surjective ideals of multilinear operators which recover, in particular, different results previously established in the literature.

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1. Introduction

The study of compact operators and weakly compact operators is an important topic in the theory of operators and Banach spaces. A helpful tool to investigate each one of these ideals of linear operators is to consider a functional that shows the deviation of a bounded linear operator $T : E \to F$ from the corresponding ideal. Examples of it are the (ball) measure of non-compactness (in case of compact operators), defined as (see [25], [20] and [11])

$$\gamma(T) = \inf\Big\{\varepsilon > 0 : T(B_E) \subseteq \bigcup_{k=1}^{m} \{y_k + \varepsilon B_F\}, y_k \in F, m \in \mathbb{N}\Big\},$$

and the measure of weak non-compactness (in case of weakly compact operators), given by (see [16])

$$w(T) = \inf\left\{\varepsilon > 0 : T(B_E) \subseteq \varepsilon B_F + W,\, W \subseteq F \text{ weakly compact}\right\}.$$

Compact operators and weakly compact operators are relevant examples of closed surjective ideals of linear operators. Other important examples of linear operator ideals which are also closed and surjective are strictly cosingular operators, Rosenthal operators, Banach-Saks operators, operators of separable range, limited operators, Grothendieck operators and Asplund operators (also called decomposing operators).

Given any linear operator ideal $\mathcal{I}$, the following functional introduced by Astala [1] in 1980

$$\gamma_{\mathcal{I}}(T) := \inf\{\varepsilon > 0 \,:\, T(B_E) \subseteq \varepsilon B_F + R(B_Z), \\ \text{for some Banach space } Z \text{ and operator } R \in \mathcal{I}(Z;F)\},$$

characterizes the operators T that belong to the closed surjective hull $\overline{\mathcal{I}}^{sur}$ (i.e. the smallest closed surjective ideal containing $\mathcal{I}$) of $\mathcal{I}$ as those for which $\gamma_{\mathcal{I}}(T) = 0$ (see [1, Theorem 3.11]). In particular, the so-called *outer measure* $\gamma_{\mathcal{I}}$ satisfies that $\gamma_{\mathcal{I}}(T) = 0$ if and only if $T \in \mathcal{I}$, whenever $\mathcal{I}$ is a closed surjective ideal.

Astala also proved that for $\mathcal{I} = \mathcal{K}$, the ideal of compact operators, $\gamma_{\mathcal{K}}(T)$ coincides with the (ball) measure of non-compactness of T (see [1, Example 3.2(a)]). On the other hand, when $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators, $\gamma_{\mathcal{W}}(T)$ coincides with the measure of weak non-compactness introduced by De Blasi (see [1, Example 3.2(b)]).

The research of different properties of $\gamma_{\mathcal{I}}$, specially its behaviour under interpolation, has been very useful and it has enabled to give an unified point of view for previous results regarding certain linear operator ideals which are closed and surjective (see for example [13], [15] and [14] and references therein).

As far as we know, in the literature there is not a similar notion to the measure $\gamma_{\mathcal{I}}$ in the setting of multilinear operators. In this paper we introduce (Definition 3.2) a function for multilinear operators that can be considered as an extension of the outer measure $\gamma_{\mathcal{I}}$ introduced by Astala. In addition to showing some properties of this new measure, we prove that it allows to characterize the operators that belong to a closed surjective ideal of multilinear operators as those having measure equal to zero (see Theorem 3.1). We also obtain interpolation formulas for the new measure (Theorems 4.1 and 4.4). Although we use similar ideas to the linear case to establish them, as a consequence of these formulas we deduce interpolation results for arbitrary closed surjective ideals of multilinear operators. Despite there are many relevant examples where interpolation techniques have been used in the study of concrete ideals of multilinear operators and related questions (see for instance [10], [8], [22], [12] and [35]), as far as we know, interpolation results established in the setting of general closed surjective ideals of multilinear operators, such as those shown in this paper, are almost non-existent. We would like to point out that some basic facts and techniques that are essential in the linear operator theory cannot be translated to multilinear operators. In this sense, let us mention for example that the kernel or the range of a multilinear operator is not necessarily a linear subspace.

In our interpolation results (Theorems 4.2 and 4.5 and Corollaries 4.6 and 4.7) we consider intermediate spaces of class $C_K(\theta;\bar{E}_j)$ and bounded multilinear operators acting from a product $\Sigma(\bar{E}_1)\times\cdots\times\Sigma(\bar{E}_n)$ of sums of spaces from Banach couples $\bar{E}_1,\ldots,\bar{E}_n$, into a fixed Banach space F (see notation, definitions and precise results in Section 4). The most relevant interpolation methods, that is, the classical real method $\bar{E}_{\theta,q}$ and the complex method $\bar{E}_{[\theta]}$, produce interpolation spaces of class $C_K(\theta;\bar{E})$. As a result of Corollary 2.8, we show that each n-ideal $[\mathcal{I}_1,\ldots,\mathcal{I}_n]$, where $\mathcal{I}_1,\ldots,\mathcal{I}_n$ are closed surjective linear operator ideals, turns out to be closed and surjective, and so our results can be applied to a good number of concrete closed surjective ideals of multilinear operators (see Example 1). This also allows to give an extension to the multilinear case of different well-known interpolation results regarding important linear operator ideals (see Remark 4.8). As another consequence, by considering the particular case of the ideal of compact multilinear operators, we obtain for the classical real method and the complex method some known compactness results of Lions-Peetre type established for bilinear operators (see Remark 4.10). We find it convenient to highlight the interest that interpolation of compact bilinear (multilinear) operators is receiving in last years by the applications that these operators have in harmonic analysis (see for instance [21], [22], [23], [34], [12], [35] and references therein).

This paper continues the research started by the authors in [32], but now dealing with the case of general closed surjective ideals of multilinear operators and their interpolation properties.

2. Closed surjective ideals of multilinear operators

Next let us recall some basic definitions we will need throughout the paper. We consider real or complex Banach spaces without distinction. If $E_1,\ldots,E_n$ and F are Banach spaces, then $\mathcal{L}(E_1,\ldots,E_n;F)$ stands for the Banach space of all continuous n-linear operators $T: E_1\times\cdots\times E_n\to F$ with the norm

$$\|T\| := \sup\{\|T(x_1,\ldots,x_n)\|_F \,:\, x_1\in B_{E_1},\ldots,x_n\in B_{E_n}\},$$

where B_{E_j} is the closed unit ball of E_j, $j=1,\ldots,n$. In particular, $\mathcal{L}(E;F)$ is the Banach space of all continuous linear operators from E into F.

Definition 2.1. Let $n\in\mathbb{N}$ be fixed. An *ideal of n-linear operators*, or an *n-ideal*, is a class $\mathcal{M}_n$ of n-linear maps such that for all Banach spaces $E_1,\ldots,E_n$ and F, the components $\mathcal{M}_n(E_1,\ldots,E_n;F) := \mathcal{L}(E_1,\ldots,E_n;F)\cap\mathcal{M}_n$ satisfy

(i) $\mathcal{M}_n(E_1,\ldots,E_n;F)$ is a linear subspace of $\mathcal{L}(E_1,\ldots,E_n;F)$ that contains the n-linear maps of finite type.
(ii) If $R\in\mathcal{L}(F;H)$, $T\in\mathcal{M}_n(E_1,\ldots,E_n;F)$ and $S_j\in\mathcal{L}(G_j;E_j)$, for $j=1,\ldots,n$, then $R\circ T\circ(S_1,\ldots,S_n)\in\mathcal{M}_n(G_1,\ldots,G_n;H)$.

If for each $n \in \mathbb{N}$, $\mathcal{M}_n$ is an ideal of n-linear operators, the class

$$\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is called an *ideal of multilinear operators* or a *multi-ideal.*

The symbol $\mathcal{L}$ will stand for the multi-ideal of all continuous multilinear operators. To avoid confusions, we will use the letter $\mathcal{I}$ to denote a *linear* operator ideal (instead of $\mathcal{M}_1$ or $\mathcal{I}_1$).

There are standard methods to construct multi-ideals from linear operator ideals. One of them is based on considering the linear mappings associated to a multilinear operator and has its origin in the paper by Pietsch [38]. Next let us review its construction. We will use the notational convention $\overset{[i]}{\cdots}$ to mean that the i-th term, or the i-th coordinate, does not appear.

