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ORIGINAL PAPER

On weakly compact multilinear operators and interpolation

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Abstract
We study weakly compact multilinear operators. We prove a variant of Gantmacher's weak compactness theorem for multilinear operators. We also present Lions–Peetre type results on weak compactness interpolation for multilinear operators. Furthermore, we provide an analogue of Persson's result on interpolation of weakly compact operators under the assumption that the target Banach couple satisfies a certain weakly compact approximation property.

Keywords Multilinear operator · Weak compactness · Interpolation · Factorization · Tensor product

Mathematics Subject Classification 46B70 · 46B28 · 47H60

1 Introduction

In recent years, the theory of multilinear operators has been intensively developed. Interest of multilinear interpolation theorems arose of their important applications in analysis. For instance, we refer to the articles by Bényi and Torres [4], Bényi and Oh [3] for some applications in harmonic analysis, and König's paper [20] on the study of the tensor stability of some operator ideals.

In Memory of Albrecht Pietsch

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We note that the behaviour under interpolation of compactness of bilinear operators has been studied intensively in the last years (see, e.g., [9, 16, 18, 19]). General interpolation results on the stability of compactness of bilinear operators acting on products of Banach spaces generated by abstract methods of interpolation, in the sense of Aronszajn and Gagliardo, were proved in [25]. Moreover, quantitative results in terms of interpolation estimates for the measure of non-compactness of bilinear operators between general spaces obtained by real methods have been proved in [24] (for Banach spaces) and in [6] (for the case of quasi-Banach spaces).

The rich theory of ideals of linear operators (see, e.g., [13, 14, 28]) and their important applications have naturally motivated the study of classes multilinear operators that become ideals in a multilinear setting. For example, with respect to interpolation, results for general ideals of multilinear operators can be found in [22, 23]. These interpolation results apply in particular to the ideal of weakly compact multilinear operators. However, it can be said that the specific study of the interpolation properties of weakly compact multilinear operators has started very recently in the papers by Cobos, Fernández-Cabrera and Martínez [10] and by Cobos and Fernández-Cabrera [8] (see also [11]).

In this work we investigate weakly compact multilinear operators acting on Banach spaces. Next, we provide a brief overview of the main results. In Sect. 2, we review the key definitions and notation used throughout the paper. In Sect. 3 we present a factorization theorem in the multilinear setting, which is a generalization of the famous theorem of Davis, Figiel, Johnson and Pełczyński [12] on the factorization of a weakly compact operator by a reflexive Banach space. Applying this result, we prove a variant of Gantmacher's theorem for a weakly compact multilinear operator in terms of the weak compactness of its generalized adjoint. In Sects. 4 and 5 we study interpolation of weakly compact multilinear operators. In particular, multilinear Lions–Peetre interpolation results for this class of operators are obtained in Sect. 4. Furthermore, in Sect. 5, we establish a result of Persson type on the stability of the weak compactness of interpolated operators, under the assumption that the target Banach couple satisfies a weakly compact approximation property.

2 Background and notation

Throughout the paper we will use standard notation of theory of Banach spaces and operators. Given a Banach space X, we denote by U_X the closed unit ball of X and by X^* and X^{**} the dual and bidual, respectively, of X. The mapping κ_X stands for the canonical embedding of X into its bidual X^{**}. For $1 \leq p < \infty$, the Banach space of all strongly p-summable sequences $x = \{x_k\}_{k\in\mathbb{N}}$ in X is denoted by $\ell_p(X)$ and it is equipped with a natural norm given by

$$\|x\| := \left(\sum_{k=1}^{\infty} \|x_k\|_X^p\right)^{\frac{1}{p}}.$$

If $X_1, \ldots, X_n$ $(n \geq 2)$ are Banach spaces, then $X_1 \times \cdots \times X_n$ is their product endowed with the standard norm $\|x\| := \max\{\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n}\}$, for any $x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. Let $X_1 \otimes \cdots \otimes X_n$ denote the tensor product of the Banach spaces $X_1, \ldots, X_n$ and let π be the projective norm given by

$$\pi(u) := \inf \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\|, \quad u \in X_1 \otimes \cdots \otimes X_n,$$

where the infimum is taken over all possible representations of u of the form $u = \sum_{j=1}^{m} x_1^j \otimes \cdots \otimes x_n^j$, $x_i^j \in X_i$ $(i = 1, \ldots, n)$. The projective tensor product of $X_1, \ldots, X_n$, i.e., the completion of $(X_1 \otimes \cdots \otimes X_n, \pi)$, will be denoted by $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n$.

As usual, if Y is another Banach space, an operator $S \colon X \to Y$ is a continuous linear mapping. Analogously, by an n-linear operator $T \colon X_1 \times \cdots \times X_n \to Y$ we mean a multilinear mapping that is continuous. The Banach space of all n-linear operators $T \colon X_1 \times \cdots \times X_n \to Y$ with the usual norm

$$\|T\| := \sup \left\{ \|T(x_1, \ldots, x_n)\|_Y;\ x_1 \in U_{X_1}, \ldots, x_n \in U_{X_n} \right\},$$

will be represented by $\mathcal{L}(X_1, \ldots, X_n; Y)$. In the case when Y is the scalar field $\mathbb{R}$ or $\mathbb{C}$, we will write $\mathcal{L}(X_1, \ldots, X_n)$.

