Journal of Physics: Conference Series **360** (2012) 012010 doi:10.1088/1742-6596/360/1/012010

3d gravity and quantum deformations: a Drinfel'd double approach

A Ballesteros[1], F J Herranz[1] and C. Meusburger[2]

[1] Departamento de Física, Universidad de Burgos, 09001 Burgos, Spain

[2] Department Mathematik, FAU Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany

E-mail: angelb@ubu.es, fjherranz@ubu.es, catherine.meusburger@math.uni-erlangen.de

Abstract. The constant curvature spacetimes of 3d gravity and their associated symmetry algebras are shown to arise from the 6d Drinfel'd double that underlies the two-parametric 'hybrid' quantum deformation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Moreover, the quantum deformation supplies the additional structures (star structure and pairing) that enter in the Chern–Simons formulation of the theory, thus establishing a direct link between quantum $\mathfrak{sl}(2,\mathbb{R})$ algebras and 3d gravity models. In this approach the flat spacetimes and Newtonian models arise as Lie algebra contractions that are governed by two dimensionful $\mathfrak{sl}(2,\mathbb{R})$ deformation parameters, which are directly related to the cosmological constant and to the speed of light.

1. Introduction

Several arguments suggest that the low energy limit of a quantum theory of gravity could be invariant under a certain quantum deformation of the Poincaré group (see [1] and references therein). In the particular case of 3d gravity this viewpoint is strongly supported by the fact that Poisson–Lie groups (which are just the semiclassical limit of quantum groups) arise as relevant symmetries of the classical theory [2, 3, 4]. There is evidence [6, 4] suggesting that the relevant symmetries in 3d quantum gravity are certain quantum doubles associated with the isometry groups of Lorentzian and Euclidean constant curvature spacetimes (see [5, 6, 7, 8, 9]).

Nevertheless, so far only quantum deformations based on a single deformation parameter have been explored in this context. However, multiparametric quantum deformations do exist (see [10] for a generic discussion and [11] for the specific quantum $gl(2)$ classification) and they are canonically associated to multiparametric classical Drinfel'd doubles.

In this contribution we show that the two-parametric (η, z) quantum $\mathfrak{sl}(2,\mathbb{R})$ approach presented in [12] provides a common framework for the Chern–Simons formulation of 3d gravity, since all the possible constant curvature spacetimes and isometry groups relevant in the theory arise naturally from the classical Drinfel'd double that characterizes the semiclassical limit of this 'hybrid' deformation. In fact, the two deformation parameters have a direct physical meaning: η corresponds to the cosmological constant, while z is related to the speed of light. Moreover, both the nonrelativistic [13] and the flat limit can be obtained as Lie algebra contractions.

2. Spacetimes and symmetries in 3d gravity

Due to the absence of local gravitational degrees of freedom, any solution of the 3d vacuum Einstein equations is locally isometric to one of six standard spacetimes (see Table 1), where the

Table 1. Constant curvature spacetimes and isometry groups in 3d gravity.

	$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$
Lorentzian	$\mathbf{dS}^{2+1} = SO(3,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{dS}^{2+1}) = SO(3,1)$	$\mathbf{M}^{2+1} = ISO(2,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{M}^{2+1}) = ISO(2,1)$	$\mathbf{AdS}^{2+1} = SO(2,2)/SO(2,1)$ $\mathrm{Isom}(\mathbf{AdS}^{2+1}) = SO(2,2)$
Euclidean	$\mathbf{S}^3 = SO(4)/SO(3)$ $\mathrm{Isom}(\mathbf{S}^3) = SO(4)$	$\mathbf{E}^3 = ISO(3)/SO(3)$ $\mathrm{Isom}(\mathbf{E}^3) = ISO(3)$	$\mathbf{H}^3 = SO(3,1)/SO(3)$ $\mathrm{Isom}(\mathbf{H}^3) = SO(3,1)$

cosmological constant is Λ. For Euclidean signature these are the three-sphere $\mathbf{S}^3$ ($\Lambda > 0$), 3d hyperbolic space $\mathbf{H}^3$ ($\Lambda < 0$) and 3d Euclidean space $\mathbf{E}^3$ ($\Lambda = 0$). The Lorentzian cases are the 3d dS space $\mathbf{dS}^{2+1}$ ($\Lambda > 0$), AdS space $\mathbf{AdS}^{2+1}$ ($\Lambda < 0$) and Minkowski space $\mathbf{M}^{2+1}$ ($\Lambda = 0$).