Definition 2.2. Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be linear operator ideals. The n-ideal $[\mathcal{I}_1, \dots, \mathcal{I}_n]$ is defined as follows: Let $T \in \mathcal{L}(E_1, \dots, E_n; F)$,

$$T \in [\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; F) \text{ if, and only if, } T_i \in \mathcal{I}_i(E_i; \mathcal{L}(E_1, \overset{[i]}{\dots}, E_n; F))$$

for all $i = 1, 2, \dots, n$, where $T_i : E_i \to \mathcal{L}(E_1, \overset{[i]}{\dots}, E_n; F)$ is defined as

$$T_i(x_i)(x_1, \overset{[i]}{\dots}, x_n) := T(x_1, \dots, x_n), \; x_1 \in E_1, \dots, x_n \in E_n.$$

As usual it is understood that if $n = 1$, then $T_1 = T$ for $T \in \mathcal{L}(E; F)$ and $[\mathcal{I}] = \mathcal{I}$ for each linear operator ideal $\mathcal{I}$.

Another way to construct multi-ideals is the *composition method*, that produces a multi-ideal $\mathcal{I} \circ \mathcal{L}$ formed by all multilinear operators $A = u \circ B$, where B is a continuous multilinear operator and u belongs to the linear operator ideal $\mathcal{I}$. In a similar way to the multilinear case, ideals of polynomials can be constructed. In particular, the composition ideal of polynomials $\mathcal{I} \circ \mathcal{P}$ is defined in an analogous manner. An *m-homogeneous polynomial* is a mapping $P : E \to F$ for which there exists an m-linear mapping $T : E \times \cdots \times E \to F$ so that $P(x) = T(x, \dots, x)$ for every $x \in E$. Given a continuous m-homogeneous polynomial $P : E \to F$, there is a unique continuous symmetric m-linear operator $\check{P}$ such that $\check{P}(x, \dots, x) = P(x)$. In [9, Proposition 3.2(b)], it is proved that $P \in \mathcal{I} \circ \mathcal{P}$ if, and only if, $\check{P} \in \mathcal{I} \circ \mathcal{L}$.

We also recall that $Q \in \mathcal{L}(E; F)$ is called a *metric surjection* if Q trasforms the open unit ball of E onto the open unit ball of F, that is, if $Q(\overset{\circ}{B}_E) = \overset{\circ}{B}_F$. Given a Banach space E, $Q_E : \ell_1(B_E) \to E$ will stand for the natural metric surjection given by $Q_E((\lambda_w)_{w \in B_E}) := \sum_{w \in B_E} \lambda_w w$.

Definition 2.3. Let $\mathcal{M}_n$ be an n-ideal. It will be denoted by $\overline{\mathcal{M}_n}$ the class of n-linear operators formed by components $\overline{\mathcal{M}_n}(E_1, \dots, E_n; F)$ that are given by the closure of $\mathcal{M}_n(E_1, \dots, E_n; F)$ in $\mathcal{L}(E_1, \dots, E_n; F)$. An n-ideal $\mathcal{M}_n$ is said to be *closed* when $\overline{\mathcal{M}_n} = \mathcal{M}_n$.

On the other hand, $\mathcal{M}_n^{sur}(E_1,\dots,E_n;F)$ will denote the following class of operators:

$$\mathcal{M}_n^{sur}(E_1,\dots,E_n;F) := \Big\{T\in\mathcal{L}(E_1,\dots,E_n;F)\colon \text{there are Banach spaces } Z_j \text{ and } P\in\mathcal{M}_n(Z_1,\dots,Z_n;F) \text{ s.t. } T(B_{E_1}\times\cdots\times B_{E_n})\subseteq P(B_{Z_1}\times\cdots\times B_{Z_n})\Big\}.$$

By a *surjective* n-ideal $\mathcal{M}_n$ we mean an n-ideal $\mathcal{M}_n$ such that, for any Banach spaces $E_1,\dots,E_n$ and F, it holds that $T\in\mathcal{M}_n(E_1,\dots,E_n;F)$ whenever $T\in\mathcal{M}_n^{sur}(E_1,\dots,E_n;F)$.

In [5] the concept of surjectivity of ideals of polynomials, and so surjectivity of multi-ideals, has been considered from a different point of view. According to [5, Corollary 4.3], for a surjective operator ideal $\mathcal{I}$, the composition ideal $\mathcal{I}\circ\mathcal{P}$ is surjective in the sense of [5, Definition 2.1]. However, it is not clear that the composition multi-ideal $\mathcal{I}\circ\mathcal{L}$ is surjective in the sense of Definition 2.3 (see [5, Proposition 3.4] that relates both definitions in the case of polynomials). Examples of composition ideals of polynomials are compact polynomials and weakly compact polynomials, which turn out to be composition of continuous polynomials with compact operators and weakly compact operators, respectively. Analogously, compact and weakly compact multilinear operators are examples of composition multi-ideals.

Remark 2.4. Note that if $T\in\mathcal{L}(E_1,\dots,E_n;F)$ is an operator such that $T\circ(Q_{E_1},\dots,Q_{E_n})\in\mathcal{M}_n(\ell_1(B_{E_1}),\dots,\ell_1(B_{E_n});F)$, then it holds that $T\in\mathcal{M}_n^{sur}(E_1,\dots,E_n;F)$. This directly follows from the fact that Q_{E_j} ($j=1,\dots,n$) is a metric surjection and thus, for any $\eta>0$, it holds that

$$T(B_{E_1}\times\cdots\times B_{E_n})\subseteq(1+\eta)(T\circ(Q_{E_1},\dots,Q_{E_n}))(B_{\ell_1(B_{E_1})}\times\cdots\times B_{\ell_1(B_{E_n})}).$$

When $n=1$, it is well-known that, given $T\in\mathcal{L}(E;F)$, then $T\circ Q_E\in\mathcal{I}(\ell_1(B_E);F)$ if, and only if, $T\in\mathcal{I}^{sur}(E;F)$. The assertion "if" can be proved by using a (fundamental) fact: if two operators $S\in\mathcal{L}(\ell_1(I);F)$ and $R\in\mathcal{L}(Z;F)$ satisfy that $S(B_{\ell_1(I)})\subseteq R(B_Z)$, then there is $u\in\mathcal{L}(\ell_1(I);Z)$ such that $S=R\circ u$ (see for instance [37, Lemma 8.5.4]).

The following lemma, inspired by [5, Proposition 3.3], provides a certain extension of the last fact mentioned in Remark 2.4 when S and R are multilinear operators.

In what follows, given an arbitrary set I and any $i\in I$, by e_i we mean the element $(\lambda_k)_{k\in I}\in\ell_1(I)$ defined as $\lambda_i=1$ and $\lambda_k=0$ for $k\neq i$.

Lemma 2.5. *Let $I_1,\dots,I_n$ be sets and $S\in\mathcal{L}(\ell_1(I_1),\dots,\ell_1(I_n);F)$. If $R\in\mathcal{L}(Z_1,\dots,Z_n;F)$ is such that for each $j=1,\dots,n$ and every $i_j\in I_j$ there are $z_{i_j}\in B_{Z_j}$ so that*

$$S(e_{i_1},\dots,e_{i_n})=R(z_{i_1},\dots,z_{i_n})$$

for any $i_1\in I_1,\dots,i_n\in I_n$, then there are $u_j\in\mathcal{L}(\ell_1(I_j);Z_j)$, $j=1,\dots,n$, such that $S=R\circ(u_1,\dots,u_n)$.