From now on, the n-linear operator $J \colon X_1 \times \cdots \times X_n \to \mathcal{L}(X_1, \ldots, X_n)^*$ is such that $(Jx)\Phi := \Phi x$, for $x \in X_1 \times \cdots \times X_n$ and $\Phi \in \mathcal{L}(X_1, \ldots, X_n)$. If $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$, following [29], we consider the *generalized adjoint* operator $T^\times \colon Y^* \to \mathcal{L}(X_1, \ldots, X_n)$, defined as $(T^\times y^*)x := y^*(Tx)$, for $y^* \in Y^*$ and $x \in X_1 \times \cdots \times X_n$. It will be useful later to keep in mind that (see [17, Lemma 2.1])

$$(T^\times)^* \circ J = \kappa_Y \circ T. \tag{2.1}$$

Let $X_1, \ldots, X_n$ $(n \geq 2)$ and Y be Banach spaces. Recall that for a given multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ there exists a unique continuous linear operator $\widetilde{T} \colon X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Y$ such that T admits the factorization

$$T \colon X_1 \times \cdots \times X_n \xrightarrow{\otimes} X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \xrightarrow{\widetilde{T}} Y,$$

where $\otimes(x_1, \ldots, x_n) := x_1 \otimes \cdots \otimes x_n$ for all $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$.

Now we are going to recall some basic definitions and specific notation that will be used for our purposes in Sects. 4 and 5. We denote by Φ the set of all functions $\varphi \colon (0, \infty) \times (0, \infty) \to (0, \infty)$, which are non-decreasing in each variable and positively homogeneous (i.e., $\varphi(\lambda s, \lambda t) = \lambda \varphi(s, t)$ for all $\lambda, s, t > 0$). Note that any function *interpolation function* $\varphi \in \Phi$ is continuous and can be extended by continuity to $[0, \infty) \times [0, \infty)$. We denote this extension by the same symbol φ. The simplest examples of interpolation functions are $as + bt$, $\max\{as, bt\}$, and $\min\{as, bt\}$, where $a, b > 0$, and the power functions $s^{1-\theta} t^\theta$ with $0 \leq \theta \leq 1$.

Let $\vec{X} = (X_0, X_1)$ be a *Banach couple*, that is, X_0 and X_1 are two Banach spaces which are continuously embedded in some Hausdorff topological vector space. The sum $X_0 + X_1$ and the intersection $X_0 \cap X_1$ of X_0 and X_1 become Banach spaces when endowed with the norms

$$\|x\|_{X_0+X_1} := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1};\ x = x_0 + x_1,\ x_i \in X_i\}, \quad x \in X_0 + X_1,$$

and

$$\|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}, \quad x \in X_0 \cap X_1.$$

A Banach space X is called an *intermediate space* with respect to $\vec{X} = (X_0, X_1)$ if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$, where "$\hookrightarrow$" means continuous inclusion.

An intermediate space X with respect to $\vec{X} = (X_0, X_1)$ is said to be *of class* $C_K(\theta; \vec{X})$, where $0 < \theta < 1$, if there is a constant $C > 0$ such that for all $t > 0$ and $x \in X$,

$$K(1, t, x; \vec{X}) \le Ct^\theta \|x\|_X.$$

Here $K(s, t, x; \vec{X})$ is defined, for every $s, t > 0$ and any $x \in X_0 + X_1$, as

$$K(s, t, x; \vec{X}) := \inf\{s\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1,\ x_i \in X_i, i = 0, 1\}.$$

The real interpolation space $(X_0, X_1)_{\theta,q}$ and the complex interpolation space $[X_0, X_1]_\theta$ are examples of spaces of class $C_K(\theta; \vec{X})$ (see [5]).

If X is an intermediate space X with respect to $\vec{X} = (X_0, X_1)$, we will consider the following functions, which belong to Φ:

$$\phi_X(s,t) := \sup\{\|x\|_X;\ x \in X_0 \cap X_1, \|x\|_{X_0} \le s, \|x\|_{X_1} \le t\},\ \ s, t > 0,$$
$$\psi_X(s,t) := \sup\{K(s, t, x; \vec{X});\ x \in U_X\},\ \ s, t > 0.$$

For given Banach couples $\vec{X}_1 = (X_0^1, X_1^1), \ldots, \vec{X}_n = (X_0^n, X_1^n)$ and $\vec{Y} = (Y_0, Y_1)$, we put $T \in \mathcal{L}(\vec{X}_1, \ldots, \vec{X}_n; \vec{Y})$ if $T\colon (X_0^1 + X_1^1) \times \cdots \times (X_0^n + X_1^n) \to Y_0 + Y_1$ is an n-linear operator whose restrictions $T\colon X_i^1 \times \cdots \times X_i^n \to Y_i$ are continuous $(i = 0, 1)$.