Moreover, 3d gravity can be formulated as a Chern–Simons (CS) gauge theory [14, 15] with gauge group given by the isometry group of the associated standard spacetime. This requires the choice of a symmetric, non-degenerate, Ad-invariant bilinear form $\langle\, , \,\rangle$ on the corresponding Lie algebra. The Lie algebras of the six isometry groups of 3d gravity can be written as [15]

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = \lambda \epsilon_{abc} J^c, \tag{1}$$

where $a = 0, 1, 2$ and indices are raised with either the 3d Minkowskian or the 3d Euclidean metric. Note that both the signature and the cosmological constant arise as structure constants ($\lambda = \Lambda$ for Euclidean signature and $\lambda = -\Lambda$ for the Lorentzian one). As shown in [15] (see also [8, 16]), the relevant bilinear form for the Chern-Simons formulation of 3d gravity is

$$\langle J_a, P_b \rangle = g_{ab}, \qquad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0, \tag{2}$$

where g_{ab} is either the Euclidean or the Minkowskian metric. With this bilinear form, the Einstein-Hilbert action for 3d gravity can be written as a Chern-Simons action by combining the triad e and the spin connection ω into a CS gauge field $A = e^a P_a + \omega^a J_a$. This is a connection with values in the Lie algebra (1), which has to be equipped with the star structure

$$J_a^* = -J_a, \qquad P_a^* = -P_a. \tag{3}$$

As we shall see in the sequel (see [12] for details), all the spacetimes and the symmetry algebras (1) of 3d gravity, together with the specific bilinear form (2) can be straightforwardly related to a two-parametric quantum deformation of the real Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$.

3. The 'hybrid' quantum deformation of $\mathfrak{sl}(2,\mathbb{R})$

The $\mathfrak{sl}(2,\mathbb{R})$ Lie bracket and undeformed coproduct $\Delta_{(0)} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R}) \otimes \mathfrak{sl}(2,\mathbb{R})$ are

$$[J_3, J_\pm] = \pm 2 J_\pm, \qquad [J_+, J_-] = J_3, \tag{4}$$

$$\Delta_{(0)}(J_i) = J_i \otimes 1 + 1 \otimes J_i, \qquad i = 3, \pm, \tag{5}$$

and we assume the star structure $J_3^* = -J_3, J_\pm^* = -J_\pm$. A quantum $\mathfrak{sl}(2,\mathbb{R})$ algebra is a power series deformation of the previous commutation rules and coproduct map (see [10]). If we expand the deformed coproduct map Δ_φ in the quantum deformation parameter(s) φ we get $\Delta_\varphi = \sum_{k=0}^{\infty} \varphi^k \, \delta_{(k)} = \Delta_{(0)} + \varphi\, \delta_{(1)} + o[\varphi^2]$ and each quantum deformation is uniquely characterized (see [10]) by the skew-symmetric part δ of the first-order deformation $\delta_{(1)}$. As this cocommutator δ always generates a Lie bialgebra structure, the classification of quantum deformations of a given Lie algebra reduces to the classification of its Lie bialgebra structures. In the case of $\mathfrak{sl}(2,\mathbb{R})$ such classification is well-known (see [10, 11, 17]).

Journal of Physics: Conference Series **360** (2012) 012010 doi:10.1088/1742-6596/360/1/012010

In particular, the so-called 'hybrid' quantum algebra $\mathfrak{sl}_{\eta,z}(2,\mathbb{R})$ is a two-parametric quantum deformation generated by the two-parametric classical r-matrix

$$r = r_\eta + r_z = \eta\, J_+ \wedge J_- + \frac{z}{2} J_3 \wedge J_+, \tag{6}$$

which is a solution of the modified classical Yang–Baxter equation. The associated δ is

$$\delta(J_+) = \eta J_+ \wedge J_3, \qquad \delta(J_3) = z J_3 \wedge J_+, \qquad \delta(J_-) = \eta J_- \wedge J_3 + z J_- \wedge J_+. \tag{7}$$

As we will show in the following, in the context of 3d gravity both parameters η and z play essential roles and have a clear physical interpretation.