Proof. For each $j = 1, \dots, n$, we define $u_j : \ell_1(I_j) \to Z_j$ as $u_j(e_{i_j}) := z_{i_j}$. Then, for any finite subset $F_j \subseteq I_j$ we have:

$$\Big\| u_j\Big(\sum_{i_j \in F_j} a_{i_j} e_{i_j}\Big)\Big\|_{Z_j} = \Big\| \sum_{i_j \in F_j} a_{i_j} z_{i_j} \Big\|_{Z_j} \leq \sum_{i_j \in F_j} |a_{i_j}| = \Big\| \sum_{i_j \in F_j} a_{i_j} e_{i_j} \Big\|_{\ell_1(I_j)}.$$

As finite sums are dense in $\ell_1(I_j)$, we can extend u_j continuously to the whole of $\ell_1(I_j)$. Hence,

$$R \circ (u_1, \dots, u_n) \left(\sum_{i_1 \in F_1} a_{i_1} e_{i_1}, \dots, \sum_{i_n \in F_n} a_{i_n} e_{i_n} \right)$$

$$= R \left(\sum_{i_1 \in F_1} a_{i_1} z_{i_1}, \dots, \sum_{i_n \in F_n} a_{i_n} z_{i_n} \right) = \sum_{i_1 \in F_1} \cdots \sum_{i_n \in F_n} a_{i_1} \cdots a_{i_n} R(z_{i_1}, \dots, z_{i_n})$$

$$= \sum_{i_1 \in F_1} \cdots \sum_{i_n \in F_n} a_{i_1} \cdots a_{i_n} S(e_{i_1}, \dots, e_{i_n}) = S \left(\sum_{i_1 \in F_1} a_{i_1} e_{i_1}, \dots, \sum_{i_n \in F_n} a_{i_n} e_{i_n} \right).$$

By continuity we can conclude that $R \circ (u_1, \dots, u_n) = S$. $\square$

From Lemma 2.5, we obtain the next proposition that yields a necessary and sufficient condition for $T \circ (Q_{E_1}, \dots, Q_{E_n})$ to belong to an n-ideal $\mathcal{M}_n$. When in particular $n = 1$, Proposition 2.6 recovers the equivalence $T \circ Q_E \in \mathcal{I}$ if and only if $T \in \mathcal{I}^{sur}$, previously mentioned in Remark 2.4.

Proposition 2.6. *Let $\mathcal{M}_n$ be an n-ideal and let $T \in \mathcal{L}(E_1, \dots, E_n; F)$. The following assertions are equivalent:*

(a) $T \circ (Q_{E_1}, \dots, Q_{E_n}) \in \mathcal{M}_n(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$.

(b) *There are Banach spaces Z_j and an operator $R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$ so that for each $j = 1, \dots, n$ and every $i_j \in B_{E_j}$ there is an element $z_{i_j} \in B_{Z_j}$ in such a way that*

$$T(i_1, \dots, i_n) = R(z_{i_1}, \dots, z_{i_n}), \ \textit{for any } i_1 \in B_{E_1}, \dots, i_n \in B_{E_n}.$$

Proof. (a) $\Rightarrow$ (b) If $T \circ (Q_{E_1}, \dots, Q_{E_n}) \in \mathcal{M}_n(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$, by the definition of Q_{E_j} $(j = 1, \dots, n)$, the assertion (b) follows by taking the operator $R = T \circ (Q_{E_1}, \dots, Q_{E_n})$.

(b) $\Rightarrow$ (a) By hypothesis, there are Banach spaces Z_j and an n-linear operator $R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$ so that for every $i_j \in B_{E_j}$ there is an element $z_{i_j} \in B_{Z_j}$, $j = 1, \dots, n$, in such a way that

$$(T \circ (Q_{E_1}, \dots, Q_{E_n}))(e_{i_1}, \dots, e_{i_n}) = T(i_1, \dots, i_n) = R(z_{i_1}, \dots, z_{i_n}),$$

for all $i_1 \in B_{E_1}, \dots, i_n \in B_{E_n}$. Lemma 2.5 allows to find operators $u_j \in \mathcal{L}(\ell_1(B_{E_j}); Z_j)$, $j = 1, \dots, n$, such that $T \circ (Q_{E_1}, \dots, Q_{E_n}) = R \circ (u_1, \dots, u_n)$, and so (a) holds. $\square$

Remark 2.7. For the class of n-ideals $[\mathcal{I}_1, \dots, \mathcal{I}_n]$, it holds that $[\mathcal{I}_1, \dots, \mathcal{I}_n]^{sur} \subset [\mathcal{I}_1^{sur}, \dots, \mathcal{I}_n^{sur}]$. In fact, if there exist Banach spaces Z_j $(j = 1, \dots, n)$ and an operator $P \in [\mathcal{I}_1, \dots, \mathcal{I}_n](Z_1, \dots, Z_n; F)$ such that $T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq P(B_{Z_1} \times \cdots \times B_{Z_n})$, then in particular it follows that $T_i(B_{E_i}) \subseteq$

$P_i(B_{Z_i})$, with $P_i \in \mathcal{I}_i(Z_i; \mathcal{L}(Z_1, \overset{[i]}{\ldots}, Z_n; F))$, for each $i = 1, \ldots, n$. Then $T_i \in \mathcal{I}_i^{sur}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)), i = 1, 2, \ldots, n$, and thus $T \in [\mathcal{I}_1^{sur}, \ldots, \mathcal{I}_n^{sur}]$. On the other hand, we also note that $\overline{[\mathcal{I}_1, \ldots, \mathcal{I}_n]} \subset [\overline{\mathcal{I}_1}, \ldots, \overline{\mathcal{I}_n}]$ for any linear operator ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ (see for example [7, Section 3] and [32, Lemma 3.1(b)]).

As a straightforward consequence we obtain the following result.

Corollary 2.8. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be linear operator ideals.*

(a) *If $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are surjective, then $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is a surjective n-ideal.*
(b) *If $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are closed, then $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is a closed n-ideal.*

Observe that if $\mathcal{M}_n$ is a surjective n-ideal (Definition 2.3), according to Remark 2.4, for each $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ such that $T \circ (Q_{E_1}, \ldots, Q_{E_n}) \in \mathcal{M}_n(\ell_1(B_{E_1}), \ldots, \ell_1(B_{E_n}); F)$, it follows that $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$. Thus, Corollary 2.8(a) gives [24, Hilfssatz in p.154] without any hypothesis of injectivity on the linear operator ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$.

Next we show some examples of closed surjective n-ideals.

Example 1. The n-ideal $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is closed and surjective whenever the linear operator ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ vary among any of the following classes (to mix different classes is allowed):

1. Compact operators.
2. Weakly compact operators.
3. Strictly cosingular operators.
4. Rosenthal operators.
5. Banach-Saks operators.
6. Operators of separable range.
7. Limited operators.
8. Grothendieck operators.
9. Asplund operators.

A table that summarizes the aforementioned properties for the linear operator ideals enumerated above can be found in [26]. We refer to the classical books [17], [19], [29] and [37] for wide information about operator theory.

3. A measure associated to ideals of multilinear operators

A natural question is if it is possible to consider a similar notion to the outer measure $\gamma_{\mathcal{I}}$ for multilinear operators. With the aim to give an answer to this question, we establish the following theorem, which provides a necessary and sufficient condition for a multilinear operator to belong to an n-ideal which is closed and surjective. It extends to the multilinear case a well-known result for linear operator ideals. To prove our result we use some ideas inspirated by [27, Proposition 1.7] and [31, Teorema 1.4.2].

Theorem 3.1. *Let $\mathcal{M}_n$ be an n-ideal which is closed and surjective. Let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$. The following assertions are equivalent:*

(a) $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$.
(b) *For each* $\varepsilon > 0$, *there are Banach spaces* $Z_1, \dots, Z_n$ *and an* n*-linear operator* $R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$ *such that*

$$T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq \varepsilon B_F + R(B_{Z_1} \times \cdots \times B_{Z_n}).$$

Proof. (a) $\Rightarrow$ (b) trivially.

(b) $\Rightarrow$ (a) By hypothesis, for any natural number k, it holds that

$$T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq \frac{1}{k} B_F + P_k(B_{Z_1^k} \times \cdots \times B_{Z_n^k}), \tag{3.1}$$

for some Banach spaces $Z_1^k, \dots, Z_n^k$ and an operator $P_k \in \mathcal{M}_n(Z_1^k, \dots, Z_n^k; F)$. Let $\Lambda := T \circ (Q_{E_1}, \dots, Q_{E_n})$, i.e. $\Lambda \in \mathcal{L}(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$ is given by

$$\begin{aligned}\Lambda((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}) &= T\Big(\sum_{w^1 \in B_{E_1}} \lambda_{w^1} w^1, \dots, \sum_{w^n \in B_{E_n}} \lambda_{w^n} w^n \Big) \\ &= \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} T(w^1, \dots, w^n).\end{aligned}$$