When $X_1, \ldots, X_n$ and Y are intermediate spaces with respect to the couples $\vec{X}_1, \ldots, \vec{X}_n$ and $\vec{Y}$, respectively, we will write $(X_1, \ldots, X_n; Y) \in \mathcal{L}_\varphi(\vec{X}_1, \ldots, \vec{X}_n; \vec{Y})$ if there exists a function $\varphi \in \Phi$ such that for every $T \in \mathcal{L}(\vec{X}_1, \ldots, \vec{X}_n; \vec{Y})$ the restriction $T\colon X_1 \times \cdots \times X_n \to Y$ is continuous and

$$\|T\|_{X_1\times\cdots\times X_n\to Y} \le \varphi\big(\|T\|_{X_0^1\times\cdots\times X_0^n\to Y_0}, \|T\|_{X_1^1\times\cdots\times X_1^n\to Y_1}\big).$$

Finally let us recall that, in a similar way to the linear case, an n-linear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be *weakly compact* if $T(U_{X_1} \times \cdots \times U_{X_n})$ is a relatively weakly compact subset in Y.

3 Gantmacher's theorem for weakly compact multilinear operators

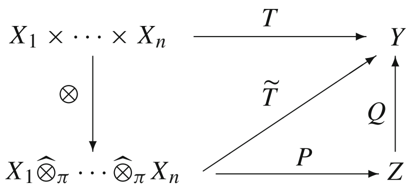
We will derive a version of Gantmacher's theorem for multilinear operators from the following extension to the multilinear setting of the famous factorization result established by Davis, Figiel, Johnson and Pełczyński. Although this factorization result is surely well-known to specialists (see [10, Theorem 5.1] for bilinear operators), we include a possible proof. Let us remind that the linear version of this theorem states that any weakly compact operator $T: X \to Y$ between Banach spaces factors through a reflexive Banach space (see [12]).

Proposition 3.1 *Let $X_1, \ldots, X_n, Y$ be Banach spaces and let $T \in \mathcal{L}(X_1, \ldots X_n; Y)$ be weakly compact. Then there exist a reflexive Banach space Z and an n-linear operator $R: X_1 \times \cdots \times X_n \to Z$ and a linear operator $S: Z \to Y$ such that $T = S \circ R$.*

Proof Similarly as in the case $n = 2$ (see [21, Chapter 8, Section 41.4] or [30, Proposition 2.2]) we have a version of Grothendieck's representation theorem for each $n > 2$, which states that the closed unit ball of $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n$ is the closed convex hull of the set $U_{X_1} \otimes \cdots \otimes U_{X_n}$. Combining this fact with the mentioned factorization $\widetilde{T} \circ \otimes = T$, we conclude that

$$\widetilde{T}\big(U_{X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n}\big) \subset \overline{\mathrm{conv}}(T(U_{X_1} \times \cdots \times U_{X_n})).$$

Since the closed convex hull of a weakly compact subset is also weakly compact (see, e.g., [15, Theorem V.6.4]), the operator $\widetilde{T}: X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Y$ is weakly compact. Applying to the linear operator operator $\widetilde{T}$ the factorization theorem due to Davis, Figiel, Johnson and Pełczyński, there are a reflexive Banach space Z and weakly compact operators $P: X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Z$ and $Q: Z \to Y$ so that $\widetilde{T} = Q \circ P$. Thus, we have the following diagram:



It shows that the operators $P \circ \otimes: X_1 \times \cdots \times X_n \to Z$ and $Q: Z \to Y$ provide a factorization of the multilinear operator T as $T = Q \circ (P \circ \otimes)$. □

From Proposition 3.1 and (2.1) we deduce a variant of Gantmacher's theorem for multilinear operators. This result has been proved for bilinear operators in [10, Lemma 2.3].

Corollary 3.2 *Let $X_1, \ldots, X_n, Y$ be Banach spaces. Then an n-linear operator $T: X_1 \times \cdots \times X_n \to Y$ is weakly compact if and only if the linear operator $T^\times: Y^* \to \mathcal{L}(X_1, \ldots, X_n)$ is weakly compact.*

Proof Assume that $T\colon X_1 \times \cdots \times X_n \to Y$ is weakly compact. Applying Proposition 3.1, there exist a reflexive Banach space Z and operators $R\colon X_1 \times \cdots \times X_n \to Z$ and $S\colon Z \to Y$ with $T = S \circ R$. Then, given any $y^* \in Y^*$ and $x \in X_1 \times \cdots \times X_n$ it holds that

$$\begin{aligned}(T^\times y^*)x = y^*(Tx) = y^*(S(Rx)) = (S^* y^*)(Rx) \\ = (R^\times(S^* y^*))x = [(R^\times \circ S^*)y^*]x\,.\end{aligned}$$

This implies that $T^\times = R^\times \circ S^*$, and $R^\times \circ S^*$ is a weakly compact operator.

Now suppose that the linear operator $T^\times\colon Y^* \to \mathcal{L}(X_1, \ldots, X_n)$ is weakly compact. Gantmacher's theorem (see, e.g., [15, Theorem VI.4.8]) gives that $(T^\times)^*$ is also weakly compact. By (2.1), we have that $\kappa_Y \circ T\colon X_1 \times \cdots \times X_n \to Y^{**}$ is weakly compact. Since κ_Y is a metric injection, it follows that $T\colon X_1 \times \cdots \times X_n \to Y$ is weakly compact. □

4 On weak compactness results of Lions–Peetre type for multilinear operators

In this section we establish multilinear Lions–Peetre interpolation results on weak compactness, namely Theorems 4.2 and 4.4, in a similar line that those on compactness (of bilinear operators) given in [25, Lemmata 2.4 and 2.5]. Before of stating Theorem 4.2, we mention the following characterization of a weakly compact n-linear operator.