4. 3d AdS gravity from the 'hybrid' Drinfel'd double

Each coboundary Lie bialgebra gives rise to a Drinfel'd double Lie algebra [17, 18]. In our case the classical Drinfel'd double $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)$ is the 6d Lie algebra spanned by $\{J_i\}_{i=3,\pm}$ and its dual basis $\{j^i\}_{i=3,\pm}$, with Lie brackets given by (4) and dual brackets induced from δ (7)

$$[j^3, j^+] = -\eta j^+ + z j^3, \qquad [j^3, j^-] = -\eta j^-, \qquad [j^+, j^-] = -z j^-. \tag{8}$$

The remaining brackets are the "crossed" or "mixed" Lie brackets, which read

$$\begin{array}{lll} [J_3, j^3] = z J_+, & [J_3, j^+] = -z J_3 - 2j^+, & [J_3, j^-] = 2j^-, \\ [J_+, j^3] = -\eta J_+ - j^-, & [J_+, j^+] = \eta J_3 + 2j^3, & [J_+, j^-] = 0, \\ [J_-, j^3] = -\eta J_- + j^+, & [J_-, j^+] = -z J_-, & [J_-, j^-] = \eta J_3 + z J_+ - 2j^3. \end{array} \tag{9}$$

The pairing between the basis $\{J_i\}_{i=3,\pm}$ and the dual basis $\{j^i\}_{i=3,\pm}$ is

$$\langle J_i, j^k\rangle = \langle j^k, J_i\rangle = \delta_i^k, \qquad \langle J_i, J_k\rangle = \langle j^i, j^k\rangle = 0, \qquad i,k = 3,\pm. \tag{10}$$

If both deformation parameters are real, $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)$ inherits the star structure

$$J_3^* = -J_3, \qquad J_\pm^* = -J_\pm, \qquad j^{3*} = -j^3, \qquad j^{\pm *} = -j^\pm. \tag{11}$$

Now we introduce in $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)$ the 'Chern–Simons basis' [12] J_a, P_a $(a = 0,1,2)$ through

$$\begin{array}{lll} J_0 = \frac{1}{2}(J_+ - J_-), & J_1 = \frac{z}{2} J_3, & J_2 = \frac{z}{2}(J_+ + J_-), \\ P_0 = \eta(J_+ + J_-) - \frac{z}{2} J_3 + j^- - j^+, & P_1 = -z^2 J_+ + 2z j^3, & P_2 = \eta z (J_+ - J_-) + \frac{z^2}{2} J_3 + z(j^+ + j^-), \end{array} \tag{12}$$

and we find that the Lie brackets of $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)$ in the CS basis are just

$$\begin{array}{lll} [J_0, J_1] = -J_2, & [J_0, J_2] = J_1, & [J_1, J_2] = z^2 J_0, \\ [J_0, P_0] = 0, & [J_0, P_1] = -P_2, & [J_0, P_2] = P_1, \\ [J_1, P_0] = P_2, & [J_1, P_1] = 0, & [J_1, P_2] = z^2 P_0, \\ [J_2, P_0] = -P_1, & [J_2, P_1] = -z^2 P_0, & [J_2, P_2] = 0, \\ [P_0, P_1] = -4\eta^2 J_2, & [P_0, P_2] = 4\eta^2 J_1, & [P_1, P_2] = 4\eta^2 z^2 J_0. \end{array} \tag{13}$$

This means that $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta) \simeq \mathfrak{so}(2,2)$, provided that η, z are non-zero real numbers. If J_0, J_b, P_0, P_b, $(b = 1,2)$ are interpreted as the generators of rotations, boosts, time translations and spatial translations, then $\mathfrak{so}(2,2)$ is the symmetry algebra of the 3d AdS space, in which J_0, J_1, J_2 span the $\mathfrak{so}(2,1)$ subalgebra and the AdS spacetime is $\mathbf{AdS}^{2+1} = SO(2,2)/SO(2,1)$.

Thus, the deformation parameters of the hybrid deformation $\mathfrak{sl}_{\eta,z}(2,\mathbb{R})$ are directly related to the cosmological constant $\Lambda = -\lambda$ and the speed of light c through $\lambda = 4\eta^2$ and $c^2 = 1/z^2$, which means that the signature of the metric is $g = \mathrm{diag}(-1, z^2, z^2)$. Moreover, by using the relations (12), we find the star structure (3) and the pairing (10) (z^2 can be rescaled to 1)

$$\langle J_0, P_0\rangle = -1, \qquad \langle J_1, P_1\rangle = z^2, \qquad \langle J_2, P_2\rangle = z^2. \tag{14}$$

Journal of Physics: Conference Series **360** (2012) 012010 doi:10.1088/1742-6596/360/1/012010

5. Analytic continuation and contractions

The previous results can be straightforwardly generalized to other signatures and values of the cosmological constant by considering imaginary values of η, z as well as the limits $\eta, z \to 0$ (which can be understood, respectively, as the "flat" and "non-relativistic" Inönü–Wigner contractions). In this way we obtain nine 3d homogenous spaces $\mathbf{X}_{\eta,z}$, which are a subfamily of the Cayley–Klein spaces [19] defined through $\mathbf{X}_{\eta,z} = \langle \mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)\rangle / \langle J_0, J_1, J_2\rangle$. In this way, the quantum algebra $\mathfrak{sl}_{\eta,z}(2,\mathbb{R})$ gives rise in a unified way to the following nine homogeneous spaces:

• The Riemannian spaces for $z \in \mathrm{i}\mathbb{R}^*$: the three-sphere, the 3d hyperbolic space and the 3d Euclidean space, with $\Lambda = \lambda = 4\eta^2 = \pm 1/R^2$, where R is the radius ($R \to \infty$ for $\mathbf{E}^3$).