Since $\mathcal{M}_n$ is a surjective n-ideal, in order to check that $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ it is sufficient to prove that $\Lambda \in \mathcal{M}_n(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$.
Let $\Upsilon := T(B_{E_1} \times \cdots \times B_{E_n})$. By inclusion (3.1), it is possible to construct mappings $\varphi_k : \Upsilon \to \frac{1}{k} B_F$ and $\psi_k : \Upsilon \to P_k(B_{Z_1^k} \times \cdots \times B_{Z_n^k})$ so that $\varphi_k(\upsilon) + \psi_k(\upsilon) = \upsilon$, for every $\upsilon \in \Upsilon$. We define $R_k \in \mathcal{L}(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$ and $S_k \in \mathcal{L}(\ell_1(B_{E_1}), \dots, \ell_1(B_{E_n}); F)$, respectively, by

$$\begin{aligned}&R_k((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}) := \\ &\qquad \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} \varphi_k(T(w^1, \dots, w^n)),\end{aligned}$$

$$\begin{aligned}&S_k((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}) := \\ &\qquad \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} \psi_k(T(w^1, \dots, w^n)).\end{aligned}$$

It is clear that

$$\begin{aligned}&\Lambda((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}) \\ &= \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} T(w^1, \dots, w^n) \\ &= \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} \left[\varphi_k(T(w^1, \dots, w^n)) + \psi_k(T(w^1, \dots, w^n)) \right] \\ &= \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} \varphi_k(T(w^1, \dots, w^n)) + \\ &\quad + \sum_{w^1 \in B_{E_1}} \cdots \sum_{w^n \in B_{E_n}} \lambda_{w^1} \cdots \lambda_{w^n} \psi_k(T(w^1, \dots, w^n)) \\ &= R_k((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}) + S_k((\lambda_{w^1})_{w^1 \in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n \in B_{E_n}}).\end{aligned}$$

Moreover, for any $(\lambda_{w^1})_{w^1\in B_{E_1}} \in B_{\ell_1(B_{E_1})}, \dots, (\lambda_{w^n})_{w^n\in B_{E_n}} \in B_{\ell_1(B_{E_n})}$,

$$\begin{aligned}
&\|R_k((\lambda_{w^1})_{w^1\in B_{E_1}}, \dots, (\lambda_{w^n})_{w^n\in B_{E_n}})\|_F \\
&\qquad = \Big\| \sum_{w^1\in B_{E_1}} \cdots \sum_{w^n\in B_{E_n}} \lambda_{w^1}\cdots\lambda_{w^n}\varphi_k(T(w^1,\dots,w^n))\Big\|_F \\
&\qquad \le \sum_{w^1\in B_{E_1}} \cdots \sum_{w^n\in B_{E_n}} |\lambda_{w^1}|\cdots|\lambda_{w^n}|\,\|\varphi_k(T(w^1,\dots,w^n))\|_F \\
&\qquad \le \frac{1}{k}\sum_{w^1\in B_{E_1}} \cdots \sum_{w^n\in B_{E_n}} |\lambda_{w^1}|\cdots|\lambda_{w^n}| \le \frac{1}{k}.
\end{aligned}$$

Hence, $\|R_k\| \le \frac{1}{k}$ and $\|\Lambda - S_k\| = \|R_k\| \to 0$ as $k\to\infty$. On the other hand, since $P_k \in \mathcal{M}_n(Z_1^k,\dots,Z_n^k;F)$, $S_k(B_{\ell_1(B_{E_1})}\times\cdots\times B_{\ell_1(B_{E_n})}) \subseteq P_k(B_{Z_1^k}\times\cdots\times B_{Z_n^k})$ and $\mathcal{M}_n$ is surjective, it follows that $S_k \in \mathcal{M}_n(\ell_1(B_{E_1}),\dots,\ell_1(B_{E_n});F)$. Then, $\Lambda \in \mathcal{M}_n(\ell_1(B_{E_1}),\dots,\ell_1(B_{E_n});F)$, because $\mathcal{M}_n$ is a closed n-ideal. $\square$

Now, we introduce a function associated to an n-ideal $\mathcal{M}_n$, inspired by Theorem 3.1, that can be considered as an extension to the multilinear case of the measure $\gamma_{\mathcal{I}}$ defined by Astala.

Definition 3.2. For any $T \in \mathcal{L}(E_1,\dots,E_n;F)$, let

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T) = \gamma_{\mathcal{M}_n}(T : E_1\times\cdots\times E_n \to F) := \\
&\quad \inf\Big\{\varepsilon>0 \,:\, T(B_{E_1}\times\cdots\times B_{E_n}) \subseteq \varepsilon B_F + R(B_{Z_1}\times\cdots\times B_{Z_n}),\ \text{for some} \\
&\qquad \text{Banach spaces } Z_j \text{ and an } n\text{-linear operator } R\in\mathcal{M}_n(Z_1,\dots,Z_n;F)\Big\}.
\end{aligned}$$

It directly follows from Theorem 3.1 that $\gamma_{\mathcal{M}_n}(T) = 0$ if and only if $T \in \mathcal{M}_n$, for a closed surjective n-ideal $\mathcal{M}_n$.

It is as well clear that for any $T \in \mathcal{L}(E_1,\dots,E_n;F)$, we have $\gamma_{\mathcal{M}_n}(T) \le \|T\|_{\mathcal{L}(E_1,\dots,E_n;F)}$, because $T(B_{E_1}\times\cdots\times B_{E_n}) \subseteq \|T\|_{\mathcal{L}(E_1,\dots,E_n;F)}B_F$. Moreover, it holds that $\gamma_{\mathcal{M}_n}(S\circ T) \le \|S\|\gamma_{\mathcal{M}_n}(T)$, for each $S\in\mathcal{L}(F;G)$.

Next we show some other properties of $\gamma_{\mathcal{M}_n}$ that recover when $n=1$ those satisfied by the outer measure $\gamma_{\mathcal{I}}$.

Proposition 3.3. *Let $\mathcal{M}_n$ be an n-ideal and let $T \in \mathcal{L}(E_1,\dots,E_n;F)$.*

(a) *If $q_j \in \mathcal{L}(D_j;E_j)$ is a metric surjection ($j=1,\dots,n$), then*

$$\gamma_{\mathcal{M}_n}(T\circ(q_1,\dots,q_n)) = \gamma_{\mathcal{M}_n}(T).$$

(b) *If $j\in\mathcal{L}(F;G)$ is a metric injection, that is, $\|jy\|_G = \|y\|_F$ for all $y\in F$, then*

$$\gamma_{\mathcal{M}_n}(j\circ T) \le \gamma_{\mathcal{M}_n}(T).$$

On the other hand, if F has the metric extension property (see definition in [1, p. 18] *or* [37, C.3.1]*), for each metric injection $j\in\mathcal{L}(F;G)$, it follows that $\gamma_{\mathcal{M}_n}(j\circ T) = \gamma_{\mathcal{M}_n}(T)$.*

(c) *Let* $J_F \in \mathcal{L}(F; \ell_\infty(B_{F^*}))$ *denote the natural metric injection given by* $J_F(y) := (\langle y, y^* \rangle)_{y^* \in B_{F^*}}$. *It holds that*

$$\gamma_{\mathcal{M}_n}(J_F \circ T) = \min\{\gamma_{\mathcal{M}_n}(j \circ T) : \textit{where } j : F \to G \textit{ is a metric injection}\}.$$

Proof. (a) Since $q_j \in \mathcal{L}(D_j; E_j)$ is a metric surjection, $q_j(B_{D_j}) \subseteq B_{E_j}(j = 1, \ldots, n)$ and so $(T \circ (q_1, \ldots, q_n))(B_{D_1} \times \cdots \times B_{D_n}) \subseteq T(B_{E_1} \times \cdots \times B_{E_n})$. In order to check the inequality $\gamma_{\mathcal{M}_n}(T) \leq \gamma_{\mathcal{M}_n}(T \circ (q_1, \ldots, q_n))$, it is enough to observe that, for each $\eta > 0$, it holds that $T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq (1 + \eta)(T \circ (q_1, \ldots, q_n))(B_{D_1} \times \cdots \times B_{D_n})$. Hence, for every $\varepsilon > \gamma_{\mathcal{M}_n}(T \circ (q_1, \ldots, q_n))$, it follows that $\gamma_{\mathcal{M}_n}(T) \leq (1 + \eta)\varepsilon$ and (a) holds.