Remark 4.1 Given Banach spaces $X_1, \ldots, X_n$ and Y, an n-linear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is weakly compact if and only if, for each $\varepsilon > 0$, there are a Banach space Z and a weakly compact n-linear operator $R\colon X_1 \times \cdots \times X_n \to Z$ such that

$$\Big\|\sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j)\Big\|_Y \le \Big\|\sum_{j=1}^{m} R(x_1^j, \ldots, x_n^j)\Big\|_Z + \varepsilon \sum_{j=1}^{m} \|x_1^j\|_{X_1} \cdots \|x_n^j\|_{X_n},$$

for any $m \in \mathbb{N}$ and all $x_1^j \in X_1, \ldots, x_n^j \in X_n$, $j = 1, \ldots, m$.

The necessity of this fact obviously holds. On the other hand, it is not hard to check that the class of weakly compact n-linear operators is a closed injective n-ideal (see [7, p. 303] or [22, Definition 2.4]) and so the sufficiency of the above assertion is a consequence of [7, Theorem 2.4].

Theorem 4.2 *Let $X_1, \ldots, X_n$ be Banach spaces, and $\vec{Y} = (Y_0, Y_1)$ a Banach couple. Suppose that Y is an intermediate space with respect to $\vec{Y} = (Y_0, Y_1)$ such that $\phi_Y(t, 1) \to 0$ as $t \to 0$. If $T \in \mathcal{L}(X_1, \ldots, X_n; Y_0 \cap Y_1)$ and the operator $T\colon X_1 \times \cdots \times X_n \to Y_0$ is weakly compact, then $T\colon X_1 \times \cdots \times X_n \to Y$ is also weakly compact.*

Proof Take any $\varepsilon > 0$. Fix a sufficiently small $t > 0$ satisfying that

$$\phi_Y(t,1) \le \frac{\varepsilon}{\|T\|_{X_1\times\cdots\times X_n\to Y_1}}.$$

Due to $T\colon X_1\times\cdots\times X_n \to Y_0$ is weakly compact, by Remark 4.1 there exist a Banach space Z and a weakly compact n-linear operator $R\colon X_1\times\cdots\times X_n\to Z$ in such a way that

$$\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{Y_0} \le \Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_Z + t\,\|T\|_{X_1\times\cdots\times X_n\to Y_1}\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n},$$

for all $m\in\mathbb{N}$ and any $x_1^j\in X_1,\dots,x_n^j\in X_n$, $j=1,\dots,m$.

If we define the weakly compact n-linear operator $S\colon X_1\times\cdots\times X_n\to Z$ as

$$S(u_1,\dots,u_n) = \frac{\varepsilon}{t\,\|T\|_{X_1\times\cdots\times X_n\to Y_1}}\,R(u_1,\dots,u_n),$$

and we put $\gamma := \|T\|_{X_1\times\cdots\times X_n\to Y_1}$, then for any $m\in\mathbb{N}$ and arbitrary $x_1^j\in X_1,\dots,x_n^j\in X_n$, $j=1,\dots,m$, it holds that

$$\begin{aligned}
\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_Y &\le \phi_Y\Big(\Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{Y_0}, \Big\|\sum_{j=1}^m T(x_1^j,\dots,x_n^j)\Big\|_{Y_1}\Big)\\
&\le \phi_Y\Big(\Big\|\sum_{j=1}^m R(x_1^j,\dots,x_n^j)\Big\|_Z + t\,\gamma\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}, \gamma\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}\Big)\\
&\le \phi_Y\Big(t\Big[\Big\|\sum_{j=1}^m \Big(\frac{1}{t}R\Big)(x_1^j,\dots,x_n^j)\Big\|_Z + \gamma\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}\Big],\\
&\qquad\Big\|\sum_{j=1}^m \Big(\frac{1}{t}R\Big)(x_1^j,\dots,x_n^j)\Big\|_Z + \gamma\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}\Big)\\
&= \Big[\Big\|\sum_{j=1}^m \Big(\frac{1}{t}R\Big)(x_1^j,\dots,x_n^j)\Big\|_Z + \gamma\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}\Big]\cdot\phi_Y(t,1)\\
&\le \Big\|\sum_{j=1}^m S(x_1^j,\dots,x_n^j)\Big\|_Z + \varepsilon\sum_{j=1}^m \|x_1^j\|_{X_1}\cdots\|x_n^j\|_{X_n}.
\end{aligned}$$

Therefore, the operator $T\colon X_1\times\cdots\times X_n\to Y$ is weakly compact. □

To prove Theorem 4.4 we will use certain ideas inspired by techniques of [25, Lemma 2.5]. We will also use a well-known fact that we recall in the next remark.