• The Lorentzian spaces for $z \in \mathbb{R}^*$: 3d AdS, dS and Minkowski space. Now $\Lambda = -\lambda$ with $\lambda = 4\eta^2 = \pm 1/\tau^2$, where τ is the (time) universe radius ($\tau \to \infty$ for $\mathbf{M}^{2+1}$).

• The non-relativistic or Galilean limits of 3d gravity [13] when $z = 0$ ($c \to \infty$): the two Newton–Hooke spacetimes and the Galilean one, both with degenerate pairing and metrics.

6. Concluding remarks

We emphasize that our approach is obtained from the first order (i.e. the one containing the semiclassical information) of the 'hybrid' quantum deformation. Therefore, the construction and analysis of the complete quantum counterpart of $\mathcal{D}_{\eta,z}(\mathfrak{sl}(2,\mathbb{R}),\delta)$ is worth further investigation. The explicit construction of the quantum algebra $\mathfrak{sl}_{\eta,z}(2,\mathbb{R})$ can be traced back to [11]. A nonlinear change of basis connecting this deformation with the Drinfel'd–Jimbo one [20, 21] was found in [22]. Note also that the dual viewpoint is equally tractable: the 'hybrid' Poisson–Lie group structure on $SL(2)$ can be obtained through the Sklyanin bracket for the classical r-matrix (6), and its quantization gives rise to the 'hybrid' quantum $SL(2)$ group [11, 23].

Acknowledgments

This work was partially supported by the Spanish MICINN under grant MTM2010-18556 and by the DFG Emmy-Noether fellowship ME 3425/1-1.

References

[1] Amelino-Camelia G, Smolin L and Starodubtsev A 2004 *Class. Quantum Grav.* **21** 3095
[2] Fock V V and Rosly A A 1992 Preprint ITEP-72-92; *ibid* 1999 *Am. Math. Soc. Transl.* **191** 67
[3] Alekseev A Y and Malkin A Z 1995 *Commun. Math. Phys.* **169** 99
[4] Meusburger C and Schroers B J 2008 *J. Math. Phys.* **49** 083510
[5] Podleś P and Woronowicz S L 1990 *Commun. Math. Phys.* **130** 381
[6] Buffenoir E, Noui K and Roche P 2002 *Class. Quant. Grav.* **19** 4953
[7] Meusburger C and Schroers B J 2003 *Adv. Theor. Math. Phys.* **7** 1003
[8] Meusburger C and Schroers B J 2009 *Nucl. Phys. B* **806** 462
[9] Majid S and Schroers B J 2009 *J. Phys. A: Math. Theor.* **42** 425402
[10] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[11] Ballesteros A, Herranz F J and Parashar P 1999 *J. Phys. A: Math. Gen.* **32** 2369
[12] Ballesteros A, Herranz F J and Meusburger C 2010 *Phys. Lett. B* **687** 375
[13] Schroers B J and Papageorgiou G 2009 *JHEP* **11** 009
[14] Achucarro and Townsend P K 1986 *Phys. Lett. B* **180** 89
[15] Witten E 1988 *Nucl. Phys. B* **311** 46
[16] Bonzom V and Livine E R 2008 *Class. Quantum Grav.* **25** 195024
[17] Gomez X 2000 *J. Math. Phys.* **41** 4939
[18] Ballesteros A, Celeghini E and del Olmo M A 2005 *J. Phys. A: Math. Gen.* **38** 3909
[19] Ballesteros A, Herranz F J, Ragnisco O and Santander M 2008 *Int. J. Theor. Phys.* **47** 649
[20] Drinfel'd V G 1987 *Quantum Groups*, Proc. Int. Cong. Math. Berkeley 1986, ed A V Gleason (Providence: AMS) p 798
[21] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; *ibid* 1986 **11** 247
[22] Aneva B L, Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2001 *J. Math. Phys.* **42** 1236
[23] Kupershmidt B A 1994 *J. Phys. A: Math. Gen.* **27** L47