(b) If $j \in \mathcal{L}(F; G)$ is a metric injection, then $\gamma_{\mathcal{M}_n}(j \circ T) \leq \|j\| \gamma_{\mathcal{M}_n}(T) \leq \gamma_{\mathcal{M}_n}(T)$. Now, suppose that F has the metric extension property and let $j \in \mathcal{L}(F; G)$ be a metric injection. Then, it is possible to find $\phi \in \mathcal{L}(G; F)$ such that $\|\phi\| = 1$ and $\phi \circ i_{j(F)} = I_F \circ j^{-1}_{|_{j(F)}}$, where $i_{j(F)}$ stands for the natural inclusion from $j(F)$ into G and I_F the identity operator in F:

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[r, "T"] & F \arrow[r, shift left, "j"] \arrow[d, "I_F"'] & j(F) \arrow[l, shift left, "j^{-1}_{|_{j(F)}}"] \arrow[r, "i_{j(F)}"] & G \arrow[dll, "\phi"] \\
 & F & &
\end{tikzcd}$$

Therefore,

$$\gamma_{\mathcal{M}_n}(T) = \gamma_{\mathcal{M}_n}(\phi \circ i_{j(F)} \circ j \circ T) \leq \|\phi \circ i_{j(F)}\| \gamma_{\mathcal{M}_n}(j \circ T) \leq \gamma_{\mathcal{M}_n}(j \circ T).$$

(c) Let $j \in \mathcal{L}(F; G)$ be any metric injection. An analogous argument as in the second part of the proof of (b) allows to see that $\gamma_{\mathcal{M}_n}(J_F \circ T) \leq \gamma_{\mathcal{M}_n}(j \circ T)$. In fact, since $\ell_\infty(B_{F^*})$ has the metric extension property, there exists $\phi \in \mathcal{L}(G; \ell_\infty(B_{F^*}))$ with $\|\phi\| = 1$ and $\phi \circ i_{j(F)} = J_F \circ j^{-1}_{|_{j(F)}}$, and thus

$$\gamma_{\mathcal{M}_n}(J_F \circ T) = \gamma_{\mathcal{M}_n}(J_F \circ j^{-1}_{|_{j(F)}} \circ j \circ T) \leq \|\phi \circ i_{j(F)}\| \gamma_{\mathcal{M}_n}(j \circ T) \leq \gamma_{\mathcal{M}_n}(j \circ T).$$

□

4. Some interpolation results and related consequences

Let us recall some basic definitions on interpolation theory. Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, A_0 and A_1 are two Banach spaces which are continuously embedded in some Hausdorff topological vector space. The sum $\Sigma(\bar{A}) := A_0 + A_1$ and the intersection $\Delta(\bar{A}) := A_0 \cap A_1$ of A_0 and A_1 become Banach spaces when endowed with the norms $K(1, \cdot)$ and $J(1, \cdot)$, respectively, where the K- and J-*functionals* are defined, for $t > 0$, by

$$K(t, a) = K(t, a; \bar{A}) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_i \in A_i\}, a \in \Sigma(\bar{A}).$$

$$J(t, a) = J(t, a; \bar{A}) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \; a \in \Delta(\bar{A}).$$

A Banach space A is called an *intermediate space* with respect to $\bar{A} = (A_0, A_1)$ if $\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A})$, where "$\hookrightarrow$" means continuous inclusion. An intermediate space A with respect to $\bar{A} = (A_0, A_1)$ is said to be *of class* $C_K(\theta; \bar{A})$, where $0 < \theta < 1$, if there is a constant $C > 0$ such that for all $t > 0$ and $a \in A$,

$$K(t, a) \leq Ct^\theta \|a\|_A.$$

The real interpolation space $(A_0, A_1)_{\theta,q}$ and the complex interpolation space $(A_0, A_1)_{[\theta]}$ are important examples of spaces of class $C_K(\theta; \bar{A})$. We refer to the books [4] and [39] for full information about fundamentals of interpolation theory.

In Theorems 4.1 and 4.4 we will investigate interpolation properties of the function $\gamma_{\mathcal{M}_n}$, associated to an arbitrary n-ideal $\mathcal{M}_n$. Our techniques are inspired by ideas used in [13, Theorem 3.2] and [15, Theorem 3.2] (see also [14]) for the linear case.

Theorem 4.1. *Let $\mathcal{M}_n$ be an n-ideal. Let $E_1, \dots, E_n, F$ be Banach spaces. Take any $i = 1, \dots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and assume that X is an intermediate space of class $\mathcal{C}_K(\theta, \bar{X})$ with constant C. If $T \in \mathcal{L}(E_1, \dots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \dots, E_n; F)$, then*

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\
&\leq C(1-\theta)^{\theta-1}\theta^{-\theta}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_0 \times E_{i+1} \times \cdots \times E_n \to F)^{1-\theta} \cdot \\
&\qquad \cdot \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_1 \times E_{i+1} \times \cdots \times E_n \to F)^{\theta}.
\end{aligned}$$

Proof. Let $\varepsilon_k > \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_k \times E_{i+1} \times \cdots \times E_n \to F)$, $k = 0, 1$. There are Banach spaces Z_j^k and an n-linear operator $R_k \in \mathcal{M}_n(Z_1^k, \dots, Z_n^k; F)$ such that (for $k = 0, 1$)

$$T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{X_k} \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \subseteq \varepsilon_k B_F + R_k(B_{Z_1^k} \times \cdots \times B_{Z_n^k}).$$

Take any $\eta > 0$ and $t > 0$. Put $Z_j := (Z_j^0 \oplus Z_j^1)_\infty, j = 1, \dots, n$, and let $R \in \mathcal{L}(Z_1, \dots, Z_n; F)$ defined by

$$\begin{aligned}
&R\big((z_1^0, z_1^1), \dots, (z_n^0, z_n^1)\big) := \\
&\qquad\qquad C(1+\eta)t^\theta R_0(z_1^0, \dots, z_n^0) + C(1+\eta)t^{\theta-1} R_1(z_1^1, \dots, z_n^1).
\end{aligned}$$

Since $R = C(1+\eta)t^\theta R_0 \circ (\phi_1^0, \dots, \phi_n^0) + C(1+\eta)t^{\theta-1} R_1 \circ (\phi_1^1, \dots, \phi_n^1)$, with $\phi_j^k : Z_j \to Z_j^k$ the natural projection, we have that $R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$. Moreover, due to X is of class $C_K(\theta; \bar{X})$, given $x \in B_X$, it is possible to find $x_0 \in X_0$ and $x_1 \in X_1$ in such a way that $x = x_0 + x_1$ and

$$\|x_0\|_{X_0} + t\|x_1\|_{X_1} \leq Ct^\theta + C\eta t^\theta = C(1+\eta)t^\theta.$$

This implies

$$\|x_0\|_{X_0} \leq C(1+\eta)t^\theta, \qquad \|x_1\|_{X_1} \leq C(1+\eta)t^{\theta-1}.$$

Therefore,

$$
\begin{aligned}
&T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_X \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\quad \subseteq C(1+\eta)t^{\theta}\, T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{X_0} \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\qquad + C(1+\eta)t^{\theta-1}\, T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{X_1} \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\quad \subseteq C(1+\eta)t^{\theta}\big(\varepsilon_0 B_F + R_0(B_{Z_1^0} \times \cdots \times B_{Z_n^0})\big) \\
&\qquad + C(1+\eta)t^{\theta-1}\big(\varepsilon_1 B_F + R_1(B_{Z_1^1} \times \cdots \times B_{Z_n^1})\big) \\
&\quad \subseteq C(1+\eta)\big(t^{\theta}\varepsilon_0 + t^{\theta-1}\varepsilon_1\big)B_F \\
&\qquad + C(1+\eta)t^{\theta} R_0(B_{Z_1^0} \times \cdots \times B_{Z_n^0}) + C(1+\eta)t^{\theta-1} R_1(B_{Z_1^1} \times \cdots \times B_{Z_n^1}) \\
&\quad \subseteq C(1+\eta)\big(t^{\theta}\varepsilon_0 + t^{\theta-1}\varepsilon_1\big)B_F + R(B_{Z_1} \times \cdots \times B_{Z_n}).
\end{aligned}
$$

Then,

$$
\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \le C(1+\eta)\big(t^{\theta}\varepsilon_0 + t^{\theta-1}\varepsilon_1\big),
$$

for any $\eta > 0$ and $t > 0$. Thus, for each $t > 0$,

$$
\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\
&\qquad \le C t^{\theta} \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_0 \times E_{i+1} \times \cdots \times E_n \to F) + \\
&\qquad\quad + C t^{\theta-1} \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_1 \times E_{i+1} \times \cdots \times E_n \to F).
\end{aligned}
$$

It is clear that this implies that $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) = 0$, when $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_k \times E_{i+1} \times \cdots \times E_n \to F) = 0$ for $k = 0$ or $k = 1$.