Remark 4.3 A subset D of a Banach space Y is relatively weakly compact if and only if, for each $\varepsilon > 0$, there are a Banach space Z and a weakly compact operator $R\colon Z \to Y$ such that

$$D \subset R(U_Z) + \varepsilon U_Y. \tag{4.1}$$

In fact, a result of Grothendieck (see [1, Theorem 3.44]) establishes that a subset D of a Banach space Y is relatively weakly compact if and only if, for every $\varepsilon > 0$, there exists a weakly compact subset W of Y satisfying that

$$D \subset W + \varepsilon U_Y.$$

In that case, it is possible to find a reflexive Banach space Z and a weakly compact operator $R\colon Z \to Y$ such that $W \subset R(U_Z)$ (see [1, Theorem 5.37]) and so

$$D \subset R(U_Z) + \varepsilon U_Y.$$

Of course if, for each $\varepsilon > 0$, (4.1) holds for some Banach space Z and weakly compact operator $R\colon Z \to Y$, then D is relatively weakly compact subset.

Theorem 4.4 *Let $\vec{X}_1 = (X_0^1, X_1^1), \dots, \vec{X}_n = (X_0^n, X_1^n)$ be Banach couples and let Y be a Banach space. Suppose that X_j is an intermediate space with respect to $\vec{X}_j = (X_0^j, X_1^j)$ such that $\psi_{X_j}(t, 1) \to 0$ as $t \to 0$, for $j = 1, \dots, n$. If $T \in \mathcal{L}(X_0^1 + X_1^1, \dots, X_0^n + X_1^n; Y)$ and the restriction $T\colon X_0^1 \times \cdots \times X_0^n \to Y$ is weakly compact, then $T\colon X_1 \times \cdots \times X_n \to Y$ is also weakly compact.*

Proof For simplicity in notation we prove the result for $n = 3$, but an analogous reasoning works for any fixed natural number n.

We may assume without loss of generality the following:

$$\begin{aligned} &\text{the inclusion map } X_j \hookrightarrow X_0^j + X_1^j \text{ has a norm less than or equal to } 1, \\ &\text{for } j = 1, 2, 3, \ \text{and also } \|T\|_{(X_0^1+X_1^1)\times(X_0^2+X_1^2)\times(X_0^3+X_1^3)\to Y} \le 1. \end{aligned} \tag{4.2}$$

If we put $\psi_j(t) := \psi_{X_j}(1, t)$, the assumption about ψ_{X_j} is equivalent to $\dfrac{\psi_j(t)}{t} \to 0$ as $t \to \infty$ $(j = 1, 2, 3)$.

Take $\varepsilon > 0$ arbitrarily. Choose a big enough $t > 0$ for which

$$\max\left\{\frac{\psi_j(t)}{t};\ j = 1, 2, 3\right\} \le \min\left\{\frac{\varepsilon}{2^3 \cdot 2^3}, 1\right\}. \tag{4.3}$$

Since $T\colon X_0^1 \times X_0^2 \times X_0^3 \to Y$ is weakly compact, the subset $T(U_{X_0^1} \times U_{X_0^2} \times U_{X_0^3})$ is relatively weakly compact in Y. By Remark 4.3, there are a Banach space Z and a weakly compact operator $R\colon Z \to Y$ satisfying that

$$T(U_{X_0^1} \times U_{X_0^2} \times U_{X_0^3}) \subset R(U_Z) + \frac{1}{2^3 \cdot t^3} U_Y. \tag{4.4}$$

Given any $(x_1, x_2, x_3) \in U_{X_1} \times U_{X_2} \times U_{X_3}$, it is possible to find decompositions $x_j = a_j + b_j$, with $a_j \in X_0^j$ and $b_j \in X_1^j$, such that

$$\|a_j\|_{X_0^j} + t\|b_j\|_{X_1^j} \leq 2\psi_j(t), \quad j = 1, 2, 3. \tag{4.5}$$

According to (4.4), there exists $z \in U_Z$ so that

$$\Big\| T\Big(\frac{a_1}{\|a_1\|_{X_0^1}}, \frac{a_2}{\|a_2\|_{X_0^2}}, \frac{a_3}{\|a_3\|_{X_0^3}}\Big) - Rz \Big\|_Y \leq \frac{1}{2^3 \cdot t^3},$$

and, by (4.5),

$$\begin{aligned}
&\|T(a_1, a_2, a_3) - R(\|a_1\|_{X_0^1}\|a_2\|_{X_0^2}\|a_3\|_{X_0^3}\, z)\|_Y \\
&\quad \leq \frac{1}{2^3 \cdot t^3}\|a_1\|_{X_0^1}\|a_2\|_{X_0^2}\|a_3\|_{X_0^3} \leq \frac{\psi_1(t)}{t}\frac{\psi_2(t)}{t}\frac{\psi_3(t)}{t} \leq \frac{\varepsilon}{2^3 \cdot 2^3}.
\end{aligned} \tag{4.6}$$

On the other hand, it follows that

$$\begin{aligned}
T(x_1, x_2, x_3) &= T(a_1, x_2, x_3) + T(b_1, x_2, x_3) \\
&= T(a_1, a_2, x_3) + T(a_1, b_2, x_3) + T(b_1, x_2, x_3) \\
&= T(a_1, a_2, a_3) \\
&\quad + T(a_1, a_2, b_3) + T(a_1, b_2, x_3) + T(b_1, x_2, x_3).
\end{aligned}$$