On the other hand, if $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_k \times E_{i+1} \times \cdots \times E_n \to F) > 0$ for $k = 0, 1$, it holds that

$$
\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\
&\le C \inf_{t>0}\big\{t^{\theta}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_0 \times E_{i+1} \times \cdots \times E_n \to F) + \\
&\qquad\quad + t^{\theta-1}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_1 \times E_{i+1} \times \cdots \times E_n \to F)\big\} \\
&= C(1-\theta)^{\theta-1}\theta^{-\theta}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_0 \times E_{i+1} \times \cdots \times E_n \to F)^{1-\theta} \cdot \\
&\qquad\quad \cdot\, \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_1 \times E_{i+1} \times \cdots \times E_n \to F)^{\theta}.
\end{aligned}
$$

□

Theorem 4.1 recovers [13, Theorem 3.2] when in particular $n = 1$. The following result provides a multilinear version of [27, Proposition 1.7] and it follows from Theorems 4.1 and 3.1.

Theorem 4.2. *Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $E_1, \dots, E_n, F$ be Banach spaces. Take any $i = 1, \dots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and assume that X is an intermediate space of class $\mathcal{C}_K(\theta, \bar{X})$. For every operator $T \in \mathcal{L}(E_1, \dots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \dots, E_n; F)$, it follows that $T \in \mathcal{M}_n(E_1, \dots, E_{i-1}, X, E_{i+1}, \dots, E_n; F)$ whenever, for $k = 0$ or $k = 1$, $T \in \mathcal{M}_n(E_1, \dots, E_{i-1}, X_k, E_{i+1}, \dots, E_n; F)$.*

Before next remark let us note that if (Ω, μ) is a σ-finite measure space, for $1 \leq p_0 \neq p_1 \leq \infty, 1 \leq q_0, q_1, q \leq \infty, 0 < \eta < 1$ and $1/p = (1-\eta)/p_0 + \eta/p_1$, it holds with equivalence of norms that (see [4, Theorem 5.3.1])

$$(L_{p_0,q_0}(\Omega), L_{p_1,q_1}(\Omega))_{\eta,q} = L_{p,q}(\Omega).$$

In particular,

$$(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\eta,\, q} = L_{p,q}(\Omega). \tag{4.1}$$

As usual the Lorentz space for the case $\Omega = [0,1]$, with Lebesgue measure, will be denoted by $L_{p,q}[0,1]$.

Remark 4.3. According to [6, Counterexample 2.5] we note that even for $n = 1$ an estimate as that given in Theorem 4.1 does not hold in general if $T \in \mathcal{L}(E; \Delta(\bar{F}))$, where $\bar{F} = (F_0, F_1)$ is a Banach couple and E is a Banach space. For the sake of completeness, we include the details of that technical counterexample.

First we recall that if $1 < p < \infty$ and T stands for the identity operator, then $T : L_p[0,1] \to L_1[0,1]$ is not a strictly singular operator because, according to Khintchine's inequality, the span of the Rademacher functions in $L_p[0,1]$ and $L_1[0,1]$ is isomorphic to ℓ_2. Thus, the restriction of T to this subspace of $L_p[0,1]$ is an isomorphism into $L_1[0,1]$. Now, let $\mathcal{I}$ be the closed surjective ideal of strictly cosingular operators (see [36] and [37]), and consider $E = L_\infty[0,1]$ and the Banach couple $\bar{F} = ((L_\infty[0,1])^*, L_\infty[0,1])$. Since $T^* : L_\infty[0,1] \to (L_\infty[0,1])^*$ is weakly compact (due to the identity operator $T : L_\infty[0,1] \to L_1[0,1]$ is weakly compact) and $(L_\infty[0,1])^*$ has the Dunford-Pettis property (see [18, Section 19]), the operator $T^* : L_\infty[0,1] \to (L_\infty[0,1])^*$ belongs to $\mathcal{I}$ (see [36, Proposition (4b)]). However, as it was pointed out by Beucher [6, Counterexample 2.5], if $T^*_{\theta,p'}$ belongs to the ideal $\mathcal{I}$, where $T^*_{\theta,p'}$ denotes the interpolated operator by applying the real method $(\cdot,\cdot)_{\theta,p'}$ with $\theta = 1/p$ and $1/p + 1/p' = 1$, then it follows from [4, Theorem 3.7.1] and (4.1) that the operator $T^* : L_\infty[0,1] \to (L_p[0,1])^*$ also belongs to $\mathcal{I}$. But it implies that $T : L_p[0,1] \to L_1[0,1]$ is a strictly singular operator (see [36, Proposition (3a)]) which, as mentioned above, is a contradiction. In terms of the measure $\gamma_{\mathcal{I}}$, this means that $\gamma_{\mathcal{I}}(T^* : E \to F_0) = 0$, but $\gamma_{\mathcal{I}}(T^*_{\theta,p'} : E \to (F_0, F_1)_{\theta,p'}) > 0$.

The next result provides an estimate for the measure of the interpolated operator in terms of the corresponding restrictions of the operator when the extreme intermediate spaces are considered, that is, in terms of the restrictions $T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F$ and $T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F$.

Theorem 4.4. *Let $\mathcal{M}_n$ be an n-ideal. Let $E_1, \ldots, E_n, F$ be Banach spaces. Take any $i = 1, \ldots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and assume that X is an intermediate space of class $\mathcal{C}_K(\theta, \bar{X})$ with constant C. If $T \in$*

$\mathcal{L}(E_1, \ldots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \ldots, E_n; F)$, *then*

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\
&\qquad \leq 4C\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)^{\Theta} \cdot \\
&\qquad\qquad \cdot \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)^{1-\Theta},
\end{aligned}$$

where $\Theta = \min\{\theta, 1-\theta\}$.

Proof. Let $\eta > 0$. We take any $0 < t \leq 1$ such that

$$t^{\theta} \leq \eta \quad \text{and} \quad t^{1-\theta} \leq \eta. \tag{4.2}$$

Let $\sigma > \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)$. There exist Banach spaces G_j and an n-linear operator $R \in \mathcal{M}_n(G_1, \ldots, G_n; F)$ such that

$$\begin{aligned}
T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{\Delta(\bar{X})} &\times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\subseteq \sigma B_F + R(B_{G_1} \times \cdots \times B_{G_n}).
\end{aligned} \tag{4.3}$$

On the other hand, if $\delta > \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)$ then, for certain Banach spaces H_j and n-linear operator $S \in \mathcal{M}_n(H_1, \ldots, H_n; F)$, it holds that

$$\begin{aligned}
T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{\Sigma(\bar{X})} &\times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\subseteq \delta B_F + S(B_{H_1} \times \cdots \times B_{H_n}).
\end{aligned} \tag{4.4}$$

Let $\varepsilon > 0$. We put $Z_j := (G_j \oplus H_j)_\infty, j = 1, \ldots, n$, and define the n-linear operator $P \in \mathcal{L}(Z_1, \ldots, Z_n; F)$ as

$$\begin{aligned}
&P\big((u_1, v_1), \ldots, (u_n, v_n)\big) := \\
&\qquad 2C(1+\varepsilon)\eta t^{-1} R(u_1, \ldots, u_n) + 2C(1+\varepsilon)\eta S(v_1, \ldots, v_n).
\end{aligned}$$

In other words, $P = 2C(1+\varepsilon)\eta t^{-1} R \circ (\phi_1^0, \ldots, \phi_n^0) + 2C(1+\varepsilon)\eta S \circ (\phi_1^1, \ldots, \phi_n^1)$, where $\phi_j^0 : Z_j \to G_j$ and $\phi_j^1 : Z_j \to H_j$ are the natural projections. Whence $P \in \mathcal{M}_n(Z_1, \ldots, Z_n; F)$.