Taking into account that $a_2 = x_2 - b_2$ and $a_1 = x_1 - b_1$, it holds for the second line on the right hand side of the last equality that

$$\begin{aligned}
&T(a_1, a_2, b_3) + T(a_1, b_2, x_3) + T(b_1, x_2, x_3) \\
&\quad = T(a_1, x_2, b_3) - T(a_1, b_2, b_3) + T(a_1, b_2, x_3) + T(b_1, x_2, x_3) \\
&\quad = T(x_1, x_2, b_3) - T(b_1, x_2, b_3) - T(x_1, b_2, b_3) + T(b_1, b_2, b_3) \\
&\qquad + T(x_1, b_2, x_3) - T(b_1, b_2, x_3) + T(b_1, x_2, x_3).
\end{aligned}$$

Keeping in mind the above and arranging addends,

$$\begin{aligned}
T(x_1, x_2, x_3) &= T(a_1, a_2, a_3) + T(a_1, a_2, b_3) + T(a_1, b_2, x_3) + T(b_1, x_2, x_3) \\
&= T(a_1, a_2, a_3) + T(b_1, x_2, x_3) + T(x_1, b_2, x_3) + T(x_1, x_2, b_3) \\
&\quad - T(b_1, b_2, x_3) - T(b_1, x_2, b_3) - T(x_1, b_2, b_3) + T(b_1, b_2, b_3).
\end{aligned}$$

Then

$$\begin{aligned}\|T(x_1,x_2,x_3) &- R(\|a_1\|_{X_0^1}\|a_2\|_{X_0^2}\|a_3\|_{X_0^3}\,z)\|_Y \\ &\le \|T(a_1,a_2,a_3) - R(\|a_1\|_{X_0^1}\|a_2\|_{X_0^2}\|a_3\|_{X_0^3}\,z)\|_Y \\ &\quad + \|T(b_1,x_2,x_3)\|_Y + \|T(x_1,b_2,x_3)\|_Y + \|T(x_1,x_2,b_3)\|_Y \\ &\quad + \|T(b_1,b_2,x_3)\|_Y + \|T(b_1,x_2,b_3)\|_Y + \|T(x_1,b_2,b_3)\|_Y \\ &\quad + \|T(b_1,b_2,b_3)\|_Y.\end{aligned}$$

Using (4.2), (4.5) and (4.6), we have that

$$\begin{aligned}&\|T(x_1,x_2,x_3) - R(\|a_1\|_{X_0^1}\|a_2\|_{X_0^2}\|a_3\|_{X_0^3}\,z)\|_Y \le \frac{\varepsilon}{2^3\cdot 2^3} \\ &\quad + 2\Big(\frac{\psi_1(t)}{t} + \frac{\psi_2(t)}{t} + \frac{\psi_3(t)}{t}\Big) \\ &\quad + 2^2\Big(\frac{\psi_1(t)}{t}\frac{\psi_2(t)}{t} + \frac{\psi_1(t)}{t}\frac{\psi_3(t)}{t} + \frac{\psi_2(t)}{t}\frac{\psi_3(t)}{t}\Big) \\ &\quad + 2^3\,\frac{\psi_1(t)}{t}\frac{\psi_2(t)}{t}\frac{\psi_3(t)}{t} \\ &\le \frac{\varepsilon}{2^3\cdot 2^3} + 3\frac{\varepsilon}{2^2\cdot 2^3} + 3\frac{\varepsilon}{2\cdot 2^3} + \frac{\varepsilon}{2^3} \le (1+3+3+1)\frac{\varepsilon}{2^3} = 2^3\frac{\varepsilon}{2^3} = \varepsilon.\end{aligned}$$

Note that the number of addends on the right hand side of the first inequality of the last string of inequalities corresponds, respectively, to the binomial coefficients $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$. An analogous model occurs for an arbitrary natural number n, but then involving k-combinations (without repetition) from a set of n elements.

Therefore, we have proved that

$$T(U_{X_1}\times U_{X_2}\times U_{X_3}) \subset R(U_{\widehat{Z}}) + \varepsilon U_Y,$$

where $\widehat{Z}$ is the space Z endowed with the norm $\dfrac{1}{2^3\psi_1(t)\psi_2(t)\psi_3(t)}\|\cdot\|_Z$. It implies that $T\colon X_1\times X_2\times X_3 \to Y$ is a weakly compact 3-linear operator. □

5 A variant of Persson's result related to weak compactness

The following definition is, in a certain way, motivated by a well-known approximation property introduced by Persson [27] that has also been useful in the study of interpolation properties of compact bilinear operators (see [16, 18]).