Using that X is of class $C_K(\theta; \bar{X})$, given any $x \in B_X$, there are decompositions of x as $x = x_0 + x_1 = x_0' + x_1'$, with $x_k, x_k' \in X_k$ $(k = 0, 1)$, and

$$\|x_0\|_{X_0} + t\|x_1\|_{X_1} \leq K(t, x) + C\varepsilon\eta \leq Ct^{\theta} + C\varepsilon\eta \leq C(1+\varepsilon)\eta,$$

$$\|x_0'\|_{X_0} + t^{-1}\|x_1'\|_{X_1} \leq K(t^{-1}, x) + C\varepsilon\eta \leq Ct^{-\theta} + C\varepsilon\eta \leq C(1+\varepsilon)\eta t^{-1}.$$

Therefore,

$$\|x_k\|_{X_k} \leq C(1+\varepsilon)\eta t^{-k}, \quad \|x_k'\|_{X_k} \leq C(1+\varepsilon)\eta t^{k-1}, \quad k = 0, 1. \tag{4.5}$$

Now let $y := x_0' - x_0 = x_1 - x_1' \in \Delta(\bar{X})$. It follows from (4.5) that

$$\begin{aligned}
\|y\|_{\Delta(\bar{X})} &\leq \max\{\|x_0'\|_{X_0} + \|x_0\|_{X_0}, \|x_1\|_{X_1} + \|x_1'\|_{X_1}\} \\
&\leq \max\{C(1+\varepsilon)\eta t^{-1} + C(1+\varepsilon)\eta, C(1+\varepsilon)\eta t^{-1} + C(1+\varepsilon)\eta\} \\
&\leq 2C(1+\varepsilon)\eta t^{-1},
\end{aligned} \tag{4.6}$$

and

$$\|x - y\|_{\Sigma(\bar{X})} \leq \|x_0\|_{X_0} + \|x_1'\|_{X_1} \leq 2C(1+\varepsilon)\eta. \tag{4.7}$$

By (4.6) and (4.7), we have that

$$B_X \subseteq 2C(1+\varepsilon)\eta t^{-1} B_{\Delta(\bar{X})} + 2C(1+\varepsilon)\eta B_{\Sigma(\bar{X})}.$$

Then, according to (4.3) and (4.4),

$$\begin{aligned}
&T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_X \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\quad \subseteq 2C(1+\varepsilon)\eta t^{-1}\, T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{\Delta(\bar{X})} \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\qquad + 2C(1+\varepsilon)\eta\, T(B_{E_1} \times \cdots \times B_{E_{i-1}} \times B_{\Sigma(\bar{X})} \times B_{E_{i+1}} \times \cdots \times B_{E_n}) \\
&\quad \subseteq 2C(1+\varepsilon)\eta t^{-1}\big(\sigma B_F + R(B_{G_1} \times \cdots \times B_{G_n})\big) \\
&\qquad + 2C(1+\varepsilon)\eta\big(\delta B_F + S(B_{H_1} \times \cdots \times B_{H_n})\big) \\
&\quad \subseteq 2C(1+\varepsilon)\eta\big(\sigma t^{-1} + \delta\big) B_F \\
&\qquad + 2C(1+\varepsilon)\eta t^{-1} R(B_{G_1} \times \cdots \times B_{G_n}) + 2C(1+\varepsilon)\eta S(B_{H_1} \times \cdots \times B_{H_n}) \\
&\quad \subseteq 2C(1+\varepsilon)\eta\big(\sigma t^{-1} + \delta\big) B_F + P(B_{Z_1} \times \cdots \times B_{Z_n}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \leq \\
&2C\eta\big[t^{-1}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) + \\
&\qquad + \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)\big].
\end{aligned} \tag{4.8}$$

Since η is arbitrary, if $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) = 0$, then $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) = 0$.

On the contrary, when $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) > 0$, it also holds that $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) > 0$, because

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) \\
&\qquad\qquad \leq \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F).
\end{aligned}$$

In that case, choose

$$\begin{aligned}
\eta := \max\Bigg\{ &\left(\frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}\right)^{\theta}, \\
&\left(\frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}\right)^{1-\theta} \Bigg\}.
\end{aligned}$$

The real number

$$t := \frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)} \leq 1$$

satisfies (4.2). If we denote $\Theta := \min\{\theta, 1-\theta\}$ and substitute these concrete choices of η and t in (4.8), we have that

$$\begin{aligned}
&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\
&\quad\le 4C\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)\cdot \\
&\qquad \cdot \max\left\{ \left(\frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)} \right)^{\theta}, \right. \\
&\qquad\qquad \left. \left(\frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)} \right)^{1-\theta} \right\} \\
&\quad= 4C\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)\cdot \\
&\qquad\qquad \cdot \left(\frac{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)}{\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)} \right)^{\Theta} \\
&\quad= 4C\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F))^{\Theta}\cdot \\
&\qquad\qquad \cdot \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)^{1-\Theta}.
\end{aligned}$$

□

Note that it holds that $\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F) \le \|T\|_{E_1,\ldots,E_{i-1},\bar{X},E_{i+1},\ldots,E_n;F} := \max\{\|T\|_{\mathcal{L}(E_1,\ldots,E_{i-1},X_k,E_{i+1},\ldots,E_n;F)} : k = 0,1\}$ and so Theorem 4.4 extends [15, Theorem 3.2] to the case of ideals of n-linear operators.

Theorems 4.4 and 3.1 yield the following result.

Theorem 4.5. *Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $E_1, \ldots, E_n, F$ be Banach spaces. Take any $i = 1, \ldots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let X be of class $\mathcal{C}_K(\theta, \bar{X})$. If $T \in \mathcal{L}(E_1, \ldots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \ldots, E_n; F)$, it follows that $T \in \mathcal{M}_n(E_1, \ldots, E_{i-1}, X, E_{i+1}, \ldots, E_n; F)$ if, and only if, $T \in \mathcal{M}_n(E_1, \ldots, E_{i-1}, \Delta(\bar{X}), E_{i+1}, \ldots, E_n; F)$.*

As an application of our previous results we deduce interpolation results on closed surjective n-ideals and operators acting from $\Sigma(\bar{E}_1) \times \cdots \times \Sigma(\bar{E}_n)$ into F, being $\bar{E}_1, \ldots, \bar{E}_n$ arbitrary Banach couples.

Corollary 4.6. *Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $\bar{E}_j = (E_{0j}, E_{1j})$ be a Banach couple, $j = 1, \ldots n$, and let F be a Banach space. Suppose that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j)$, $j = 1, \cdots, n$. For any operator $T \in \mathcal{L}(\Sigma(\bar{E}_1), \ldots, \Sigma(\bar{E}_n); F)$, it holds that $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$ whenever $T \in \mathcal{M}_n(E_{01}, \ldots, E_{0n}; F)$ or $T \in \mathcal{M}_n(E_{11}, \ldots, E_{1n}; F)$.*

Proof. Assume for example that $T \in \mathcal{M}_n(E_{01}, \ldots, E_{0n}; F)$. Taking into account that $T \in \mathcal{L}(\Sigma(\bar{E}_1), E_{02}, \ldots, E_{0n}; F)$, we have the situation that illustrates the next diagram and so, applying Theorem 4.2, we obtain that $T \in \mathcal{M}_n(E_1, E_{02}, \ldots, E_{0n}; F)$:

$$\begin{array}{ll} E_{01}\times E_{02}\times\cdots\times E_{0n} \xrightarrow{T\in\mathcal{M}_n} & \\ & F \Longrightarrow E_1\times E_{02}\times\cdots\times E_{0n} \xrightarrow{T\in\mathcal{M}_n} F. \\ E_{11}\times E_{02}\times\cdots\times E_{0n} \xrightarrow{T} & \end{array}$$

Now we can consider T as an operator of $\mathcal{L}(E_1,\Sigma(\bar{E}_2),E_{03},\ldots,E_{0n};F)$. If we again use Theorem 4.2 according to the situation that shows the following diagram, then it follows that $T\in\mathcal{M}_n(E_1,E_2,E_{03},\ldots,E_{0n};F)$:

$$\begin{array}{ll} E_1\times E_{02}\times E_{03}\times\cdots\times E_{0n} \xrightarrow{T\in\mathcal{M}_n} & \\ & F \Longrightarrow E_1\times E_2\times E_{03}\times\cdots\times E_{0n} \xrightarrow{T\in\mathcal{M}_n} F. \\ E_1\times E_{12}\times E_{03}\times\cdots\times E_{0n} \xrightarrow{T} & \end{array}$$