We say that a Banach couple $\vec{Y} = (Y_0, Y_1)$ has the *property* $\mathcal{WP}_1$ if for any weakly compact subset $D \subset Y_0$ there is a constant $C > 0$ such that, for each $\varepsilon > 0$, there exists an operator $P\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ satisfying the following conditions:

(i) $\|P\|_{Y_i\to Y_i} \le C,\ \ i = 0, 1,$

(ii) $\sup_{y\in D}\|y - Py\|_{Y_0} \le \varepsilon.$

We note that when $\vec{Y} = (Y_0, Y_1)$ satisfies the condition $(\mathfrak{h})$ (see [18, p. 1668]) and Y_0 has Schur property (i.e., if the weakly convergent sequences in Y_0 are norm convergent), then $\vec{Y} = (Y_0, Y_1)$ has $\mathcal{WP}_1$.

Under the assumption that the Banach couple in the target has $\mathcal{WP}_1$, we next establish a one-sided interpolation result for weakly compact n-linear operators.

Theorem 5.1 *Let $\vec{X}_1 = (X_0^1, X_1^1), \dots, \vec{X}_n = (X_0^n, X_1^n)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Assume that X_j is an intermediate space of class $C_K(\theta_j, \vec{X}_j)$, $j = 1, \dots, n$ $(0 < \theta_j < 1)$, and let Y be an intermediate space with respect to $\vec{Y}$ in such a way that $(X_1, \dots, X_n; Y) \in \mathcal{L}_\varphi(\vec{X}_1, \dots, \vec{X}_n; \vec{Y})$ with $\varphi(t, 1) \to 0$ as $t \to 0$. For any $T \in \mathcal{L}(\vec{X}_1, \dots, \vec{X}_n; \vec{Y})$ such that the restriction $T\colon X_0^1 \times \cdots \times X_0^n \to Y_0$ is weakly compact, it holds that $T\colon X_1 \times \cdots \times X_n \to Y$ is also weakly compact whenever $\vec{Y}$ has $\mathcal{WP}_1$.*

Proof Consider the subset $D = \overline{T(U_{X_0^1} \times \cdots \times U_{X_0^n})}$, which is weakly compact in Y_0. There is a constant $C > 0$ such that, for any natural number n, there exists $P_n\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ such that $\|P_n\|_{Y_i \to Y_i} \le C$ $(i = 0, 1)$ and $\sup_{y \in D} \|y - P_n y\|_{Y_0} \le 1/n$. Hence,

$$\|T - P_n \circ T\|_{X_0^1 \times \cdots \times X_0^n \to Y_0} \le 1/n,$$
$$\|T - P_n \circ T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1} \le (1 + C)\|T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1}.$$

Therefore,

$$\begin{aligned}
&\|T - P_n \circ T\|_{X_1 \times \cdots \times X_n \to Y} \\
&\quad \le \varphi(\|T - P_n \circ T\|_{X_0^1 \times \cdots \times X_0^n \to Y_0}, \|T - P_n \circ T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1}) \\
&\quad \le \varphi(1/n, (1 + C)\|T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1}) \\
&\quad = (1 + C)\|T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1}\, \varphi\Big(\frac{1}{n(1 + C)\|T\|_{X_1^1 \times \cdots \times X_1^n \to Y_1}}, 1\Big) \to 0, \qquad (5.1)
\end{aligned}$$

as $n \to \infty$.

Since $P_n \circ T \in \mathcal{L}(X_0^1 + X_1^1, \dots, X_0^n + X_1^n; Y)$ and weakly compact n-linear operators are a closed surjective n-ideal in the sense of [23], applying [23, Corollary 4.6] we have that $P_n \circ T\colon X_1 \times \cdots \times X_n \to Y$ is weakly compact. It follows from (5.1) that $T\colon X_1 \times \cdots \times X_n \to Y$ is also weakly compact. □

We also define the next property, which is stronger than $\mathcal{WP}_1$. A Banach couple $\vec{Y} = (Y_0, Y_1)$ has the *property* $\mathcal{WP}_2$ if there is a constant $C > 0$ such that, for all weakly compact subset $D \subset Y_0$ and any $\varepsilon > 0$, there exists an operator $P\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ satisfying the following conditions:

(i') $\|P\|_{Y_0 + Y_1 \to Y_i} \le C$, $i = 0, 1$,
(ii) $\sup_{y \in D} \|y - Py\|_{Y_0} \le \varepsilon$.

Notice that the constant C in the definition of property $\mathcal{WP}_2$ is the same independently of the weakly compact subset $D \subset Y_0$. Moreover, condition (i') clearly implies

(i) of the definition of $\mathcal{WP}_1$. Thus, if $\vec{Y} = (Y_0, Y_1)$ has $\mathcal{WP}_2$, then the Banach couple has also property $\mathcal{WP}_1$.

On the other hand, when Y_0 and Y_1 are the same Banach space Y, with Y having the weakly compact approximation property in the sense of [26, p. 1125] (see also [2]), the couple $\vec{Y} = (Y, Y)$ has $\mathcal{WP}_2$. We refer to the articles [26, 31], which show examples of Banach spaces with the weakly compact approximation property.

Next we show that if a Banach couple $\vec{Y} = (Y_0, Y_1)$ has $\mathcal{WP}_2$, the Banach couple formed by the corresponding spaces $\ell_p(Y_i)$ of strongly p-summable sequences ($i = 0, 1$) also enjoys this property.

Proposition 5.2 *Let $\vec{Y} = (Y_0, Y_1)$ be a Banach couple satisfying $\mathcal{WP}_2$. Then, the Banach couple $(\ell_p(Y_0), \ell_p(Y_1))$ has also $\mathcal{WP}_2$ for any $1 \le p < \infty$.*

Proof For each $n \in \mathbb{N}$, we consider the natural projection $R_n \colon \ell_p(Y_0) + \ell_p(Y_1) \to Y_0 + Y_1$ given, for $y = y^0 + y^1$ with $y^i = \{y_k^i\}_{k\in\mathbb{N}} \in \ell_p(Y_i)$ $(i = 0, 1)$, by

$$R_n(y) := y_n^0 + y_n^1$$

and the inclusion map $S_n \colon Y_0 \cap Y_1 \to \ell_p(Y_0) \cap \ell_p(Y_1)$ defined, for $y \in Y_0 \cap Y_1$, as

$$S_n(y) := \{\delta_k^n y\}_{k\in\mathbb{N}},$$

where δ_k^n is the Kronecker delta.

Take a weakly compact subset $\Lambda \subset \ell_p(Y_0)$ and fix $\varepsilon > 0$. The restriction of R_n to $\ell_p(Y_0)$ acts from $\ell_p(Y_0)$ into Y_0 continuously. Then, for each $n \in \mathbb{N}$, the set $R_n(\Lambda)$ is weakly compact in Y_0. By assumption, there exists $C > 0$ (which is independent of n) and an operator $P_n \colon Y_0 + Y_1 \to Y_0 \cap Y_1$ such that

$$\|P_n\|_{Y_0+Y_1\to Y_i} \le C, \quad i = 0, 1. \tag{5.2}$$

and

$$\sup_{y^0\in\Lambda} \|R_n y^0 - P_n(R_n y^0)\|_{Y_0} \le \frac{\varepsilon}{2^{n/p}}. \tag{5.3}$$

Now consider the linear mapping $\otimes P$ defined on $\ell_p(Y_0) + \ell_p(Y_1)$ by

$$\otimes P y := \sum_{n=1}^{\infty} S_n(P_n(R_n y)), \quad y \in \ell_p(Y_0) + \ell_p(Y_1).$$

We claim that $\otimes P \colon \ell_p(Y_0) + \ell_p(Y_1) \to \ell_p(Y_0) \cap \ell_p(Y_1)$ is a bounded operator. In fact, if $y = y^0 + y^1$, with $y^i = \{y_k^i\}_{k\in\mathbb{N}} \in \ell_p(Y_i)$, $i = 0, 1$, keeping in mind (5.2), we have

$$\begin{aligned}\| \otimes Py\|_{\ell_p(Y_0)\cap\ell_p(Y_1)} &= \max_{i=0,1}\Big\{\Big(\sum_{n=1}^{\infty}\Big\| P_n(R_n y)\Big\|_{Y_i}^p\Big)^{1/p}\Big\}\\ &\le \max_{i=0,1}\Big\{\Big(\sum_{n=1}^{\infty}\Big(\|P_n\|_{Y_0+Y_1\to Y_i}\|y_n^0+y_n^1\|_{Y_0+Y_1}\Big)^p\Big)^{1/p}\Big\}\\ &\le C\Big(\sum_{n=1}^{\infty}(\|y_n^0\|_{Y_0}+\|y_n^1\|_{Y_1})^p\Big)^{1/p}\\ &\le C\left(\Big(\sum_{n=1}^{\infty}\|y_n^0\|_{Y_0}^p\Big)^{1/p}+\Big(\sum_{n=1}^{\infty}\|y_n^1\|_{Y_1}^p\Big)^{1/p}\right)\\ &= C\Big(\|y^0\|_{\ell_p(Y_0)}+\|y^1\|_{\ell_p(Y_1)}\Big).\end{aligned}$$

Taking infimum over all representations $y = y^0 + y^1$ one obtains

$$\| \otimes Py\|_{\ell_p(Y_0)\cap\ell_p(Y_1)} \le C\|y\|_{\ell_p(Y_0)+\ell_p(Y_1)}$$

and this proves the claim. Clearly, the above estimate yields

$$\| \otimes P\|_{\ell_p(Y_0)+\ell_p(Y_1)\to\ell_p(Y_i)} \le C, \quad i = 0, 1.$$

To conclude we note that, given any $y^0 \in \Lambda$, it follows from (5.3) that

$$\begin{aligned}\|y^0 - \otimes Py^0\|_{\ell_p(Y_0)} &= \Big\|y^0 - \sum_{n=1}^{\infty} S_n(P_n(R_n y^0))\Big\|_{\ell_p(Y_0)}\\ &= \Big(\sum_{n=1}^{\infty}\|R_n y^0 - P_n(R_n y^0)\|_{Y_0}^p\Big)^{1/p} \le \Big(\sum_{n=1}^{\infty}\frac{\varepsilon^p}{2^n}\Big)^{1/p} = \varepsilon,\end{aligned}$$

and whence

$$\sup_{y^0\in\Lambda}\|y^0 - \otimes Py^0\|_{\ell_p(Y_0)} \le \varepsilon.$$

This completes the proof. □

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