The proof concludes repeating the same argument. □

Example 2. As a concrete example related to Corollary 4.6, we mention that, for any closed surjective n-ideal $\mathcal{M}_n$, any Banach couples of Lebesgue spaces $(L_{p_0^1}(\Omega_1),L_{p_1^1}(\Omega_1))$, $\ldots,(L_{p_0^n}(\Omega_n),L_{p_1^n}(\Omega_n))$, $1\le p_0^j\ne p_1^j\le\infty\,(j=1,\ldots,n)$, and every bounded multilinear operator T from $L_{p_0^1}(\Omega_1)+L_{p_1^1}(\Omega_1)\times\cdots\times L_{p_0^n}(\Omega_n)+L_{p_1^n}(\Omega_n)$ into any Banach space F, it follows from Corollary 4.6 and (4.1) that if $1\le q^j\le\infty, 0<\eta^j<1$ and $1/p^j=(1-\eta^j)/p_0^j+\eta^j/p_1^j$, then

$$T:L_{p^1,q^1}(\Omega_1)\times\cdots\times L_{p^n,q^n}(\Omega_n)\to F \ \text{ belongs to the } n\text{-ideal } \mathcal{M}_n$$

when, for $k=0$ or $k=1$, the restriction

$$T:L_{p_k^1}(\Omega_1)\times\cdots\times L_{p_k^n}(\Omega_n)\to F \ \text{ belongs to } \mathcal{M}_n.$$

Let us also notice that Sobolev spaces H_p^s and Besov spaces $B_{p,q}^s$, which play an important role in different branches of mathematics, can be obtained as interpolation spaces by the real and complex methods (we refer to [4, Chapter 6] and [39, Chapter 2] for detailed information). Here we just mention as an illustration that if $0<\theta<1$, $s_\theta=(1-\theta)s_0+\theta s_1$, $1/p_\theta=(1-\theta)/p_0+\theta/p_1$ and $1/q_\theta=(1-\theta)/q_0+\theta/q_1$,

$$\big(B_{p_0,q_0}^{s_0},B_{p_1,q_1}^{s_1}\big)_{\theta,p_\theta}=B_{p_\theta,q_\theta}^{s_\theta}$$
$$\text{(for } s_0\ne s_1, p_\theta=q_\theta \text{ and } 1\le p_0,p_1,q_0,q_1\le\infty\text{);}$$
$$\big(H_p^{s_0},H_p^{s_1}\big)_{\theta,q}=B_{p,q}^{s_\theta} \ \text{ and } \ \big(H_{p_0}^s,H_{p_1}^s\big)_{\theta,p_\theta}=H_{p_\theta}^s$$
$$\text{(for } s_0\ne s_1 \text{ and } 1\le p_0,p_1,p,q\le\infty\text{);}$$
$$\big(B_{p_0,q_0}^{s_0},B_{p_1,q_1}^{s_1}\big)_{[\theta]}=B_{p_\theta,q_\theta}^{s_\theta} \ \text{ and } \ \big(H_{p_0}^{s_0},H_{p_1}^{s_1}\big)_{[\theta]}=H_{p_\theta}^{s_\theta}$$
$$\text{(for } s_0\ne s_1, 1<p_0,p_1<\infty \text{ and } 1\le q_0,q_1\le\infty\text{).}$$

Corollary 4.6 complements [24, Theorem 1]. From Theorem 4.5 we now establish an interpolation result in terms of the extreme restriction $T : \Delta(\bar{E}_1) \times \cdots \times \Delta(\bar{E}_n) \to F$, which improves Corollary 4.6 .

Corollary 4.7. *Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $\bar{E}_j$ be a Banach couple, $j = 1, \dots n$, and let F be a Banach space. Suppose that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j), j = 1, \dots, n$. For any $T \in \mathcal{L}(\Sigma(\bar{E}_1), \dots, \Sigma(\bar{E}_n); F)$, it holds that $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ if, and only if, $T \in \mathcal{M}_n(\Delta(\bar{E}_1), \dots, \Delta(\bar{E}_n); F)$.*

Proof. Due to $\mathcal{M}_n$ is an n-ideal and $\Delta(\bar{E}_j) \hookrightarrow E_j, j = 1, \dots, n$, it is clear that if $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$, then $T \in \mathcal{M}_n(\Delta(\bar{E}_1), \dots, \Delta(\bar{E}_n); F)$. Reciprocally, assume that $T \in \mathcal{M}_n(\Delta(\bar{E}_1), \dots, \Delta(\bar{E}_n); F)$. Then, taking into account that $T \in \mathcal{L}(\Sigma(\bar{E}_1), \Delta(\bar{E}_2), \dots, \Delta(\bar{E}_n); F)$, Theorem 4.5 implies that $T \in \mathcal{M}_n(E_1, \Delta(\bar{E}_2), \dots, \Delta(\bar{E}_n); F)$. Applying again Theorem 4.5 to T as an operator of $\mathcal{L}(E_1, \Sigma(\bar{E}_2), \Delta(\bar{E}_3), \dots, \Delta(\bar{E}_n); F)$, it is possible to deduce that $T \in \mathcal{M}_n(E_1, E_2, \Delta(\bar{E}_3), \dots, \Delta(\bar{E}_n); F)$. The proof finishes by a repetition of this reasoning. $\square$

Remark 4.8. By Corollary 2.8, when $\mathcal{I}_1, \dots, \mathcal{I}_n$ are closed surjective linear operator ideals, we know that $\mathcal{M}_n = [\mathcal{I}_1, \dots, \mathcal{I}_n]$ is a closed surjective n-ideal and so Corollaries 4.6 and 4.7 can be applied to $\mathcal{M}_n$. Having in mind Example 1, it allows to establish results on interpolation of concrete closed surjective ideals of multilinear operators. In particular, if $\mathcal{M}_n = [\mathcal{I}, \dots, \mathcal{I}]$ and $\mathcal{I}$ is any of the ideals considered in Example 1, we obtain an extension to the multilinear case of well-known interpolation results established in the setting of linear operator ideals (see for example [4, Theorem 3.8.1(i)], [27, Proposition 1.7], [33, Theorem 2.8] for $\varphi(t) = t^\theta, 0 < \theta < 1$, [6, Proposition 2.2 and Theorem 3.4] and [30, Proposition 3]).

Let us also note that when combining Corollary 4.7 and [32, Corollary 4.4] immediately follows the next result that applies to an n-ideal which is closed, surjective and injective (see definition of injective n-ideal in [32, Definition 2.4]).

Corollary 4.9. *Let $\mathcal{M}_n$ be a closed surjective injective n-ideal. Let $\bar{E}_j, j = 1, \dots n$, and $\bar{F}$ be Banach couples. Assume that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j), j = 1, \dots, n$, and F is of class $\mathcal{C}_J(\eta, \bar{F})$. If $T \in \mathcal{L}(\Sigma(\bar{E}_1), \dots, \Sigma(\bar{E}_n); \Delta(\bar{F}))$, then $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ if, and only if, $T \in \mathcal{M}_n(\Delta(\bar{E}_1), \dots, \Delta(\bar{E}_n); \Sigma(\bar{F}))$.*

By $\mathcal{L}_\mathcal{K}$ we denote the class of compact multilinear operators. Recall that an operator $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *compact* if $T(B_{E_1} \times \cdots \times B_{E_n})$ is a relatively compact subset of F or, equivalently, if for any bounded sequence $\{(x_1^k, \dots, x_n^k)\}_{k \in \mathbb{N}}$ in $E_1 \times \cdots \times E_n$, the sequence $\{T(x_1^k, \dots, x_n^k)\}_{k \in \mathbb{N}}$ has a convergent subsequence in F (see for example [7, p. 308] or [3, p. 3610]). The problem of interpolation of compact bilinear (multilinear) operators has attracted the interest of different authors in recent years (see for instance [21], [22], [23], [34], [12], [35] and references given there). This interest is specially motivated by the fact that compact bilinear (multilinear) operators appear naturally in harmonic analysis (see for example [3], [2] and [28]).

Remark 4.10. It is well-known that $(\mathcal{L}_{\mathcal{K}})_n$ is a closed surjective n-ideal. Therefore, applying Corollary 4.6 or Corollary 4.7 (when $n = 2$) we deduce [22, Theorem 5.1], when $\Gamma_0 = \ell_{q_0}(2^{-\theta_0 m})$ and $\Gamma_1 = \ell_{q_1}(2^{-\theta_1 m})$, and also [22, Corollary 5.5] (see as well [12, Theorem 2.1]). We would like to mention that $(\mathcal{L}_{\mathcal{K}})_n$ is in addition an injective n-ideal and so [32, Corollary 4.2 or Corollary 4.4] (for $n = 2$) allows to obtain a compactness result of Lions-Peetre type when the source space is a product of fixed Banach spaces, namely [22, Theorem 5.3], in case $\Gamma = \ell_q(2^{-\theta m})$, and [22, Corollary 5.6] (see also [12, Theorem 2.2]).

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